I want to talk a little bit today about strategies of solving problems. When you present your proof, I don't want you to cite derived rule 6 or theorem 2.2 from the book. There are a few short cuts which are enough to reduce the proof to manageable size. For example, you can take tautological reasoning for granted. You can also substitute tautological equivalents within the scope of modal operators. It's important to keep in mind that the latter is a more radical short cut than the former.

Let's go through one simple example. One of your assignments was to prove the following sentence.

(1) \((\lozenge p \land \lozenge (q \supset r)) \supset (\Box(p \supset q) \supset \lozenge (p \land r))\)

A successive conditional like \((A \supset (B \supset C))\) means that if you have \(A\) and you have \(B\), then you can get \(C\). Thus, a way to approach (1) is ask yourself why – if you have (2) and (3) – you can get (4).

(2) \(\Box p \land \lozenge (q \supset r)\)
(3) \(\Box(p \supset q)\)
(4) \(\lozenge (p \land r)\)

Using semantic intuitions, we can see why rather easily. From (2) it can be deduced that \(p\) is necessary and from (3), that \((p \supset q)\) is necessary. It then follows that \(q\) must be necessary too.\(^1\) Now from (2) it also follows that \((q \supset r)\) is possible. If \(q\) is necessary and \((q \supset r)\) is possible, then \(r\) must be possible.\(^2\) If \(p\) is necessary and \(r\) is possible, then \((p \land r)\) must be possible.

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\(^1\) Here is how to prove it.

1. \(\Box p\) given
2. \(\Box(p \supset q)\) given
3. \(\Box(p \supset q) \supset (\Box p \supset \Box q)\) K
4. \(\Box p \supset \Box q\) 2, 3, Modus Ponens
5. \(\Box q\) 1, 4, Modus Ponens

\(^2\) Here is how to prove it.

1. \(\Box q\) given
2. \(\lozenge (q \supset r)\) given
3. \(q \supset ((q \supset r) \supset r)\) tautology
2. \(\Box(q \supset ((q \supset r) \supset r))\) 2, necessitation
3. \(\Box q \supset \Box((q \supset r) \supset r)\) K, Modus Ponens
4. \(\Box((q \supset r) \supset r)\) 1, 3, Modus Ponens
5. \(\Box((q \supset r) \supset r) \supset (\lozenge(q \supset r) \supset \lozenge r)\) theorem of K
6. \(\lozenge(q \supset r) \supset \lozenge r\) 4, 5, Modus Ponens
7. \(\lozenge r\) 2, 6, Modus Ponens
Let me say a few words about meta-theoretical reasoning. It is important to keep apart object language variables such as p, q etc. and meta-language variables such as φ, ψ etc. A statement like (5),

(5) \[ \vdash \Box(\phi \supset \psi) \supset (\Diamond \phi \supset \Diamond \psi) \]

is a statement about sentences of the object language. Specifically, it is a universal claim about these. It says that every sentence of the form (5) is a theorem.

In problem 2.11, you are asked to prove that the following rule preserves validity in K and D but not in T.

(7) if \( |\models \Box \phi \supset \Box \psi \) then \( |\models \phi \supset \psi \)

This rule is a universal claim about the object language. It really says this.

(8) For all sentences φ and ψ: if \( |\models \Box \phi \supset \Box \psi \) then \( |\models \phi \supset \psi \)

Now to do our proof, we can turn (8) into (9).

(9) For all sentences φ and ψ: if \( \phi \supset \psi \) is not valid, then \( \Box \phi \supset \Box \psi \) is not valid.

We prove (9) by showing that whenever we have a countermodel for \( \phi \supset \psi \), we have a countermodel for \( \Box \phi \supset \Box \psi \), with φ and ψ being any sentences.

In our proof, the existence of a countermodel for \( \phi \supset \psi \) is assumed. Thus, we take it for granted that \( \phi \) is true and \( \psi \) is false in some world a. To show that there is a countermodel for \( \Box \phi \supset \Box \psi \), we add a world b which sees only a and which a does not see. This guarantees that \( \Box \phi \) is true and \( \Box \psi \) is false in b, making the model which contains both a and b a countermodel for \( \Box \phi \supset \Box \psi \).

Note that if we make the additional assumption that b sees itself, then \( \Box \phi \supset \Box \psi \) will only be false in b if \( \phi \) is true in b. But we are not allowed to assume that. Saying that b can see itself and \( \phi \) is true in b is saying something about \( \phi \), namely that it can be true in a world which sees itself. But our argument must be about all \( \phi \), and so cannot assume that \( \phi \) might be true in a world that can see itself. That is why the argument does not work for the system T, and reflexive R.