In an proof-theoretic system, we start with a set of axioms, which are either listed individually or identified by an effective procedure (e.g. schemes, truth-tables). The transformation rules tell us how to derive sentences from other sentences. A proof is then conceived as a sequence of sentences, each of which is either an axiom or derived from previous sentences in the sequence by a transformation rule. A theorem is a sentence which appears in a proof.

A sentence $S$ is said to be **derivable** from a set $\Gamma$ of sentences – i.e. $\Gamma \vdash \phi$ – if and only if $(S \supset \phi)$ is a theorem, where $S$ is a conjunction whose conjuncts are all members of $S$.

A set $\Gamma$ of sentences is **consistent** if no contradiction is derivable from it. In other word, $\Gamma$ is consistent iff it is not the case that every sentence is derivable from it. Hence, $\Gamma$ is **inconsistent** iff every sentence is derivable from it.

Thus, consistency is defined in terms of derivability, which in turn is defined in terms of theoremhood.

On the semantic side, a set $\Gamma$ of sentences is said to be **satisfiable** iff there exists a model for $\Gamma$ – i.e. a model in which all members of $\Gamma$ are true. A sentence $S$ is a **semantic consequence** of a set $\Gamma$ of sentences – i.e. $\Gamma \models \phi$ – iff every model for $\Gamma$ is also a model in which $S$ is true.\(^1\)

A sentence is **valid** iff it is true in every model. In other word, a sentence is valid iff it is a semantic consequence of every set. Because if every model for every set is a model in which $\phi$ is true, then obviously $\phi$ is true in every model, and vice versa.

If a set $\Gamma$ is not satisfiable, then there exists no model for $\Gamma$. In that case, it is trivially true that every model for $\Gamma$ is a model in which $\phi$ is true, for any $\phi$. Thus, if $\Gamma$ is not satisfiable, then every sentence is a semantic consequence of $\Gamma$.

So we have the following cross-relation between proof-theoretic and semantic concepts.

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\(^1\) We take a model to be a sequence $<W,R,v,w^*>$, where $W$ is a non-empty set of possible worlds, $R$ is a binary relation on $W$, $v$ is a function from sentences and worlds to truth-values, and $w^*$ a designated member of $W$, i.e. the 'actual world'. A sentence $\phi$ is true in model $M = <W,R,v,w^*>$ iff $v(\phi,w^*) = 1$. Thus, the notion "true in a model" makes sense only if the model contains a designated 'actual' world $w^*$. If we do not include $w^*$ in the model, to say there exists a model for $\Gamma$ would be to say there exist a model $<W,R,v>$ such that for some $w \in W$, $v(\phi,w) = 1$ for all $\phi \in \Gamma$, and to say that $\phi$ is true in all models would be to say that $\phi$ is true in all worlds in all models. So we see that there is – at this point at least – no substantive difference between including $w^*$ in a model and not. I opt for the former, anticipating the possibility which might come later that we will want to constraint $R$ in terms of $w^*$. 
(1)  a.  $\Gamma$ is consistent iff $\Gamma$ is satisfiable  
    b.  $\phi$ is a theorem iff $\phi$ is valid  
    c.  $\Gamma \vdash \phi$ iff $\Gamma \models \phi$

It is useful to think of claims in terms of their universal vs. existential character. Saying that $\Gamma$ is satisfiable is making an existential claim: there exists a model for $\Gamma$. But saying $\Gamma$ is consistent is making an universal claim, or negating an existential claim: there does not exist a proof containing $(\phi \supset (p \land \neg p))$, where $\phi$ is a conjunction of members of $\Gamma$. To say $\phi$ is a theorem is to say that there exists a proof containing $\phi$, but to say that $\phi$ is valid is to say that $\phi$ is true in all models, a universal claim. But of course, we make these claims for all sentences, so overall it is a universal claim, which however has existential and universal parts.

It's important to pay attention to the universal or existential nature of the claim you want to prove, because the way you go about proving an existential claim is different from the way you go about proving a universal claim. The most direct way to prove an existential claim is to produce an example. Of course, to prove an existential claim which holds true for an arbitrary entity is more complex, since in effect you're proving a universal claim.

The completeness proof will have this form. We are trying to prove the following.

(2)  For any set $\Gamma$ of sentences, if $\Gamma$ is consistent, then $\Gamma$ is satisfiable

The strategy for proving this will be: take an arbitrary consistent set and show that – in terms of the properties of that set – you can define a model for that set. In other word, take any arbitrary consistent set and show that an entity can be constructed which meets the conditions of being a model: an ordered sequence consisting of a non-empty set, a binary relation on that set, a designated member of that set and a valuation function. One can construct such an entity as a function of the arbitrary consistent set $\Gamma$, and then make use of the properties of consistency in showing that the thing one constructed is in fact a model that does the work, i.e. a model which makes all sentences of $\Gamma$ true.

So to prove that for everything of some kind, there exist something of another kind, we can take an arbitrary thing of the first kind and show that we can construct, for that thing, a thing of the second kind.

Intuitively, completeness means that if you have a valid sentence, you can prove it. In other word,

(3)  For any sentence $\phi$, if $\phi$ is valid, then $\phi$ is a theorem

So a completeness proof is going from a semantic notion to a proof-theoretic notion. Because of the internal relationship between consistency and theoremhood on the one hand, and validity and satisfiability on the other, (2) actually implies (3), so proving (2) is proving (3). Here is why.

If (2) holds for sets of sentences, then it holds for $\{\neg \phi\}$, a particular set. Thus, the following is implied by (2): if $\{\neg \phi\}$ is consistent, then $\{\neg \phi\}$ is satisfiable, which by contraposition implies that if $\{\neg \phi\}$ is not satisfiable, then $\{\neg \phi\}$ is inconsistent. By definition, $\{\neg \phi\}$ is not satisfiable iff there exists no model in which $\neg \phi$ is true, i.e. if
φ is true in every model, i.e. if φ is valid. And also by definition, {∼φ} is inconsistent iff a contradiction is derivable from it, and by tautological reasoning, a contradiction is derivable from {∼φ} iff φ is a theorem.

Thus, (2) implies (3).

See handout for details of the proof.