A note on light geometric graphs

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A note on light geometric graphs

Eyal Ackerman∗ Jacob Fox† Rom Pinchasi‡

March 19, 2012

Abstract

Let $G$ be a geometric graph on $n$ vertices in general position in the plane. We say that $G$ is $k$-light if no edge $e$ of $G$ has the property that each of the two open half-planes bounded by the line through $e$ contains more than $k$ edges of $G$. We extend the previous result in [1] and with a shorter argument show that every $k$-light geometric graph on $n$ vertices has at most $O(n\sqrt{k})$ edges. This bound is best possible.

Keywords: Geometric graphs, $k$-near bipartite.

1 Introduction

Let $G$ be an $n$-vertex geometric graph. That is, $G$ is a graph drawn in the plane such that its vertices are distinct points and its edges are straight-line segments connecting corresponding vertices. It is usually assumed, as we will assume in this paper, that the set of vertices of $G$ is in general position in the sense that no three of them lie on a line.

A typical question in geometric graph theory asks for the maximum number of edges that a geometric graph on $n$ vertices can have assuming a forbidden configuration in that graph. This is a popular area of study extending classical extremal graph theory, utilizing diverse tools from both geometry and combinatorics. For example, an old result of Hopf and Pannwitz [3] and independently Sutherland [7] states that any geometric graph on $n$ vertices with no pair of disjoint edges has at most $n$ edges. This is a special case of Conway’s thrackle conjecture.

Let $e$ be an edge of $G$. We say that $G$ has a $k$-light side with respect to $e$, if one of the two open half-planes bounded by the line through $e$ contains at most $k$ edges of $G$. If $G$ has a $k$-light side with respect to every edge $e$, then we say that $G$ is $k$-light. In other words, $G$ is $k$-light if no edge of $G$ has the property that each of the two open half-planes bounded by the line through $e$ contains more than $k$ edges of $G$.

The notion of a $k$-light graph is a weakening of the notion of a $k$-near bipartite graph defined in [1]. A graph $G$ is $k$-near bipartite if every line in the plane bounds an open half plane containing at most $k$ edges of $G$. Therefore, every $k$-near bipartite graph is also a $k$-light graph. It is shown in [1] that $k$-near bipartite graphs on $n$ vertices contain $O(\sqrt{kn})$ edges. In this paper we prove the same result for $k$-light graphs, thus strengthening the result in [1]. Moreover, our proof is much shorter but on the other hand relies on other results about geometric graphs.

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2 The maximum number of edges in \( k \)-light geometric graphs

We are interested in the maximum number of edges of an \( n \)-vertex \( k \)-light geometric graph. A simple construction from [1] shows an \( n\sqrt{k} \) lower bound for \( k \leq \left( \frac{2}{k} - 1 \right)^{3/2} \), even for \( k \)-near bipartite graphs. In this construction every line contains at most \( k \) edges of \( G \) in one of the two open half-planes bounded by it. Another construction of a \( k \)-light graph with \( nk \) edges is obtained by taking the vertices of a regular \( n \)-gon and connecting by edges vertices whose cyclic distance is at most \( \sqrt{k} \). In this construction, however, it is no longer true that every line bounds an open half-plane containing at most \( k \) edges of \( G \).

Our main result shows that these constructions are essentially best possible.

**Theorem 1.** Let \( n \) and \( k \) be positive integers. Every \( n \)-vertex \( k \)-light geometric graph has at most \( O(n\sqrt{k}) \) edges.

**Proof.** Let \( G \) be an \( n \)-vertex \( k \)-light geometric graph with \( m \) edges. We orient every edge \( e \) of \( G \) in such a way that the open half-plane bounded to the left of \( e \) contains at most \( k \) edges of \( G \). Because \( G \) is \( k \)-light such an orientation exists.

We will need the following two lemmas.

**Lemma 2.1.** Let \( G \) be an oriented geometric graph on \( n \) vertices. There exists an absolute constant \( c_3 \) such that if \( G \) has more than \( c_3n \) edges, then it contains an edge \( e \) such that the open half-plane bounded to the left of \( e \) contains an edge of \( G \).

**Proof.** It is enough to show that in any (unoriented) geometric graph \( G \) with \( n \) vertices and sufficiently many (that is, at least \( c_3n \)) edges there is an edge \( e \) such that each of the two open half-planes bounded by the line through \( e \) contains an edge of \( G \). This is in fact the case \( k = 1 \) in Theorem 1 that we wish to prove. The reader is encouraged to find a simple proof of this fact. Here we will rely on a rather elaborate argument of Valtr [8] that proves a much stronger statement than what we need.

We refer the reader to [5, 4, 8]. Two edges of a geometric graph are called avoiding or sometimes parallel if no line passing through one edge meets the other edge. Equivalently, two edges are avoiding if they are opposite edges in a convex quadrilateral.

The notion of avoiding edges was first defined by Kupitz [5], who conjectured that any geometric graph on \( n \) vertices with more than \( 2n - 2 \) edges must contain a pair of avoiding edges. In [4] it is shown that if a graph \( G \) on \( n \) vertices does not contain a pair of avoiding edges, then the number of edges in \( G \) is at most \( 2n - 1 \). In [8] Valtr improved this bound by one, completing the proof of Kupitz’ conjecture. He further generalized this result, showing that for any fixed \( k \), every geometric graph with more than \( c_4n \) edges contains \( k \) pairwise avoiding edges. Here \( c_4 \) is an absolute constant that depends only on \( k \).

In fact, Valtr’s result is a bit stronger. Looking into the proof in [8] reveals that he actually shows that a geometric graph with more than \( c_4n \) edges contains \( k \) edges \( e_1, \ldots, e_k \) that are pairwise avoiding, but what is more important to our needs is that the line through \( e_i \) separates \( e_1, \ldots, e_{i-1} \) from \( e_{i+1}, \ldots, e_k \). More specifically, Valtr defines three partial orders on a set of edges in \( G \) and any chain with respect to any of the partial orders is a collection of such edges. It is then shown that if the number of edges in \( G \) is large enough, then there exists a chain of length \( k \) in one of the partial orders.

Thus, for the case \( k = 3 \) it follows that if \( G \) contains more than \( c_3n \) edges, then there are three pairwise avoiding edges \( e, f, g \) such that the line through \( f \) separates \( e \) and \( g \). This immediately implies Lemma 2.1 as in any orientation of \( f \) the half-plane bounded to the left of \( f \) will contain an edge of \( G \).

**Lemma 2.2.** Let \( G \) be an oriented geometric graph on \( n \) vertices with \( m \) edges. There exists a positive absolute constant \( d \) with the following property. If the number of edges in \( G \) is greater than \( 2c_3n \) (where \( c_3 \) is the constant from Lemma 2.1), then \( G \) contains at least \( dm^3/n^2 \) pairs of edges \((e, f)\) such that the open half-plane bounded to the left of \( e \) contains \( f \).
Proof. This is by now a quite standard consequence of the result in Lemma 2.1 and is carried out by a similar probabilistic technique used to derive a similar bound for the number of pairs of crossing edges in a geometric graph (see p. 55 in [6], also p. 45 in [2]).

Denote by \(x(G)\) the number of pairs of edges \((e, f)\) in \(G\) such that the open half-plane bounded to the left of \(e\) contains \(f\). Pick every vertex of \(G\) independently with probability \(p\), and denote by \(G' = (V', E')\) the subgraph of \(G\) that is induced by the chosen vertices. Clearly, \(E[|V'|] = pn\), \(E[|E'|] = p^2m\), and \(E[x(G')] = p^4x(G)\). On the other hand, it follows from Lemma 2.1 that \(x(G') \geq |E'| - c_3|V'|\), and this holds also for the expected values: \(E[x(G')] \geq E[|E'|] - c_3E[|V'|]\). Plugging in the expected values and setting \(p = 2c_3n/m < 1\) we get that \(x(G) \geq \frac{1}{8c_3} \frac{m}{n^2}\).

Let \(c_3\) and \(d\) be the constants from Lemmas 2.1 and 2.2. Clearly we may assume that \(G\) contains at least \(2c_3n\) edges or else we are done. By Lemma 2.2, \(G\) contains at least \(dm^3/n^2\) pairs \((e, f)\) of edges such that the open half-plane bounded to the left of \(e\) contains \(f\). However, by the choice of orientation of the edges in \(G\), an edge \(e\) can belong to at most \(k\) such pairs \((e, f)\). We conclude that \(dm^3/n^2 \leq km\). This now easily implies that \(m \leq \frac{1}{\sqrt{d}} n\sqrt{k}\) as desired.

References