BEYOND $H_0$ AND $q_0$: COSMOLOGY IS NO LONGER JUST TWO NUMBERS

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BEYOND $H_0$ AND $q_0$: COSMOLOGY IS NO LONGER JUST TWO NUMBERS

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ABSTRACT

For decades, $H_0$ and $q_0$ were the quest of cosmology, as they promised to characterize our “world model” without reference to a specific cosmological framework. Using Monte Carlo simulations, we show that $q_0$ cannot be directly measured using distance indicators with both accuracy (without offset away from its true value) and precision (small error bar). While $H_0$ can be measured with accuracy and precision, to avoid a small bias in its direct measurement (of order 5%) we demonstrate that the pair $H_0$ and $\Omega_M$ (assuming flatness and $w = -1$) is a better choice of two parameters, even if our world model is not precisely $\Lambda$CDM. We illustrate this with analysis of the Constitution set of supernovae and indirectly infer $q_0 = -0.57 \pm 0.04$. Finally, we show that it may be possible to directly determine $q_0$ with both accuracy and precision using the time dependence of redshifts (“redshift drift”).

Key words: cosmological parameters – methods: numerical – supernovae: general

1. INTRODUCTION

Cosmology has been characterized as the search for two numbers (Sandage 1970): $H_0$, the present expansion rate, and $q_0$, the present deceleration parameter. Few would argue with the statement that cosmology is a much grander enterprise today, and that our “world model” is better described by a larger set of physically motivated parameters including the energy densities of radiation, dark matter, dark energy, and the equation of state of dark energy.

In this paper we show that, in fact, the deceleration parameter cannot be directly measured using distance indicators with both accuracy and precision. That is, $q_0$ cannot be determined with statistical precision (small error bar) without incurring a bias away from the true value. While $H_0$ can be measured with both accuracy and precision, avoiding a slightly biased measurement requires a better choice of parameters than $H_0$ and $q_0$, e.g., $H_0$ and $\Omega_M$.

Sandage introduced $H_0$ and $q_0$ to provide a model-independent, kinematic description of the expansion of the universe. This description begins with a Taylor series for the cosmic scale factor $R(t)$,

$$\frac{R(t)}{R_0} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 + \cdots,$$

where $H_0 \equiv \dot{R}_0 / R_0$ and $q_0 \equiv -\ddot{R}_0 / H_0 R_0$. Using the definition of redshift, $1 + z \equiv R_0 / R$, and luminosity distance, $d_L \equiv (1 + z)r(z)R_0$, and the above expansion, the observable luminosity distance can be expressed in a Taylor series in redshift:

$$H_0d_L = z + \frac{1}{2}(1 - q_0)z^2 + O(z^3).$$

Note too, that no assumption about the validity of general relativity has been made, only that spacetime is isotropic and homogeneous and described by a metric theory. We note that the next order term—the jerk parameter $j_0$—may be added (e.g., Chiba & Nakamura 1998; Visser 2004, 2005; Weinberg 2008):

$$\frac{R(t)}{R_0} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 + \frac{1}{6}j_0 H_0^3(t - t_0)^3 + \cdots,$$

$$H_0d_L = z + \frac{1}{2}(1 - q_0)z^2 - \frac{1}{6}(1 - 3q_0^2 + j_0 + \frac{K}{H_0^2 R_0^2})z^3 + \cdots,$$

where $j_0 \equiv (\dddot{R}_0 / R_0) / H_0^3$ and $K = 0, -1, or 1$ for a flat, open, or closed universe, respectively.

The power of the $(H_0, q_0)$ (or quadratic) expansion is that, in principle, measurements of $d_L(z)$ can be used to determine the present expansion rate—arguably the most important number in all of cosmology—and the deceleration parameter. Moreover, in the simple matter-only cosmology of the time, general relativity, through the Friedmann equations, relates $q_0$ to the physical parameter, $\Omega_0 \equiv \rho_M / \rho_{crit}$, where $\rho_M$ is the present matter density and $\rho_{crit} \equiv 3H_0^2 / 8\pi G$ is the present critical density:

$$q_0 = \Omega_0 / 2.$$

Further, in this model, $q_0$ is related to the spatial curvature and destiny of the universe: if $q_0$ is greater than $1/2$, $\Omega_0$ is greater than unity and the universe is positively curved and will ultimately re-collapse; conversely, if $q_0$ is less than $1/2$, $\Omega_0$ is less than 1 and the universe is negatively curved and will expand forever. The case of $q_0 = 1/2$ is the flat universe that expands forever at an ever slowing rate.

This is all well and good; the question we address here is whether these two (or three) model-independent parameters can in fact be measured. The answer is simple: only $H_0$ can be measured with accuracy and precision. The explanation is simple as well: at low redshifts, say $z \lesssim 0.2$, where the Taylor expansion is most accurate, poor leverage on $q_0$ (and $j_0$) and peculiar velocities severely limit the precision; at higher redshifts, where the effect of peculiar velocities is negligible and the leverage is greater, the quadratic expansion does not accurately approximate $d_L(z)$ (see Figure 1) and a bias is introduced in the measurement. One can measure $q_0$ either with...
we generate mock distance modulus and redshift (μ, z) data over various redshift ranges, then study the degree to which q_0 and j_0 are constrained when these data are analyzed with the quadratic (H_0, q_0) or cubic (H_0, q_0, j_0) Hubble expansions. We then compare the degree to which H_0 is constrained using these expansions to the constraints obtained when the analysis assumes ΛCDM. In the latter case, H_0 and \( \Omega_M \) are the free parameters. We also investigate the effects of peculiar velocities and intrinsic luminosity scatter.

We model the distance indicators as a population of imperfect standard candles with an intrinsic Gaussian absolute magnitude scatter \( \sigma_{\text{int}} \) (While we are motivated by and will eventually use Type Ia SNe (SNe Ia) as the distance indicator, our results are more general.) We explore two possible values of \( \sigma_{\text{int}} = 0.15 \) mag, of order today’s state of the art for SNe Ia after light-curve fitting (e.g., Kowalski et al. 2008; Kessler et al. 2009; Hicken et al. 2009a; Rapetti et al. 2007; Conley et al. 2011); and 0.02 mag, an optimistic estimate of what distance indicators including SNe Ia might eventually be able to achieve. At this precision, peculiar velocity scatter dominates the \( \mu \) uncertainties at \( z < 0.2 \). Note the intrinsic distance modulus scatter is simply related to the luminosity scatter as \( \sigma_{\text{int}} = (d\mu/dL)\alpha_L = 1.08(\sigma_L/L) \).

Each mock measurement of \( \mu \) is chosen from a Gaussian distribution of width \( \sigma_{\text{int}} \) centered on the true value of \( \mu(z) \), which is given by standard equations (e.g., Weinberg 2008). We then apply a modest peculiar velocity scatter to the redshift data, giving the mock measurement of redshift: \((z + 1)(\zeta_{\text{pec}} + 1) - 1\), where each \( \zeta_{\text{pec}} \) is chosen from a Gaussian distribution centered at zero with \( \sigma_{\text{pec}} = 300 \) km s\(^{-1}\) (e.g., Riess et al. 2004, 2007; Rapetti et al. 2007; Hicken et al. 2009b; Lampeitl et al. 2010). (We note that there is no a priori reason why peculiar velocities should be described by an independent redshift scatter applied to each standard candle, and, in general, one would expect correlations due to bulk flows to prevent statistical uncertainties in fit parameters from decreasing as fast as \( 1/\sqrt{N} \), where \( N \) is the number of data points. However, previous surveys (e.g., Lampeitl et al. 2010) and \( N \)-body simulations (Vanderveld 2008) have suggested that such a correlation is not significant enough to pose a large issue.) We briefly explore the effect of lowering this scatter to the 150 km s\(^{-1}\) achieved by Conley et al. (2011) using the local bulk flow model of Hudson et al. (2004).

In parameter estimations, this peculiar velocity scatter can be incorporated as an additional uncertainty in \( \mu \) added in quadrature to the intrinsic scatter: \((d\mu/dL)\alpha_L = 2.17\sigma_{\text{pec}}/z\), approximating \( \mu(z) \) with the linear expansion for \( dL \).

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<td>Model</td>
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<tr>
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<td>de Sitter</td>
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<tr>
<td>Open</td>
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<tr>
<td>Einstein–de Sitter (EdS)</td>
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<td>w-slope</td>
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Note: \(^a\) Komatsu et al. (2011).

precision (using measurements extending to high redshift), or with accuracy (restricting the measurements to low redshift). We also show that including the cubic term in the model—jerk \( j_0 \)—does change this conclusion.

Moreover, while one might be tempted to simply abandon \( q_0 \) and \( j_0 \) given these complications, in fact they have found use recently to avoid a bias of order -5% that otherwise inflicts direct measurements of \( H_0 \) due to the less accurate linear expansion (Riess et al. 2009, 2011). To better avoid such biases, we explore other two- or three-parameter descriptions of \( dL \). In particular, the pair \( H_0 \) and \( \Omega_M \) (where flatness and \( w = -1 \) are assumed) eliminates this small bias in measuring \( H_0 \) and is better motivated. We use the Constitution compilation of supernovae (SNe; Hicken et al. 2009b) to concretely illustrate our findings. Finally, we briefly address how \( q_0 \) might be measured with accuracy and precision by exploiting the very small time evolution of redshifts—of the order of cm s\(^{-1}\) decade\(^{-1}\)—known as redshift drift (Sandage 1962; Loeb 1998).

Our paper is organized as follows. In Section 2, we detail our Monte Carlo distance indicator surveys and redshift drift surveys. We present the results of these simulations in Section 3, including analysis of the Constitution set of SNe to illustrate our findings. Section 4 provides some discussion and conclusions.

2. MONTE CARLO SIMULATIONS

2.1. Simulated Distance Indicator Surveys

We use a diverse set of cosmological models (Table 1) including ΛCDM, de Sitter (dS), an open model, Einstein–de Sitter (EdS), and a model with rapid dark energy evolution over redshift which we call “w-slope” (Figure 2). For each,
While others have assumed the empty universe model here (e.g., Kessler et al. 2009; Lampeitl et al. 2010), essentially taking the quadratic expansion with $q_0 = 0$, we found the difference on our parameter estimates to be negligible. Figure 3 shows the effective $\mu$ scatter $\sigma_{\mu}(z)$.

$$\sigma_{\mu}(z) = \sqrt{\sigma_{\text{int}}^2 + (2.17\sigma_{\text{pec}}/z)^2}. \quad (3)$$

as well as the $\mu$ uncertainties in the Constitution data set (Hicken et al. 2009b) with the intrinsic scatter $\sigma_{\text{int}} = 0.15$ mag added in quadrature.

For a given cosmological model, redshift range, and intrinsic standard candle scatter, we simulate 10,000 surveys, each with 500 mock objects uniformly distributed over redshift. We use a minimum redshift of 0.015 and explore several values of the maximum redshift between 0.05 and 1.5 chosen to illustrate the dependence of accuracy and precision in parameter estimates on redshift. Redshifts smaller than 0.015 are not helpful for measuring $q_0$ due to severe peculiar velocity uncertainties (see Figure 3) and the very weak leverage on that parameter at such low $z$. To eliminate simulation biases due to mock redshifts scattering out of the redshift range (0.015, $z_{\text{max}}$), but not scattering into it, we generate more than 500 data points over a slightly larger redshift range, then keep only 500 data points in the desired range.

We then explore biases in precision measurements of $H_0$ due to the inaccuracy of the $d_L$ expansion. To avoid any systematics present in high-$z$ SN data (such as evolution or dust effects), typically only nearby SNe are used to measure $H_0$. For these simulations, we take $z_{\text{max}} = 0.1$, which can be identified as the cutoff redshift of the low-$z$ segment of the Constitution set (Figure 3).

2.2. Parameter Estimation from Simulated Distance Indicator Surveys

For each mock survey, we estimate model parameters by minimizing a $\chi^2$ statistic. For instance, to estimate $H_0$ and $q_0$

$$\chi^2(H_0, q_0) = \sum_{i=1}^{500} \frac{(\mu_i - \mu_{\text{exp}}(H_0, q_0; z_i))^2}{\sigma_{\mu}(z_i)}. \quad (4)$$

Then for each cosmological model, redshift range, and intrinsic $\mu$ scatter, we plot two-dimensional (2D) contours in $(H_0, q_0)$ parameter space with 68% and 95% probability content generated from the 10,000 parameter estimations. We treat the 10,000 $(H_0, q_0)$ pairs as samples from a 2D Gaussian, compute the covariance matrix, then plot contours of that Gaussian. We have verified that the error in assuming the distribution is Gaussian is not significant for our purposes. Each contour plot shows the probability distribution of $(H_0, q_0)$ pairs in parameter space given the intrinsic $\mu$ scatter and peculiar velocity scatter from which the data are drawn, showing any bias in and correlation between the parameter estimates. The extents of the contours may be taken as error ellipses for a single measurement.

2.3. Comparison with Real Data

To compare our results with real data, we use the Constitution set (Table 1 in Hicken et al. 2009b), which combines the low-redshift ($z \lesssim 0.08$) CfA3 sample of Hicken et al. (2009a) with the Union sample of Kowalski et al. (2008), processed with the SALT light-curve fitter of Guy et al. (2005). We use the published ($z$, $\mu$, $\Delta \mu$) data, where $\Delta \mu$ includes some systematic uncertainties as discussed by Hicken et al. (2009b). Note that we use this data set only to constrain $q_0$ and $j_0$, not to make precision $H_0$ measurements.

When analyzing the Constitution set of SNe, we pick several illustrative maximum redshifts to compare results with simulations. Unlike with our mock data sets, the number of objects is not the same for each redshift range. The numbers of SNe up to $z_{\text{max}} = [0.1, 0.3, 0.5, 1.5]$ are $[141, 164, 248, 397]$. Following standard practice (e.g., Lampeitl et al. 2010; Kessler et al. 2009; Rapetti et al. 2007), we add the intrinsic standard candle scatter (0.15 mag) in quadrature to the published $\mu$ uncertainties.

2.4. Simulated Redshift Drift Surveys

The redshift of an object within the Hubble flow varies with time due to the expansion of the universe (Sandage 1962; Loeb 1998):

$$\frac{dz}{dt} = (1 + z)H_0 - H(z) \rightarrow -zq_0H_0 + O(z^2),$$

where the $(H_0, q_0)$ expansion has been used to obtain the final expression. The estimate for $q_0$ using this method, assuming $H_0$ has been determined by other means, is given by

$$q_0 \approx -\frac{1}{H_0 z} \frac{dz}{dt}. \quad (5)$$

Note that unlike in the quadratic Hubble expansion, both $H_0$ and $q_0$ appear together at lowest order (linear in $z$) here.

We assume $dz/dt$ is observed for $N$ distance indicators spaced uniformly over redshift from $z_{\text{min}} = 0.015$ to $z_{\text{max}}$, exploring $N = 500, 5000$, and various values of $z_{\text{max}}$. We assume a peculiar acceleration scatter with a mean of zero and standard deviation of 200 km$^{-1}$ s$^{-1}$ (50 Myr)$^{-1}$ = 4 cm$^{-1}$ s$^{-1}$ decade$^{-1}$, estimating this numerical value from the orbital period and circular velocity of our solar system around the Milky Way. This is of order the cosmological signal in $\Lambda$CDM.
3. RESULTS

3.1. Measuring $q_0$

Figure 4 shows $(H_0, q_0)$ contour plots from analysis of 10,000 $\Lambda$CDM simulations with the quadratic expansion, as well as contours from analysis of Constitution data. The trend with $z_{\text{max}}$ (and our basic result) is clearly illustrated: increasing $z_{\text{max}}$ improves precision of the $q_0$ estimate only at the expense of a growing bias away from the true value. This basic trend is seen for all the cosmological models (except the de Sitter model; see below), and is simply illustrated in the top panel of Figure 5, which shows the quadrature sum of statistical uncertainty and bias for all the models as a function of $z_{\text{max}}$ (all with $\sigma_{\text{int}} = 0.15$).

The “sweet spot” where both accuracy and precision are decent varies between $z_{\text{max}} \sim 0.1$ and 0.5 depending on the model. For $\Lambda$CDM, a minimum of 32% is reached at $z_{\text{max}} = 0.25$ with $\Delta z_{\text{max}} = 0.18$ (within 20% of the minimum).

The bottom panel of Figure 5 shows the effects of varying some of our data quality assumptions. Lowering $\sigma_{\text{pec}}$ to 150 km s$^{-1}$ has a negligible effect on $q_0$ precision unless $\sigma_{\text{int}}$ is lowered significantly as well. Increasing survey size and lowering $z_{\text{min}}$ help further, giving a narrow minimum of 13% at $z_{\text{max}} \sim 0.10$ with $\Delta z_{\text{max}} = 0.06$ (within 20% of the minimum). None of these improvements is a “silver bullet”; all are required to tackle the large error bars produced in these $q_0$ estimates due to the very weak statistical leverage on that parameter in the quadratic expansion.

Returning to Figure 4, we more carefully study shrinking the intrinsic $\mu$ scatter in the left panel. Lowering $\sigma_{\text{int}}$ to 0.02 mag, the regime where parameter uncertainties in the $z_{\text{max}} = 0.1$ simulations are dominated by peculiar velocity scatter, reduces the $q_0$ uncertainties by a factor of $\sim 4$, though the biases worsen slightly due to the altered distribution of uncertainties as a function of $z$ (high-$z$ data now weighted much more relative to low-$z$ data; see Figure 3). We note here that while our simulations with $\sigma_{\text{int}} = 0.15$ mag are intended to roughly approximate present-day SNe Ia surveys, the Constitution data set contains many objects with much larger uncertainties and the set is distributed non-uniformly over redshift (Figure 3). Thus, while the Constitution $(H_0, q_0)$ contours exhibit the same trends
Figure 6. Contours in \((H_0, q_0)\) space from simulated distance indicator surveys in various cosmological models (see Table 1) analyzed with quadratic expansion. The cross hairs indicate the true \((H_0, q_0)\) in each model, where \(H_0\) is in units of its true value. The improvement in precision at the expense of worsening accuracy (growing bias) is visible in all models except the de Sitter model, in which the quadratic expansion is exact. Estimates of \(q_0\) in the w-slope model are especially poor owing to its rapid evolution over redshift (see Figure 2). Note the expanded axes in the left column.

(A color version of this figure is available in the online journal.)
Similar contour plots for other cosmological models appear in Figure 6. Again we observe the expected tradeoff between accuracy at low $z$ and precision at high $z$. The open and EdS models are somewhat better approximated by their quadratic expansions than is $\Lambda$CDM (see Figure 1), and thus, they admit slightly more accurate $q_0$ estimates. Estimates of $q_0$ in the w-slope model are especially poor owing to its rapid evolution over redshift (see Figure 2). The de Sitter model is worth mentioning because it provides a check on the reliability of the Monte Carlo simulations. In that model, the quadratic expansion is exact, a fact reflected in our parameter estimates by the absence of any biases.

Figure 7 shows the results of including the cubic term when analyzing data from the $\Lambda$CDM model in the form of $(j_0, q_0)$ contour plots after marginalizing over $H_0$. The overall trend is the same as before: increasing $z_{\text{max}}$ improves precision at the expense of a growing bias. Relative to the quadratic expansion, the bias is reduced and we thus use $z_{\text{max}} = 0.7, 0.9$, and 1.5. Reducing $\sigma_{\text{int}}$ to 0.02 mag helps somewhat, but as before worsens the biases slightly. Using the entire Constitution set ($z_{\text{max}} = 1.5$), we estimate $q_0 = -0.64 \pm 0.14$, within error bars of our simulation result of $-0.45 \pm 0.09$, with slight differences possibly due to the non-uniform distribution of redshifts and non-uniform errors in the Constitution set.

Turning to our simulated redshift drift surveys, Figure 8 shows the quadrature sum of statistical uncertainty and bias in $q_0$ estimates from simulated redshift drift surveys fit to Equation (5), with $\Lambda$CDM as the underlying cosmology. Compared to our distance indicator simulations, a somewhat broader redshift “sweet spot” where both accuracy and precision are decent is observed here (cf. Figure 5). The precision is also comparatively improved at small $z$.

Using the Constitution data and the relation $q_0 = 3\Omega_M/2-1$ (assuming $\Lambda$CDM, which is parameterized by $H_0$ and $\Omega_M$), we infer $q_0 = -0.57 \pm 0.04$, with better accuracy (closer to true value) and precision (smaller error bar) than estimates derived from the quadratic or cubic expansions. Here, we have used the published Constitution uncertainties, meaning this error bar even includes some systematic uncertainties (see Section 2.3).

### 3.2. Measuring $H_0$

Next, we explore the small biases that creep into precision $H_0$ measurements due to the inaccuracy of the quadratic and cubic expansions. We simulate distance indicator surveys with $z_{\text{max}} = 0.1$ (as discussed in Section 2.1) in $\Lambda$CDM and in two models close to $\Lambda$CDM but still consistent with current data ($\Lambda$CDM may not be the exact world model): a model with $w = -0.8$ that is otherwise identical to $\Lambda$CDM; and a slightly closed model with $w = -1$, $\Omega_M = 0.30$, and $\Omega_{\text{DE}} = 0.75$ ($\Omega_K = -0.05$). We then estimate $H_0$ by fitting to

1. $(H_0, \Omega_M)$ parameterization of $\Lambda$CDM (assuming flatness and $w = -1$);
2. quadratic expansion, marginalizing over $q_0$;
3. quadratic expansion, fixing $q_0 = -0.58$;
4. cubic expansion, marginalizing over $q_0$ and $j_0$;
5. cubic expansion, fixing $q_0 = -0.58$ and $j_0 = 1$.

The results are shown in Figure 10. We note first that using the linear expansion for $d_L$ yields a bias in $H_0$ (not shown) of $-4\%$ for $\sigma_{38} = 0.15$, and $-5\%$ for $\sigma_{38} = 0.02$, in good agreement with the $-3\%$ bias identified by Riess et al. (2009) using real data. Using the quadratic or cubic expansion and marginalizing over the other parameters reduces these biases to order 0.5%, and assuming values for those parameters reduces them further (except for the $w = -0.8$ model).

Even if our world model is not $\Lambda$CDM, the $(H_0, \Omega_M)$ parameterization is the most robust, with biases of order 0.01%. With $\sigma_{38} = 0.02$, peculiar velocities dominate over $\mu$ uncertainties at these redshifts, and provide the limiting uncertainty in $H_0$. They are also the source of the lingering $\sim 0.01\%$ biases, as the peculiar velocity uncertainties are estimated using measured $\dot{z}$, not actual $\dot{z}$, so objects scattering to lower (higher) $\dot{z}$ are under-(over-) weighted in the likelihood analysis.

4. DISCUSSION

The quest to measure $H_0$ and $q_0$ and determine our world model drove cosmology for almost three decades. The Hubble constant has now been directly measured to around 5% precision (Riess et al. 2011), inferred from cosmic microwave background (CMB) anisotropy and other cosmological measurements to almost 1% (Komatsu et al. 2011), and there are aspirations to improve that to less than a percent (e.g., Freedman & Madore 2010; Reid et al. 2010; Riess et al. 2011; Sekiguchi et al. 2010; Suyu et al. 2012; Weinberg et al. 2013). $H_0$ remains arguably the most important single number in cosmology, as it sets the age and size of the universe and underpins many other measurements.

On the other hand, not only was the minus sign in the definition of $q_0$ unfounded, but $q_0$ cannot actually be measured with precision and accuracy using distance indicators! The reasons are simple: at low redshifts, where the $d_L$ expansion is accurate, the combination of peculiar velocities and the small change in $d_L$ for different values of $q_0$ (lack of leverage) makes $q_0$ impossible to measure with any precision. At higher redshift, where there is leverage and precision is possible, the inaccuracy of the $(H_0, q_0)$ expansion strongly biases estimates of $q_0$. There is no “sweet spot” at intermediate redshift that allows both precision and accuracy regardless of the world model (see Figure 5).

Moreover, the mere usage of $q_0$ as the second parameter in precision measurements of $H_0$ leads to a non-negligible bias. As we and others have shown (Riess et al. 2009), fitting to a linear Hubble law introduces a bias of order $-5\%$. One solution, used by Riess et al. (2009, 2011), is to use the $(H_0, q_0, j_0)$ expansion and by fiat impose the “correct” values of $q_0$ and $j_0$ obtained with $z \sim 1$ data. This approach has several drawbacks.

1. Neglecting the uncertainties on $q_0$ and $j_0$ priors underestimates the real uncertainty in $H_0$, providing a false sense of precision. In our simulations, going from fixing those values to freeing them increases the $H_0$ uncertainty by a factor of 10 (see Figure 10), an amount that will be important in future precision attempts to further constrain $H_0$ using this method.
2. $H_0$ estimates using this method are no longer independent of high-$z$ SNe and their potential systematics.
3. This method is not robust to small changes in $w$, acquiring a $\sim 1\%$ bias if our world model actually has $w = -0.8$ (see Figure 10), within the error bars of recent measurements.
4. A generic problem with the cubic expansion is that there are more parameters than there are coefficients to fit for (Chiba & Nakamura 1998; Visser 2005). Simply neglecting the curvature term removes this problem, however it should be noted that purported estimates of \( J_0 \) using this method are actually estimates of \( J_0 + K/(H_0 R_0)^2 \).

Instead and what we think is better, one can use the more physical two-parameter description of our cosmological model, \((H_0, \Omega_M)\). If our world model is indeed \( \Lambda \)CDM, then these are the only two parameters needed to describe \( d_L(z) \). In this case, there is essentially no bias in determining the Hubble constant and only peculiar velocities limit the precision (see Figure 10), and even they may be mitigable to some extent using bulk flow models (Hudson et al. 2004; Conley et al. 2011). The same figure shows that even if \( \Lambda \)CDM does not exactly describe our world model, the biases are still very small.

Additional parameters can be added (and may be needed) to completely characterize \( d_L \); for example, \( w \) or \( w_0 \) and \( u_\alpha \), and \( \Omega_k \), but given the closeness of our world model to \( \Lambda \)CDM, \((H_0, \Omega_M)\) is a set of two that affords accuracy and precision. Further, we note that within \( \Lambda \)CDM additional parameters are needed to characterize CMB anisotropy and other dynamical aspects of the universe. The standard six-parameter set is \( \Omega_\gamma h^2 \) (baryon density), \( \Omega_M h^2 \) (total matter density), \( \Omega_{DE} \), \( n_S \) (power-law index of density perturbations), \( \tau \) (optical depth to last scattering), and \( \Delta^2 \) (overall amplitude of the spectrum of inhomogeneity) (e.g., Komatsu et al. 2011), and even more parameters can be added. Cosmology today is much richer than the two numbers that Sandage used to characterize our world model and the goal of cosmology.

All this being said, there is a certain elegance and utility to the two parameters \( H_0 \) and \( q_0 \), even if one of them is difficult to directly measure. For example, both can be defined independently of general relativity and its Friedmann equations (all that is required is the assumption of isotropy and homogeneity and a metric theory; Turner & Riess 2002; Shapiro & Turner 2006). Together, they characterize the most basic features of the expansion—rate of expansion and whether the expansion is slowing down or speeding up. Within the parameterizations mentioned above, \( q_0 \) can be inferred; for example, with the two parameters \( \Omega_M \) and \( H_0 \):

\[
q_0 = \frac{3}{2} \Omega_M - 1 = -0.57 \pm 0.04,
\]

where the numerical value has been determined from the Constitution data set (assuming \( \Lambda \)CDM), as discussed in Section 3.1, and is in agreement with the true value of \(-0.58\) for \( \Lambda \)CDM with \( \Omega_M = 0.28 \).

Finally, we have demonstrated with simulations that surveys of redshift drift have the potential to better directly determine \( q_0 \) with accuracy and precision (see Figure 8). This method is more powerful because \( \partial z/\partial t \) depends upon the product of \( H_0 \) and \( q_0 \) and is linear in redshift, allowing good constraints over somewhat wider redshift ranges and improved precision near \( z = 0 \). Of course, this is a very challenging measurement—the effect is on the order of a few cm s\(^{-1}\) decade\(^{-1}\) (of order peculiar accelerations for \( z < 2 \))—and a reach goal for the next generation of extremely large optical telescopes (e.g., Balbi & Quercellini 2007; Corasaniti et al. 2007; Amendola et al. 2008; Liske et al. 2008; Quercellini et al. 2012).

Cosmology has changed dramatically since Sandage characterized it as the quest for two numbers. It has become a precision science characterized by a larger set of more physically motivated numbers. While, \( q_0 \) is not actually measurable using luminosity distances and a hindrance to accurately measuring \( H_0 \), it is nonetheless of interesting parameter in cosmology today—and still not directly measured.

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