Entanglement spectrum of a random partition: Connection with the localization transition

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<td><a href="http://dx.doi.org/10.1103/PhysRevB.91.220101">http://dx.doi.org/10.1103/PhysRevB.91.220101</a></td>
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<tr>
<td>Publisher</td>
<td>American Physical Society</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
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<tr>
<td>Accessed</td>
<td>Fri Feb 08 23:06:02 EST 2019</td>
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Entanglement spectrum of a random partition: Connection with the localization transition

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(Received 15 December 2014; revised manuscript received 7 April 2015; published 15 June 2015)

We study the entanglement spectrum of a translationally invariant lattice system under a random partition, implemented by choosing each site to be in one subsystem with probability \( p \in [0,1] \). We apply this random partitioning to a translationally invariant (i.e., clean) topological state, and argue on general grounds that the corresponding entanglement spectrum captures the universal behavior about its \textit{disorder-driven} transition to a trivial localized phase. Specifically, as a function of the partitioning probability \( p \), the entanglement Hamiltonian \( H_A \) must go through a topological phase transition driven by the percolation of a random network of edge states. As an example, we analytically derive the entanglement Hamiltonian for a one-dimensional topological superconductor under a random partition, and demonstrate that its phase diagram includes transitions between Griffiths phases. We discuss potential advantages of studying disorder-driven topological phase transitions via the entanglement spectra of random partitions.

DOI: 10.1103/PhysRevB.91.220101

PACS number(s): 73.43.Cd, 03.67.Mn

In recent years, systematic studies of quantum entangled states have greatly advanced our understanding of topological states of matter that cannot be adiabatically connected to a trivial product state. For example, topological entanglement entropy is directly related to the total quantum dimension of a common feature, these phases have topologically protected gapless excitations on a physical boundary. When the ground state is spatially cut into left and right halves, the low-lying part of the entanglement spectrum shares the same universal characteristics as the energy spectrum of these boundary excitations [3–9].

A recent work studied the entanglement spectrum obtained from an extensive partition that divides a system into two \textit{extensive} subsystems [10]. For topological states that support gapless edge states, the corresponding entanglement spectrum was found to encode a wealth of information about the universal quantum critical behavior that would arise at its phase transition to a trivial direct-product state, despite the fact that the system under study itself is noncritical. It has been further shown that the entanglement spectra of extensive partitions can be directly computed from the matrix product state or tensor network representation of ground-state wavefunctions [11], which may offer insights into topological phase transitions [12–15].

In this Rapid Communication, we study the entanglement spectrum generated from a random partition that spatially bipartitions a system in a \textit{probabilistic} manner: Each physical site is chosen to be in subsystem \( B \) (or \( A \)) with a probability \( p \in [0,1] \) (or \( 1-p \)). We apply this random partition to a translationally invariant (i.e., clean) topological state, and argue on general grounds that the corresponding entanglement spectrum reproduces the universal behavior about its \textit{disorder-driven} transition to a trivial localized phase. As an example, we analytically derive the form of the entanglement Hamiltonian for a clean one-dimensional topological superconductor under random partition, and establish the entanglement phase diagram as a function of probability \( p \), finding agreement with the physical phase diagram of a disordered superconductor [16].

We begin by considering a translationally invariant topological state, which can be either a topological insulator or superconductor or a bosonic symmetry-protected topological phase. It has been shown [10] that upon varying the geometry of \( A \) and \( B \) subsystems in an extensive partition, the corresponding entanglement Hamiltonian undergoes a gap-closing transition that lies in the same universality class as the transition to a topologically trivial state realized by tuning the \textit{physical} Hamiltonian. This intriguing connection is absent when applying the extensive partitioning to a topologically trivial wavefunction, and follows from the nature of topological phase transitions, which are driven by the percolation of gapless edge states. For example, the transition from a quantum Hall insulator to a trivial insulator is described by the quantum percolation of chiral edge states in a Chalker-Coddington network model [17]. Extensive partitioning of a quantum Hall insulator precisely creates, in the low-lying part of the entanglement spectrum, a network of chiral edge states moving along the percolating borders between \( A \) and \( B \). This mapping explains why topological phase transitions and entanglement spectra of extensive partitions are intimately related. It further motivates us to study the random partitioning of a topological ground state, for which the entanglement spectrum is expected to mimic the network model with randomness and thus connect with the localization transition.

We define the probabilistic partitioning of a clean, topological state \( |\Psi\rangle \) as follows. We independently choose each physical site in the full system to be in the \( B \) subsystem with probability \( p \); the remaining sites are defined to be in subsystem \( A \). A partial trace of the density matrix over sites in subsystem \( B \) yields a reduced density matrix for the \( A \) subsystem \( \rho_A \equiv \text{Tr}_B |\Psi\rangle \langle \Psi| \), which can be interpreted as the \textit{thermal} density matrix at temperature \( T=1 \) for an entanglement Hamiltonian \( H_A; \rho_A \equiv e^{-H_A/T} \).

Our goal is to determine the phase diagram of \( H_A \) as a function of the partitioning probability \( p \). First, when \( p \to 0 \), the probabilistic partitioning yields a vanishingly small \( B \) subsystem, with most sites belonging to the \( A \) subsystem. In this limit, the ground state of the entanglement Hamiltonian,
denoted by $|\psi_A\rangle$, must share the same topological index as the original ground state $|\Psi\rangle$. As $p \to 1$, however, most sites become part of the $B$ subsystem, so that $|\psi_A\rangle$ becomes a trivial product state over the disjoint regions of the $A$ subsystem, and hence must be topologically trivial. Since the topological character of $|\psi_A\rangle$ changes as we tune the partitioning probability $p$, we conclude that the entanglement Hamiltonian $H_A$ must go through a phase transition at some critical partitioning probability $p = p_c$. Physically, the transition is driven by the percolation of a random network of gapless edge states propagating around traced-out regions of the $B$ subsystem, as in the case of the aforementioned checkerboard-type extensive partition. Even though the original state $|\Psi\rangle$ is translationally invariant, the probabilistic partitioning procedure introduces randomness into the entanglement Hamiltonian $H_A$, with the probability $p$ effectively tuning disorder strength.

The phase diagram of the entanglement Hamiltonian as a function of $p$ satisfies additional constraints. For a given bipartition, the eigenvalue spectra of the reduced density matrices $\rho_A$ and $\rho_B$ are identical, though their Hilbert spaces are distinct. By definition, for a fixed partitioning probability $p$, the $B$ subsystem is, on average, equivalent to the $A$ subsystem obtained with a partitioning probability $1 - p$. Therefore, the ensemble-averaged spectra of the entanglement Hamiltonians $H_A(p)$ and $H_B(1 - p)$ must be identical. As a result, the presence of a phase transition in the entanglement Hamiltonian with partition probability $p$ implies another transition at probability $1 - p$. In the case where the topological index of $|\Psi\rangle$ cannot be evenly divided between two subsystems, as is the case for topological insulators with a $Z_2$ index or quantum Hall insulators with an odd Chern number, we further expect that $H_A$ exhibits at least a topological phase transition at partitioning probability $p = 1/2$, when the two subsystems are equivalent on average.

For the remainder of this Rapid Communication, we apply our random partitioning procedure to Kitaev’s model [18] for a clean one-dimensional $p$-wave superconductor, extract the phase diagram of the entanglement Hamiltonian as a function of partitioning probability, and demonstrate its correspondence with a disordered superconductor. The Kitaev model is described by the Hamiltonian

$$H = -w \sum_n \left(c_{n+1}^{\dagger} c_n + \text{H.c.}\right) + \sum_n \left(\Delta c_{n+1}^{\dagger} c_n + \text{H.c.}\right)$$

$$- \mu \sum_n \left(c_n^{\dagger} c_n - \frac{1}{2}\right).$$

(1)

with fermion operators $c_n, c_n^{\dagger}$ satisfying canonical anticommutation relations. To simplify the calculation below, we restrict ourselves to the case where $w = \Delta \in \mathbb{R}$. Introducing two species of Majorana fermions at each lattice site $\gamma_n \equiv c_n + c_n^{\dagger}$ and $\chi_n \equiv (c_n - c_n^{\dagger})/i$, we may write the Hamiltonian as

$$H = \frac{i w}{2} \sum_n \left[\eta \gamma_n \chi_n + \chi_n \gamma_{n+1}\right].$$

(2)

where the dimensionless parameter $\eta \equiv \mu/2w$ distinguishes the topologically trivial strong pairing phase ($|\eta| > 1$) and topologically nontrivial weak pairing phase ($|\eta| < 1$). As shown in Fig. 1, in the topological superconductor (TSC) phase, Majorana fermions couple more strongly across two adjacent lattice sites than within a lattice site, leading to two unpaired Majorana fermions at the ends of the chain. In the extreme limit when $\eta = 0$ in the topological phase, pairs of Majorana fermions within lattice sites completely decouple.

We will refer to this special point $\eta = 0$ as the Kitaev limit of the Hamiltonian (2), which proves to be a useful starting point for our analysis below.

Let $|\Psi(\eta)\rangle$ be the ground state of the Kitaev Hamiltonian with parameter $|\eta| < 1$ in the topological regime. Applying the random partitioning procedure to this ground state, we trace over physical lattice sites (each of which contains two Majorana fermions) with probability $p$, and obtain an entanglement Hamiltonian $H_A(\eta; p)$. Clearly, $H_A(\eta; p)$ contains random couplings between sites in the $A$ subsystem, and mimics the physical Hamiltonian of a disordered superconductor, with the partitioning probability $p$ playing the role of disorder strength.

To derive $H_A(\eta; p)$, we note that the entanglement Hamiltonian of a free fermion system such as the Kitaev model can only contain fermion bilinear terms [19]. The spectrum of the entanglement Hamiltonian over an $N$-site subsystem $A$ can be determined from the $N \times N$ correlation matrices in the original ground state $C_{nm} \equiv \langle \Psi | c_n^{\dagger} c_m | \Psi \rangle$ and $F_{nm} \equiv \langle \Psi | b_n^{\dagger} b_m | \Psi \rangle$, by solving the eigenvalue problem [19],

$$(2\hat{C} - 2\hat{F} - 1)(2\hat{C} + 2\hat{F} - 1)\phi_\ell = \tan^2\left(\frac{\epsilon_\ell}{2}\right) \phi_\ell,$$

(3)

where $\epsilon_\ell$ is an eigenvalue of the entanglement Hamiltonian, with eigenvector $\phi_\ell$. Rewriting the complex fermions in terms of Majorana operators, we may define the $2N \times 2N$ skew-symmetric correlation matrix $\Gamma$ for the Majorana fermions which has eigenvalues $\pm \tan(\epsilon_\ell/2)$ [20,21]. Since we are interested in the low-lying part of the entanglement spectrum $\epsilon_\ell \to 0$, $\tan(\epsilon_\ell/2) \to \epsilon_\ell/2$, and hence $\Gamma$ satisfies

$$2\Gamma \phi_\ell \approx \pm \epsilon_\ell \phi_\ell.$$

(4)

Therefore, the correlation matrix for Majorana fermions in the original ground state is equivalent to the entanglement Hamiltonian acting on low-lying states $\epsilon_\ell \to 0$ in the entanglement spectrum. By building the correlation matrix for Majorana fermions in the ground state $|\Psi(\eta)\rangle$, we may now construct the entanglement Hamiltonian for the Kitaev model after a random partition.

For arbitrary $\eta$, performing a random partition will generally produce an entanglement Hamiltonian with highly non-local couplings, due to the nonvanishing correlations between distant Majorana fermions in the ground state. However, for a sufficiently small $|\eta|$, i.e., when the system is close to the
Kitaev limit, we may derive the form of \( H_A(\eta; p) \) analytically. Let us first consider the case \( \eta = 0 \), when pairs of Majorana fermions decouple. A single cut between two adjacent lattice sites then produces, in the entanglement spectrum of the \( A \) subsystem, an unpaired Majorana fermion at the end of \( A \). Aside from this, the entanglement spectrum at \( \eta = 0 \) is identical to the energy spectrum (properly normalized) of decoupled Majorana pairs in the Kitaev Hamiltonian. Therefore, performing a random partition with several cuts will yield an \( A \) subsystem that consists of disjoint segments, each of which hosts unpaired Majorana fermions at the two ends.

We now explicitly construct the entanglement Hamiltonian \( H_A(\eta; p) \) near the Kitaev limit by perturbing away from \( \eta = 0 \). As one may expect, a small \( \eta \) induces a small coupling between the unpaired Majorana fermions at ends of disjoint segments with the rest of the \( A \) subsystem. By an analytical calculation \[21\], we find that couplings between two Majorana fermions in \( H_A(\eta; p) \) decrease exponentially with their separation in the original lattice. Therefore, it suffices to include nearest-neighbor couplings within subsystem \( A \) only in \( H_A(\eta; p) \).

Two types of nearest-neighbor couplings appear in \( H_A(\eta; p) \). First, to leading order in \( \eta \), couplings belonging to a connected sequence of sites in the \( A \) subsystem are identical to those appearing in the original Kitaev Hamiltonian, after a proper normalization. Second, couplings between Majorana fermions belonging to different segments in the \( A \) subsystem are computed from their two-point correlation function. If a series of \( N \) consecutive lattice sites—between sites \( m - 1 \) and \( m + N \)—are determined to be within the \( B \) subsystem and traced over in the random partition, a coupling will be induced between the Majorana fermions at the right and left edges of the two lattice sites, which is found to be proportional to \( C(N) \equiv \langle \Psi(\eta)|l\Xi_{m-1}\gamma_{m+N}|\Psi(\eta) \rangle \). An explicit calculation \[21\] yields the result that at long distances, i.e., large \( N \),

\[
C(N) \sim \eta^N/\sqrt{N} + O(\eta^{N+1}).
\]  

We then conclude that the entanglement Hamiltonian takes the form

\[
H_A(\eta; p) = \frac{i}{2} \sum_{n \in A} \eta \gamma_n \chi_n + \frac{i}{2} \sum_{m > n \in A} f_{nm} \chi_n \gamma_m.
\]  

where \( f_{nm} \) has nonzero elements \( f_{n,m+1} = 1 \) and \( f_{nm} = C(m-n-1) \) if \( n \) and \( m \) label lattice sites at the right and left edges of two adjacent segments in the \( A \) subsystem. The couplings in the entanglement Hamiltonian are illustrated in Fig. 2.

![Figure 2](https://example.com/figure2.png)

**FIG. 2.** (Color online) The entanglement Hamiltonian \( H_A(\eta; p) \), derived in the Kitaev limit \( |\eta| \ll 1 \), consists of several couplings between adjacent Majorana fermions in the subsystem \( A \). Blue and green hoppings between nearest-neighbor Majorana fermions appear with dimensionless coupling 1 and \( \eta \), respectively. Tracing over \( N \) lattice sites induces a coupling \( O(\eta^N) \) between the dangling Majorana modes on the adjacent chains in the \( A \) subsystem.

Now, if we consider the amplitude at the end of the chain, we see that \( \psi_N \) takes the form \[ |\psi_N \rangle = \eta^{\ell_k+\ell_{k+1}} |0 \rangle_{C(d_k)} |0 \rangle_{C(d_{k+1})} \ldots \eta^{\ell_{N-1}} |0 \rangle_{C(d_{N-1})} |0 \rangle_{C(d_N)} \] in a basis of Majorana sites in \( A \). From the form of the two-point function computed previously we see that the amplitude for the state on the first Majorana site in the \( N \)th cluster of the \( A \) subsystem is given by

\[
\psi_N = \prod_{k=1}^{N-1} \frac{\eta^{\ell_k}}{C(d_k)} = \eta^{\sum_{j=1}^{N-1} \ell_j - d_k}.
\]

When \( p < 1/2 \), we observe that the state \[7\] is an exact zero-energy eigenstate of the Hamiltonian Hamiltonian. When \( p > 1/2 \), however, the amplitude at the end of the chain diverges and the above state becomes non-normalizable for an infinite set of clusters.

The above calculation of an edge state immediately implies that the entanglement Hamiltonian \( H_A \) changes from being topologically nontrivial at partitioning probability \( p < 1/2 \) to trivial when \( p > 1/2 \), and hence must be critical at the point
This transition can also be understood by integrating out the Majorana fermions in the interior of the clusters in the $A$ subsystem and constructing an effective entanglement Hamiltonian $H_{A}^{\text{eff}}$ acting exclusively on the dangling Majorana modes at the ends of each cluster. In this case, $H_{A}^{\text{eff}}$ will describe a dimerized Majorana fermion chain, in which two adjacent Majorana fermions correspond to sites separated by lengths $\ell_{k}, \ell_{d_{k}}, \ell_{k+1}, \ell_{d_{k+1}} \ldots$ in the original lattice. The nearest-neighbor hopping in $A$, which is proportional to the corresponding correlation function in the ground state, is determined by the lengths $\ell_{d_{k}}$ for intracluster hoppings or $\{\ell_{k+1}\}$ for intercluster hoppings. At $p = 1/2$, the $A$ and $B$ subsystems are equivalent on average, so that the length distributions $\{\ell_{k}\}$ and $\{d_{k}\}$ are identical, and the ensemble of $H_{A}^{\text{eff}}$ is translationally invariant, instead of dimerized. The corresponding ground state of a one-dimensional Majorana fermion chain is well known to be critical [22].

We now demonstrate that in the vicinity of $p = 1/2$, the entanglement Hamiltonian is in Griffiths phases, characterized by a singularity in the density of states at zero energy due to the proliferation of segments of the topologically ordered or trivial phase, respectively. Recall that when $p < 1/2$, near $p \approx 1/2$, the characteristic size of the clusters in the $A$ subsystem is larger than that of the $B$ subsystem. Then, the dangling Majorana modes on adjacent clusters in the $A$ subsystem, separated by distance $x$, will mix to form localized bound states with finite energy $\epsilon \sim \exp[c x \ln |\eta|]$, with $c > 0$ a constant. Since the probability of such a configuration of sites in the $A$ subsystem is $p^{2}(1-p)^{2}$, the contribution of these low-energy modes to the density of states in the entanglement ground state is [16]

$$\rho(\epsilon) = \int_{0}^{\infty} dx (1-p)^{2} p^{2} \delta(\epsilon - e^{x \ln |\eta|}) \propto \frac{1}{\epsilon^{1-\beta(p)}},$$

(10)

with the nonuniversal exponent $\beta(p) \equiv \ln(p)/c \ln |\eta|$. The power-law singularity in the density of states signals the presence of a Griffiths region for an entanglement ground state with $p$ near $1/2$, due to the proliferation of low-energy configurations of Majorana edge modes dimerizing across lattice sites. $p > 1/2$ also correspond to a Griffiths phase, with exponent $\beta(1-p)$ due to Majorana modes at the ends of the same chain forming bound states with exponentially small energy. The two Griffiths phases at $p < 1/2$ and $p > 1/2$ are both characterized by a power-law singularity in the density of states at zero energy, but are topologically distinct, as shown by the presence and absence of zero-energy Majorana fermions at the boundary.

To summarize, we have introduced a random partitioning scheme to study the disorder-driven quantum critical behavior of a topological phase; applying this procedure to the one-dimensional $p$-wave superconductor yields an interesting phase diagram, consisting of two topologically distinct Griffiths phases separated by a critical point. In addition to its theoretical unconventional, studying a disorder-driven topological phase transition via quantum entanglement poses distinct advantages over conventional numerics, especially when dealing with interacting systems. First, our approach only requires knowledge of a single translationally invariant ground state in the absence of disorder, while a numerical study of a disordered quantum critical point requires knowledge of the full low-lying spectrum, for every disorder realization. Second, the entanglement spectrum obtained from a random partition exhibits a duality between the partitioning probability $p$ and $1-p$. This guarantees a topological phase transition takes place at $p = 1/2$ in topological states with an irreducible topological index [10], as shown by the above example. In contrast, identifying the location of the phase transition point is highly nontrivial in direct numerical studies of disordered Hamiltonians.

The random partitioning scheme can be used straightforwardly to study spin chains [23,24], as well as higher-dimensional systems to numerically extract critical exponents of disorder-driven phase transitions, such as localization transitions in all Altland-Zirnbauer symmetry classes of noninteracting topological phases [25,26]. It will be interesting to see whether our method applies to topological crystalline insulators, whose entanglement spectra show nontrivial features [27]. It might also be interesting to study the entanglement spectrum of fractional topological phases under a random partition, although the connection with topological phase transitions appears to be less direct.

We thank Tim Hsieh for helpful discussions. This work is supported by the DOE Office of Basic Energy Sciences, Division of Materials Sciences and Engineering under Award No. DE-SC0010526.


