**Gapped Domain Walls, Gapped Boundaries, and Topological Degeneracy**

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Gapped Domain Walls, Gapped Boundaries, and Topological Degeneracy

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Gapped domain walls, as topological line defects between (2 + 1)D topologically ordered states, are examined. We provide simple criteria to determine the existence of gapped domain walls, which apply to both Abelian and non-Abelian topological orders. Our criteria also determine which (2 + 1)D topological orders must have gapless edge modes, namely, which (1 + 1)D global gravitational anomalies ensure gaplessness. Furthermore, we introduce a new mathematical object, the tunneling matrix $W$, whose entries are the fusion-space dimensions $W_{ia}$, to label different types of gapped domain walls. By studying many examples, we find evidence that the tunneling matrices are powerful quantities to classify different types of gapped domain walls. Since a gapped boundary is a gapped domain wall between a bulk topological order and the vacuum, regarded as the trivial topological order, our theory of gapped domain walls inclusively contains the theory of gapped boundaries. In addition, we derive a topological ground state degeneracy formula, applied to arbitrary orientable spatial 2-manifolds with gapped domain walls, including closed 2-manifolds and open 2-manifolds with gapped boundaries.

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Introduction.—The insulator has a finite energy gap, which is rather trivial at low energy. Nonetheless, domain walls, separating different symmetry-breaking insulating regions, can enrich the physics of a trivial insulator, such as some paramagnets [1]. Topological order [2–4], on the other hand, as a new kind of many-body quantum ordering, has a gapped bulk with exotic properties: some have (i) gapless edge modes, (ii) anyonic excitations with fractional or non-Abelian statistics [5], such as fractional quantum Hall states, and (iii) long-range entanglement [6–8]. In this Letter, we would like to investigate the gapped domain walls of topological orders, and how gapped domain walls further enrich their physics.

It was conjectured that the (2 + 1)D topological orders are completely classified by the gauge connection on the moduli space of the degenerate ground states [4,9]. The non-Abelian part of the gauge connection is the non-Abelian geometric phase [10] characterized by the $S,T$ matrices, which also encode the anyon statistics. The Abelian part is related to the gravitational Chern-Simons term in the effective theory and is described by the chiral central charge $c_-$ of the edge state. Nonzero $c_-$ implies robust gapless edge modes.

By now we understand how to label a 2D topological order by a set of “topological order parameters” $(S,A,c_+)$, analogous to “symmetry-breaking order parameters” for spontaneous symmetry breaking systems [11,12]. However, it is less known how different topological orders are related. To this end, it is important to investigate the following circumstance: there are several domains in the system and each domain contains a topological order, while the whole system is gapped. In this case, different topological orders are connected by gapped domain walls. Our work addresses two primary questions: (Q1) Under what criteria can two topological orders be connected by a gapped domain wall, and how many different types of gapped domain walls are there? Since a gapped boundary is a gapped domain wall between a nontrivial topological order and the vacuum, we also address the question “under what criteria can topological orders allow gapped boundaries?” (Q2) When a topologically ordered system has a gapped bulk, gapped domain walls, and gapped boundaries, how can one calculate its ground state degeneracy (GSD) [2,3,13–15], on any orientable manifold?

Main result.—Consider two topological orders, phases $A$ and $B$, described by $(S^A,T^A,c^A_+)$ and $(S^B,T^B,c^B_+)$. Suppose there are $N$ and $M$ types of anyons in phase $A$ and phase $B$, then the ranks of their modular matrices are $N$ and $M$, respectively. If $A$ and $B$ are connected by a gapped domain wall, first, their central charges must be the same $c^A_+ = c^B_+$. Next we find that the domain wall can be labeled by an $M \times N$ tunneling matrix $W$ whose entries are fusion-space dimensions $W_{ia}$ satisfying the commuting condition, Eq. (2), and the stable condition, Eq. (3):

$$W_{ia} \in \mathbb{N}, \quad (1)$$

$$S^B W = W S^A, \quad T^B W = W T^A, \quad (2)$$

$$W_{ia} W_{jb} \leq \sum_{kc} (N^B)^i_c (N^A)^j_b, \quad (3)$$

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\[
N_{ab}^c = \sum_m S_{am} S_{hm} S_{cm} / S_{1m} \in \mathbb{N}.
\] (4)

Gapped domain walls.—Below we demonstrate the physical meanings of the gapped domain wall conditions, Eqs. (1)–(3). First we put phase A and phase B on a sphere \(S^2\), separated by a gapped domain wall. Note that there can be many types of domain walls separating the same pair of phases A and B. What data characterize those different types of domain walls? We fix the domain-wall type, labeled by \(W\), and trap [23] an anyon \(a^*\) in phase A, and an anyon \(i\) in phase B. This configuration is denoted by \((S^2, i, W, a^*)\). The states with such a configuration may be degenerate and the degenerate subspace is the fusion space \(\mathcal{V}(S^2, i, W, a^*)\). Here we propose using the fusion-space dimensions \(\mathcal{W}_{ia} = \dim(\mathcal{V}(S^2, i, W, a^*))\) to characterize the gapped domain wall \(W\).

There are nonlocal operators \(O_{W,ia}\) that create a pair \(aa^*\) in phase A, and then tunnel \(a\) through the domain wall to an anyon \(i\) in phase B, \(O_{W,ia} |\psi_{W,ia}^\prime\rangle = \mathcal{V}(S^2, i, W, a^*)\), where \(|\psi_{W,ia}^\prime\rangle\) is the ground state. Since we care about the fusion states rather than the operators themselves, we would take the equivalent class \([O_{W,ia}] = \{U_{W,ia} |O_{W,ia} - O_{W,ia}^\dagger| \psi_{W,ia}^\prime\rangle = 0\}\). We refer to \([O_{W,ia}]\) as tunneling channels, which correspond to fusion states in \(\mathcal{V}(S^2, i, W, a^*)\). Therefore, the fusion space \(\mathcal{W}_{ia}\) is the number of linearly independent tunneling channels. So we also refer to \(\mathcal{W}\) as the “tunneling matrix.”

The commuting condition Eq. (2) dictates the consistency of anyon statistics in the presence of gapped domain walls. Since modular \(S, T\) matrices encode the anyon statistics, we require that \(W\) should commute with them as Eq. (2): \(S^B W = W S^A, T^B W = W T^A\).

We may as well create a pair \(ii^*\) in phase B and tunnel \(i^*\) to \(a^*\). \(W^\dagger\) describes such tunneling in the opposite direction (i.e., \(W : A \rightarrow B, V^\dagger : B \rightarrow A\)). \(V^\dagger\) and \(W\) contains the same physical data. To be consistent, tunneling \(i^*\) to \(a^*\) should give the same fusion-space dimension, \(\dim(\mathcal{V}(W^\dagger))_{ia'} = \mathcal{W}_{ia'} = \mathcal{W}_{ia}\). This is guaranteed by \(\mathcal{V}(S^2)^2 = (S^B)^2 W\) and \((S^2)_{ab} = \delta_{a'b}\).

The fusion spaces with four anyons further provide us consistency conditions of \(W\). To see this, first notice that there are generalized tunneling channels, \(O_{W,ia'x}\), which, in addition to tunneling \(a\) to \(i\), also create the quasiparticle \(x\) on the domain wall. If we combine the tunneling channels \(O_{W,ia'x}\) and \(O_{W,jb'x'}\), we can create fusion states with a domain wall \(W\) and four anyons \(i, j, a^*, b^*\), as in Fig. 1(a). In other words, \(O_{W,ia'x} O_{W,jb'x'}\) form a basis of the fusion space \(\mathcal{V}(S^2, i, j, W, a^*, b^*)\). Let \(K^c_{ia'}\) denote the number of tunneling channels \(O_{W,ia'x}\), and we know that \(\dim(\mathcal{V}(S^2, i, j, W, a^*, b^*)) = \sum_c K^c_{ia'}.K^c_{ia'}\). However, the tunneling process as shown in Fig. 1(b), i.e., fusion \(a, b\) to \(c\), using \(O_{W,ka'}\) to tunnel \(c\) to \(k\) and splitting \(k\) to \(i, j\), forms
Eq. (2) requires that stable gapped domain walls we have walls, i.e., the GSD cannot be reduced no matter what $P_W$.

Therefore, we must have

$$\sum_{x} K_{iab}^x K_{jb}^x = \sum_{ke} (N^B)_{ij} W_{ke} (N^A)_{ab}^c. \quad (5)$$

We are interested in classifying stable gapped domain walls, i.e., the GSD cannot be reduced no matter what small perturbations are added near the domain wall. For stable gapped domain walls we have $W_{ia} = K_{iab}$. Unstable gapped domain walls $U$ split as the sum of stable ones $W^{(1)}, W^{(2)}, ..., W^{(N)}$, and $U_{ia} = \sum_{i=1}^{N} W_{ia}^{(n)}$, for $N \geq 2$.

Now if a gapped domain wall $W$ is stable, Eq. (5) becomes $\sum_{ke} (N^B)_{ij} W_{ke} (N^A)_{ab}^c = W_{ia} W_{jb} + \sum_{i \neq 1} K_{iab}^x K_{jb}^x \geq W_{ia} W_{jb}$. We know that Eq. (3) is necessary for a gapped domain wall to be stable. Furthermore, setting $i = j = a = b = 1$ we know that $W_{11} = W_{11}^T$ and Eq. (2) requires that $W_{11} > 0$, thus $W_{11} = 1$ and $W$ cannot be the sum of more than one stable tunneling matrix; it must be stable itself. Therefore Eq. (3) with Eq. (2) is also sufficient for a gapped domain wall to be stable.

**Stability of composite domain walls.**—Let us consider two stable domain walls, $W^{(1)}$ between phases $A$ and $B$, and $W^{(2)}$ between phases $B$ and $C$, as in Fig. 1(c). When the two domain walls are far separated, they are both stable. Any small perturbations added near $W^{(1)}$, or near $W^{(2)}$, cannot reduce the GSD.

We then shrink the size of the middle phase $B$, such that the two domain walls are near enough to be regarded as a single domain wall. This way we obtain a composite domain wall, whose tunneling matrix is the composition $W^{(2)} W^{(1)}$, as in Fig. 1(d). However, this composite domain wall $W^{(2)} W^{(1)}$ may no longer be stable. Unless phase $B$ is a vacuum, we allow more perturbations to $W^{(2)} W^{(1)}$ than when $W^{(1)}$ and $W^{(2)}$ are far separated. Some operators simultaneously acting on both $W^{(1)}$ and $W^{(2)}$ may reduce the GSD, in which case, the composite domain wall $W^{(2)} W^{(1)}$ is not stable.

In the special case when phase $B$ is a vacuum, the composite $W^{(2)} W^{(1)}$ remains stable. One can explicitly check this with Eq. (3).

**GSD in the presence of gapped domain walls.**—Below we derive the GSD, for a 2D system containing several topological orders separated by looplike gapped domain walls. Domain walls cut a whole 2D system into several segments. Without losing generality, let us consider an example in Fig. 2 with topological orders, phases $A$, $B$, $C$, $D$, and four nontrivial domain walls, $W^{(1)}$, $W^{(2)}$, $W^{(3)}$, $W^{(4)}$, on a manifold, Fig. 2(e). We first add extra trivial domain walls $W = I$, so that all segments between domain walls are reduced to simpler topologies: caps, cylinders, or pants. We also add oriented skeletons to the manifold, and put anyon indices on both sides of the domain walls, as shown in Fig. 2(e). Next, see Figs. 2(a)–2(d), for the segments with oriented skeletons and anyon indices, we associate certain tensors: caps with $\delta_{\alpha \beta}$, cylinders with $\delta_{\alpha \beta}$, pants with $N^k_{ij}$ in the corresponding topological order, and domain walls with their tunneling matrices $W_{ia}$. We may reverse the orientation and at the same time replace the index $a$ by $b$.

**FIG. 1 (color online).** (a), (b) Tunneling channels. (c) Separated domain walls $W^{(1)}$ and $W^{(2)}$. (d) Composite domain wall $W^{(2)} W^{(1)}$. (a) Several domain walls on the torus. (b) A cylinder with two gapped boundaries. (c) A pair of pants with three gapped boundaries.
associating the tensor also be calculated, by putting the anyon can be a vacuum. Dimensions of generic fusion spaces can connected to the boundaries, e.g., phases on caps thus the GSD. of anyons, which exactly counts the number of ground states, contract all the anyon indices. Physically such tensor with two gapped boundaries ¼ with two gapped boundaries, and in Fig. 3(c): GSD = \sum_{i,j,k} W_{ij}^{(1)} W_{jk}^{(2)} W_{ik}^{(3)} \frac{1}{2}. The rocket graph in Fig. 2(e): GSD = \sum_{a,i,j,k,r} W_{ij}^{(1)} W_{jk}^{(2)} W_{ik}^{(3)} W_{ij}^{(4)}. We apply our formalism to several topological orders. Details of our examples are organized in the Supplemental Material [17]. Part of our result is listed in Table I (the number of gapped domain-wall types) and Table II (GSD).

**Conclusion.**—Given \( S, T \) matrices of topological orders with the same central charge, we have provided simple criteria, Eqs. (1)–(3), to check the existence of gapped domain walls. We want to mention that, a gapped domain wall can be related to a gapped boundary by the folding trick [25]. By studying gapped boundaries, we can also obtain all the information of gapped domain walls. But, to compute the GSD, gapped domain walls allow more configurations on 2D surfaces than gapped boundaries.

The gapped domain walls and boundaries can be explicitly realized in lattice models [25–27]. Levin-Wen string-net models [28] are exactly solvable models for topological orders. Recently, it was found that a topological order can be realized by a Levin-Wen model if and only if it has gapped boundaries [25,27]. Thus our work provides the criteria of whether a topological order has a Levin-Wen realization.

2D Abelian topological orders can be described by Chern-Simons field theories. The boundary of a Chern-Simons theory is gappable, if and only if there exists a Lagrangian subgroup [13,14,29–32]. Our tunneling matrix criteria, Eqs. (1)–(3), are equivalent to the Lagrangian subgroup criteria for Abelian topological orders (a detailed proof is given in the Supplemental Material [17]), but are more general and also apply to non-Abelian topological orders.

One can also use the anyon condensation approach [33–39] to determine the gapped boundaries of (non-Abelian) topological orders, by searching for the Lagrangian condensable anyons (mathematically, Lagrangian algebras [35,36]), whose condensation will break the topological order to vacuum. However, we use only an integer vector \( W_{1a} \) to determine the anyon \( q_W \), while in the anyon condensation approach, besides the multiplicity \( W_{1a} \), there are many additional data satisfying a series of formulas. These formulas put certain constraints on the condensable anyon, but not in a simple and explicit manner. Our claim that Eqs. (1)–(3) are necessary and sufficient for a gapped domain wall to exist means that, Lagrangian condensable anyons must satisfy Eqs. (1)–(3), and, for the anyon \( q_W \) satisfying Eqs. (1)–(3), there must exist solutions to the additional data in the anyon condensation approach.

We know that the effective \((1 + 1)D\) edge theory of a \((2 + 1)D\) topological order has a gravitational anomaly.

### Table I

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<th>No. gapped DW</th>
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<td>Double semion</td>
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<td>2</td>
</tr>
<tr>
<td>Doubled Fibonacci</td>
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<tr>
<td>Doubled Ising</td>
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<td>(D(5))</td>
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### Table II

GSD of a single topological order (the first column) on a sphere with a number of punctures (the first row). Each puncture has a gapped boundary. The last three orders allow only one type of gapped boundary, so its GSD is unique for a given topology. Toric code allows two types of gapped boundaries, and its GSD varies, which depends on boundary types associated to each puncture. This agrees with Refs. [13,24].

<table>
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<th>(No. of punctures)</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tr>
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<td>2</td>
<td>2</td>
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<td>Doubled Fibonacci</td>
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<td>Doubled Ising</td>
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The gravitational anomalies are classified by the bulk topological order \((S, T, c_-)\) \([40, 41]\). When \(c_- \neq 0\), the edge effective theory has a perturbative gravitational anomaly which leads to a topological gapless edge (i.e., the gaplessness of the edge is robust against any change of the edge Hamiltonian). Even in the absence of a perturbative gravitational anomaly, \(c_- = 0\), certain global gravitational anomalies \([42]\) characterized by \((S, T, 0)\) can also lead to a topological gapless edge \([13, 30]\). Our work points out that such global gravitational anomalies are described by \(S, T\) which do not allow any nonzero solution \(W\) of Eqs. (1)–(3). The corresponding 2D topological order \((S, T, 0)\) will have a topological gapless edge.

Since a domain wall sits on the border between two topological orders, our study on domain walls can also guide us to better understand the phase transitions of topological orders.

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Note added.—We recently became aware that related work, Ref. [39], has independently obtained part of our results using a different approach: anyon condensation. The comparison between our new approach and anyon condensation is explained in the Conclusion section.

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[23] The concepts of trapping anyons, composite anyon types, and fusion spaces are discussed in Ref. [23].