1. Frame, model, possible world

A frame is a triple \( <W,R,@> \) where \( W \) is a set of possible worlds, \( R \) is a binary relation on \( W \) (the 'accessibility relation') and \( @ \) is a designated member of \( W \) (the 'actual world'). The intuitive idea behind the formal notion of the accessibility relation is that necessity itself is something that can vary from world to world.

As an example, think of a temporal notion of necessity. The options that are open to me now are different from the ones available to me tomorrow. I may take the plane or the train now, but tomorrow I can only take the train, because the plane has left. We see how time and modality are related.

Ability can also be understood as a modal notion: if the world were different, my capacity would be different. If only I hadn't had that skiing accident and broken my leg, I would be able to run, but in fact I can't run because I really had had the skiing accident. So even if we fix the time, the possibilities that are open in one possible world might be different from those that are open in another possible world.

Knowledge might also be conceived as a modal notion: the epistemic possibilities in the actual world might have been different if things had been different. I know I have hands, but in a world where I am a brain in a vat, it will not be the case that I have hands, and so not true, in that world, that I know that I have hands.

So we allow for the possibility that a proposition may be necessary, while at the same time it is a contingent fact that the proposition is necessary.

A model is a pair \( <F,v> \), where \( F \) is a frame \( <W,R,@> \). Intuitively, a frame is the part of the model which is wholly independent from the language. It is the theorist's representation of the 'subject matter', i.e. of what is talked about.

The function \( v \) in the model links the language to what it talks about. We can think of \( v \) simply as a stipulated function which assigns values to sentence letters. The semantic rules will then extend it to all other sentences by specifying how the value of a complex sentence is to be determined by the values of its constituent sentences. Alternatively and equivalently, you can think of \( v \) as a function from all sentences to their semantic value, and take the semantic rules to be constraints on \( v \): only a function which satisfies these constraints will be considered a valuation.

Note that I include the actual world \( @ \) in the frame. This is not done by everyone. Hughes and Cresswell (1996) defines a frame to be a pair \( <W,R> \). For most of what we do, the difference is not substantive. It becomes substantive only when we want to define \( R \) with reference to the actual world.

Is there something distinctive about the actual world? You might say of course there is: it is real. But isn't it true that anything we can say about any possible world, we can say about the actual world as well? Is that not what it means to be possible: to
be possibly actual? What would a world look like which is not actual but merely possible? Woody Allen told a story in which fictional characters sit around wondering whether they are fictional or not. They reflect on the emptiness of their lives and conclude that a real person would have a much richer life than they do so they're probably fictional. What prevents us from doing the same, i.e. from looking at our world and ask ourselves whether it is one of those merely possible worlds? Look at quantum mechanics for example. Some people think it can't be really true. But it is true in our world so we must be merely possible! Or similarly consider the fact that the holocaust took place in our world: that fact might be evidence that this can't be the actual world, especially if you buy the theological claim that the actual world is the best among all possible worlds. But this kind of reasoning seems absurd.

The point is that there is nothing that is true in this world which distinguishes it from the other worlds, as the actual world. Modal concepts which treat the 'actual world' differently on the basis of what is true might thus be philosophically suspect. But on the other hand, there are linguistic operators – such as 'actually' – whose definition presupposes the notion of an actual world: 'actually p' is true in a world w iff p is true in the actual world. This definition allows us to capture certain scope facts such as that presented by (1). (I believe this sentence is from Russell.)

(1) I wish my yacht were longer than it actually is.

To evaluate (1), we have to carry the actual length of the speaker's yacht into the non-actual world of his wish and compare it with the length of his yacht there. So if you want to model the behavior of 'actually', you need the notion of a distinguished actual world.

2. The system K

Let's talk about system K now. Its axioms include all tautologies of propositional logic (PL) plus the following sentence, which is called K.

(1) □(p ⊃ q) ⊃ (□p ⊃ □q)

Notice that the tautologies and K are particular sentences. The rule of Substitution says that the result of uniformly replacing sentence letters in a theorem with arbitrary sentences is a theorem. For example, we can uniformly substitute p with (p ∨ q) and q with ∼r in (1). The result is (2), and it is a theorem.

(2) □((p ∨ q) ⊃ ∼r) ⊃ (□(p ∨ q) ⊃ □∼r)

Thus, Uniform Substitution ensures that any sentence 'of the same form' as a PL tautology or (1) is a theorem. Now instead of doing things this way, we can get rid of Substitution and work with axiom schemata instead of particular sentences as axioms. For example, instead of postulating (1) and saying that its sentence letters can be substituted for uniformly, we postulate the axiom schema in (3).

(3) □(φ ⊃ ψ) ⊃ (□φ ⊃ □ψ)

It is part of the definition of an axiom schema that the result of replacing the metalinguistic variables, φ or ψ, in (3) with any object language sentences be an axiom. So on this way of thinking, both (1) and (2) are axioms: both of them instantiate the scheme in (3). In one case, the (metalinguistic) variables φ and ψ are
assigned the values $p$ and $q$, respectively. In the other case, they are assigned the values $(p \lor q)$ and $\sim r$, respectively.

A theorem is an axiom or a sentence derived from an axiom by a derivation rule. A derivation rule of $K$ is (4), which is called 'Modus Ponens' or 'detachment'.

(4) $\vdash \phi, \vdash \phi \supset \psi \Rightarrow \vdash \psi$

The symbol '$\vdash$' represents theoremhood. Modus Ponens says the following: if $\phi$ is a theorem and $\phi \supset \psi$ is also a theorem, then $\psi$ is a theorem. The '$\supset$' is thus a metalinguistic symbol. Basically, it says that we may write $\psi$ after $\phi$ and $\phi \supset \psi$ in a proof.

Let us at this point make clear the concept of a proof. A proof is a finite sequence of sentences, every one of which is either an axiom or derived from some other previous sentences in the sequence by a derivation rule.

The second derivation rule of $K$ is (5), which is called Necessitation.

(5) $\vdash \phi \Rightarrow \vdash \Box \phi$

Necessitation says that if $\phi$ is a theorem then 'necessarily $\phi$' is also a theorem. The intuition behind Necessitation is the following: if $\phi$ is a theorem, then it is valid; if it is valid, it is true in all worlds in all models; but if that is the case, then obviously it is also necessarily true in all worlds and all models, i.e. its necessitation is also a theorem.

That's basically the system $K$. 