NOTES ON BRANCING TIME

In any multi-modal theory - any semantic theory for a language with more than one modal operator - we need, for each modal operator, a set of indices or "possible worlds" and an R relation on them. In the bimodal tense logic, the indices were moments of time rather than possible worlds, and the two operators (past and future) were interpreted relative to the same set of moments of time. The two R relations were converses of each other. The branching time theory combines tense with modality so that we can consider their interaction. The language will have, in addition to the two tense operators, also a necessity operator. The possible worlds relative to which the necessity operator is interpreted will themselves consist of sets of moments of time, arranged in a linear order.

Since there are two different indices (two different sets of entities playing the role of possible worlds) for interpreting the different operators, the theory is two dimensional as well as multi-modal. That is, sentences of the language will be interpreted relative to the two different parameters. The truth or falsity of a sentence will be defined relative to a pair consisting of a world and a time.

In all of the theories we have considered so far, the "worlds" are primitive points. They are not assumed to have any particular internal structure; the structure is in the relations between the worlds. But in the branching time theory, the possible worlds (possible histories) are not primitive, but are defined in terms of more basic elements of the structure. The accessibility relations relative to which the three modal operators are defined are also not primitive, but are defined in terms of one basic R relation.

The basic structure is a frame of the familiar kind, <K,R> where K is a nonempty set, the possible times or moments, and R is a relation that defines a tree structure. That is, R satisfies the following conditions:

- It is transitive: (x)(y)(z)((xRy ̈  yRz)  ċ xRz)
- It is backward connected: (x)(y)(z)((xRy ̈  zRy)  ċ (xRz  ̈  zRx  ̈  x=z))
- It is irreflexive: (x)~xRx

Intuitively, xRy means that y is in the possible future of x.

Possible histories are represented by complete paths through the tree. A subset h of K is a history if and only if it satisfies the following two conditions:

1. For any x and y, if x,y  Ě h, then xRy or yRx or x=y.
2. h is not a proper subset of any subset of K meeting condition (1).

Timeless propositions are sets of possible histories. Tensed propositions are sets of pairs consisting of history and a time that is a member of the history. In the object language to be defined, sentences express tensed propositions, so the value of a sentence will be a function determining a truth value as a function of a time and a possible world. If v is the valuation function, P is a sentence letter, h a history, and i  Ě h, then  \( \forall^i(P) \subset \{1,0\} \).
The language contains, in addition to the usual truth functional operators (with the usual semantic rules), three operators, F, P and □ meaning "it will be the case that," "it was the case that" and "it is settled, or fixed that." The semantic rules are as follows:

\[ v^h_i(F\phi) = 1 \text{ iff } v^j_i(\phi) = 1 \text{ for some } j \in h \text{ such that } iRj. \]

\[ v^h_i(P\phi) = 1 \text{ iff } v^j_i(\phi) = 1 \text{ for some } j \in h \text{ such that } jRi. \]

\[ v^h_i(\Box \phi) = 1 \text{ iff } v^{h'}_i(\phi) = 1 \text{ for all } h' \text{ such that } i \in h'. \]

What should the logic look like? Notice that, the rules for the tense operators define values at one time in terms of values for other times within the same history. Histories are sets of moments defined so that the relation R determines a linear ordering of all the moments in such a set. So the logic of F and P will include the basic linear tense logic. We can add the requirement that our basic R relation be dense, or alternatively that it be discrete, and we can require that it be serial (in one or both directions), or alternatively that there be a first or last moment of each history, adding the corresponding axioms to the system.

As the tenses are interpreted in terms of different moments of time within a fixed world, so the modal operator is interpreted in terms of different histories that share a fixed time. The binary relation between worlds, or histories, is a time indexed relation: what is necessary at one time is not the same as what is necessary at another time. The definition of the relation is this:

\[ h =_t h' \text{ iff } t \in h \cap h' \]

For any t, this relation is symmetric, transitive, and (for \( t \in h \)) reflexive, so the logic of this concept of necessity will be S5.

In addition to the axioms for the appropriate linear tense logic and for an S5 modal logic, there will be one additional axiom that concern the interaction of the two kinds of operators:

\[ \vdash F\Box \phi \supset \Box F\phi \]

This axiom reflect the asymmetry of the structure: if a path will be open at some future time, then it is open now.

Here are two theorems concerning the interaction of tense and modality that follow from this axiom, with the help of the axioms for the basic tense logic and S5 modal logic:

\[ \vdash \Box H\phi \supset H\Box \phi \]

\[ \vdash \Box P\phi \supset P\Box \phi \]