We have seen last time that both the metalinguistic strategy and the truth-table strategy are inadequate for developing a semantics for Modal Logic. Let us explore another approach.

Recall our aim: we want to find some feature of propositions on the basis of which we can define ' and the connectives value-functionally – that is to say we can define them in such a way that the value of a complex expression is a function of the value of its component parts.

Let us begin by acknowledging a clear intuition: propositions differ not only with respect to whether they are true or false, but also with respect to when they are true or false. In other word, propositions have not only truth-values but also truth-conditions. It is obvious that two propositions can both be true but are nevertheless different. "Obama is president" and "Berlin is the German capital" are both true. However, there are possible circumstances under which one is true and the other is false: they have different truth-conditions.

Now think of a "possible world" as a complete specification of a way things can be. A possible world thus renders a proposition true or false. We can then represent the truth-conditions of a proposition as the set of possible worlds in which that proposition is true, or equivalently, as the characteristic function of that set, namely the function from worlds to truth-values which maps a world to 1 iff the proposition is true in that world.

It turns out that taking the value of propositions to be functions from worlds to truth-values preserves the value-functionality of negation and conjunction. All we need to do is to relativize – by subscripting - the valuation function to a possible world. In other word, we move from (1) to (2).

(1)  
  a. \( v(\neg \varphi) = 1 \) iff \( v(\varphi) = 0 \)  
  b. \( v(\varphi \land \psi) = v(\varphi) \times v(\psi) \)

(2)  
  a. \( v_w(\neg \varphi) = 1 \) iff \( v_w(\varphi) = 0 \)  
  b. \( v_w(\varphi \land \psi) = v_w(\varphi) \times v_w(\psi) \)

What about '□'? In the simplest theory, a proposition is necessarily true in a given world \( w \) iff it is true in all possible worlds.

(3) \( v_w(\square \varphi) = 1 \) iff \( v_w(\varphi) = 1 \) for all \( w' \)

This says – basically - that the truth of a modal statement does not depends on the facts.

At this point, let us take note of some terminology. We will take the truth value of the proposition expressed by a sentence \( \varphi \) to be the extension of \( \varphi \), and the set of possible worlds in which that proposition has the value 1 to be the intension of \( \varphi \). Furthermore, we will identify a proposition with the set of possible worlds in which it
is true. Thus, the extension of a sentence is a truth-value and its intension is a proposition.

As we said, (3) represents the simplest semantics of '☐'. There is a way to generalize it which gives us a whole range of alternatives. The idea is the following. Intuitively, the truth of a modal claim in this world depends on how things are in some other worlds. It could be that we have to look at every world to evaluate a modal sentence in this world, but we should leave open the possibility that only a proper subset of worlds are relevant. We should also leave open the possibility that each world has its own set of relevant worlds: a modal claim is true in w if things are so and so in this and this world, and true in w' if things are so and so in that and that world.

This means that we should impose on the set of possible worlds a binary relation R such that w' is relevant for determining the truth of modal claims in w iff wRw'. Accordingly, we change (3) to (4).

\[(4) \quad v_w(☐\varphi) = 1 \text{ iff } v_{w'}(\varphi) = 1 \text{ for all } w' \text{ such that } wRw'\]

R is called the accessibility relation. If we take R to be the universal relation, (4) is equivalent to (3). But the point is that R need not be the universal relation. In fact, we have opened up the possibility of investigating the consequences for our system of R having different properties (e.g. reflexivity, transitivity etc).

We define a frame to be a triple <W, R, @>, where W is the set of all possible worlds, R is the accessibility relation, and @ is the actual world. Some definitions of frame omit the third element.

An ordered pair <F, v> is called a model, where F is a frame and v is a valuation, i.e. a function from sentence letters to truth-values.

We can now define the notion of logical truth as "true in all models".
24.244 Modal Logic
Fall 2009

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.