**Lower-critical spin-glass dimension from 23 sequenced hierarchical models**

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>As Published</td>
<td><a href="http://dx.doi.org/10.1103/PhysRevE.92.022136">http://dx.doi.org/10.1103/PhysRevE.92.022136</a></td>
</tr>
<tr>
<td>Publisher</td>
<td>American Physical Society</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
</tr>
<tr>
<td>Accessed</td>
<td>Fri Apr 28 22:52:02 EDT 2017</td>
</tr>
<tr>
<td>Citable Link</td>
<td><a href="http://hdl.handle.net/1721.1/98229">http://hdl.handle.net/1721.1/98229</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>Article is made available in accordance with the publisher’s policy and may be subject to US copyright law. Please refer to the publisher’s site for terms of use.</td>
</tr>
<tr>
<td>Detailed Terms</td>
<td></td>
</tr>
</tbody>
</table>
Lower-critical spin-glass dimension from 23 sequenced hierarchical models

Mehmet Demirtaş,1 Ashi Tuncer,2 and A. Nihat Berker1,3

1Faculty of Engineering and Natural Sciences, Sabancı University, Tuzla 34956, Istanbul, Turkey
2Department of Physics, Istanbul Technical University, Maslak 34469, Istanbul, Turkey
3Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 19 December 2014; revised manuscript received 11 July 2015; published 24 August 2015)

The lower-critical dimension for the existence of the Ising spin-glass phase is calculated, numerically exactly, as \( d_L = 2.520 \) for a family of hierarchical lattices, from an essentially exact (correlation coefficient \( R^2 = 0.999999 \)) near-linear fit to 23 different diminishing fractional dimensions. To obtain this result, the phase transition temperature between the disordered and spin-glass phases, the corresponding critical exponent \( y_T \), and the runaway exponent \( y_R \) of the spin-glass phase are calculated for consecutive hierarchical lattices as dimension is lowered.

DOI: 10.1103/PhysRevE.92.022136 PACS number(s): 64.60.De, 75.10.Nr, 05.10.Cc, 75.50.Lk

I. INTRODUCTION

Singulate phase diagram behavior as a function of spatial dimensionality \( d \) compounds the interest and challenge of the phase transitions problems, as effectively posing the “phase transition of phase transitions” problem. Most visible are the lower-critical dimensions, which are the spatial dimensional thresholds for different types of orderings. For example, the lower-critical threshold for ferromagnetic ordering in magnetic systems is \( d_L = 1 \) for one-component (Ising) spins and \( d_L = 2 \) for spins with more than one component. Similarly, the lower-critical dimensions for ferromagnetic ordering under quenched random fields [1–7] are respectively \( d_L = 2 \) and \( d_L = 4 \) for one-component spins and for spins with more than one component. The method that we use in this study gives correctly the lower-critical dimensions of the Ising and \( q \)-state Potts models \( (d_L = 1) \), of the \( (n > 1) \)-component vector spin models \( (d_L = 2) \), of the Ising model with quenched random fields \( (d_L = 2) \), as well as the algebraic order of the \( XY \) model at its lower-critical dimension \( d_L = 2 \) [7–13].

On the question of the spin-glass lower-critical dimension, \( d_L = 2.5 \) was obtained from replica symmetry-breaking mean-field theory[14]. Renormalization-group work, on a family of hierarchical lattices different from ours below, has found \( d_L \) close to 2.5 [15]. Extrapolation to lowest \( d \) of a high-\( d \) expansion on this family of hierarchical lattices has yielded \( d_L = 2.504 \) [16]. Detailed numerical fit to the spin-glass critical temperatures for integer dimensions has also suggested \( d_L = 2.5 \) [17].

In other theory, early renormalization-group work [18], on in effect two hierarchical lattices again different from ours below, has obtained \( 2 < d_L < 3 \). Other theoretical works have claimed \( d_L = 4 \) from ordered-phase stability studies [19–21], \( 2 < d_L < 3 \) from transfer-matrix studies [22], and \( d_L = 2 \) from Monte Carlo [23,24] and ground-state studies [25]. A very recent experimental study [26] on Ge:Mn films has shown the spin-glass lower-critical dimension to be \( 2 < d_L < 3 \).

As seen above, the lower-critical dimension need not be integer, in view of physical fractal systems, hierarchical lattices, and algebraic manipulations that analytically continue. To our knowledge, spin-glass ordering is the only system that exhibits this behavior. In fact, it is of interest to find the exact value of the lower-critical dimension. Our current work does this for the case of the family of hierarchical lattices studied here, with \( d_L = 2.520 \). We obtain this result from a remarkably good fit (correlation coefficient \( R^2 = 0.999999 \)) to the renormalization-group runaway exponent \( y_R \) from the numerically exact renormalization-group solution of a family of 23 hierarchical models with noninteger dimensions \( d = 2.46, 2.63, 2.77, 2.89, 3.00, 3.10, 3.18, 3.26, 3.33, 3.40, 3.46, 3.52, 3.58, 3.63, 3.68, 3.72, 3.77, 3.81, 3.85, 3.89, 3.93, 3.97, 4.00 \). Our result is also consistent with the results that are graphically displayed in Ref. [15] and with the extrapolation to lowest \( d \) of a high-\( d \) expansion in Ref. [16] for a different family of hierarchical lattices. In fact, the comparison and coincidence of spin-glass lower-critical dimensions from different families of hierarchical lattices, started here, is of continuing interest.

II. LOWER-CRITICAL DIMENSION FROM SEQUENCED HIERARCHICAL MODELS

Hierarchical models are constructed [27–31] by imbedding a graph into a bond, as exemplified in Fig. 1, and repeating this procedure by self-imbedding infinitely many times. This procedure can also be done on units with more than two external vertices, e.g., the layered Sierpinski gasket in Ref. [32]. When interacting systems are placed on hierarchical lattices, their renormalization-group solution proceeds in the reverse direction than the lattice build-up just described, each eliminated elementary graph generating a renormalized interaction strength for the ensuing elementary bond. Hierarchical lattices were originally introduced [27] as presenting exactly soluble models with renormalization-group recursion relations that are identical to those found in approximate position-space renormalization-group treatments of Euclidian lattices [8,9], identifying the latter as physically realizable approximations. However, from the above, it is clear that any graph (or graphs [31]) may be chosen in the self-imbedding procedure and one need not be faithful to any approximate renormalization-group solution. Hierarchical lattices [33–70] have been used to study a variety of spin-glass [71] and other statistical mechanics problems.

The length rescaling factor \( b \) in a hierarchical lattice is the number of bonds on the shortest distance between the external vertices of the elementary graph which is replaced by a single...
FIG. 1. (a) The construction of the family of hierarchical lattices used in this study. Each lattice is constructed by repeatedly self-imbedding the graph. The graphs here are $n$ parallel series of $b = 3$ bonds. The dimension $d = 1 + \ln n/\ln b$ of each lattice is given. The renormalization-group solution consists in implementing this process in the reverse direction for the derivation of the recursion relations of the local interactions. The lattices shown here and 19 other lattices with nearby fractional dimensions are used in our calculations. (b) The family of hierarchical lattices with $n_1$ parallel $b = 3$ series of $n_2$ parallel bonds. The resulting hierarchical models are equivalent to the family in (a) with $n = n_1n_2$, with respect to identical critical exponents and phase diagram topology including the occurrence or nonoccurrence of a spin-glass phase.
Similar previous studies, on other spin-glass systems, are in Refs. [33–42].

IV. DIMINISHING CRITICAL, RUNAWAY EXPONENTS, CRITICAL TEMPERATURES AND THE LOWER-CRITICAL DIMENSION OF THE SEQUENCE

For our chosen sequence of hierarchical systems (Fig. 1), we have calculated, at antiferromagnetic bond concentration $p = 0.5$, the phase transition temperature $1/JC$ where the renormalization-group flows bifurcate between the disordered-phase and the spin-glass-phase attractor sinks. The spin-glass sink is characterized by an interaction probability distribution $P(J_{ij})$ that is symmetric in ferromagnetism-antiferromagnetism ($J_{ij} \geq 0$) and that diverges in interaction absolute value: The average interaction strength $\langle |J| \rangle$ across the system diverges as $b^{\eta_T}$ where $n$ is the number of renormalization-group iterations and $y_R > 0$ is the runaway exponent. The spin-glass sink and simultaneously the spin-glass phase disappears when the runaway exponent $y_R$ reaches 0 [42]. The calculated spin-glass phase transition temperatures and critical and runaway exponents are given in Fig. 2 and in Table I as a function of spatial dimension $d$. The lattice with $d = 2.46$, not having a spin-glass phase, is below the lower-critical dimension. For the 22 other consecutive lattices with a spin-glass phase, we have chosen to fit the runaway exponent values, since they gives an excellent, near-linear fit with an amazingly satisfactory correlation coefficient of $R^2 = 0.999 999$. This fit gives, with a small extrapolation, $y_R = 0$ for $d = 2.520$. Note the near linearity, namely, the smallness of the quadratic coefficient in Eq. (5). (In fact, a linear fit gives $y_R = 0$ for $d = 2.516$, with a little less amazingly satisfactory correlation coefficient of $R^2 = 0.999 992$.)

Our calculated lower-critical dimension $d_L$, where the spin-glass phase disappears at zero temperature, is thus seen to be $d_L = 2.520$, for the sequence of hierarchical lattices studied here. It is noteworthy that $d_L$ is not an integer, contrary to previous examples of lower-critical dimensions (and even contrary to upper-critical dimensions, where mean-field behavior sets in) for other systems.

Another important quantity is the critical exponent $\gamma_T = 1/\nu > 0$ of the phase transition between the disordered and spin-glass phases. This exponent is calculated from the scaling behavior of small deviations of the average interaction strength from its fixed finite value at the unstable fixed distribution of the phase transition. The calculated critical exponents are also given in Fig. 2. As the spatial dimension is lowered, $\gamma_T$ also approaches 0. At the lower-critical dimension, $\gamma_T$ reaches 0. The disordered-spin-glass phase transition disappears at $d_L$, where the spin-glass phase disappears.

V. CONCLUSION

Our family of hierarchical lattices (Fig. 1) yields smooth and systematic behavior in all three quantities: the critical temperatures $1/JC$ and critical and runaway exponents $y_T$ of the phase transitions between the spin-glass and paramagnetic phases as a function of dimension $d$, for the hierarchical models with antiferromagnetic bond concentration $p = 0.5$. The runaway exponents $y_R$ of the spin-glass phase are also shown and give a perfect fit to $y_R = -1.309 08 + 0.528 513d - 0.003 548 05d^2$, leading with a small extrapolation to the lower-critical dimension $d = 2.520$ for $y_R = 0$, with a very satisfactory correlation coefficient of $R^2 = 0.999 999$.

\[ y_R = \frac{1}{1/JC} \]

\[ y_T \]

\[ y_R = \frac{1}{\nu} \]

\[ \gamma_T = \frac{1}{\nu} \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]

\[ \nu \]

\[ \gamma_T \]

\[ 1/JC \]

\[ y_T \]

\[ y_R \]

\[ d_L \]
temperatures $1/J_C$, the critical exponents $\gamma_C$, and, eminently
fitably, the runaway exponents $\gamma_R$. All three quantities yield
the lower-critical temperature of $d_L = 2.520$. It is noteworthy
that $d_L$ is not an integer, contrary to previous examples of
lower-critical dimensions (and even contrary to upper-critical
dimensions, where mean-field behavior sets in) for other systems.

ACKNOWLEDGMENTS

We thank Prof. H. Nishimori for suggesting this calculation
to us. Support by the Alexander von Humboldt Foundation,
the Scientific and Technological Research Council of Turkey
(TÜBİTAK), and the Academy of Sciences of Turkey (TÜBA)
is gratefully acknowledged.