\[ m \frac{d^2r}{dt^2} = \vec{F} \]

\[ m \omega \times \frac{d\omega}{dt} = \vec{r} \times \vec{F} = \text{torque} = \vec{\tau} \]

\[ \Rightarrow \quad \frac{d^2\vec{r}}{dt^2} = \vec{\tau} \]

\[ \frac{d}{dt} \left( m \vec{r} \times \frac{d\vec{r}}{dt} \right) = \frac{d}{dt} \vec{L} \]

\[ \vec{L} = m\vec{r} \times \vec{v} = \vec{r} \times \vec{p} = \text{angular momentum} \]

Analogue to \( \frac{d\vec{p}}{dt} = \vec{F} \) for linear momentum.

In central forces \( \vec{\tau} = 0 \Rightarrow \vec{L} \) is conserved. We used this recently (Kepler's 2nd).

Example: Gravitational capture

\[ \begin{array}{c}
\text{mass} M \\
\text{body} b \\
\end{array} \quad \begin{array}{c}
\vec{v} \\
\text{far-away body, } m \ll M \\
\end{array} \]

Choose x-axis along v, ignore CM reduction

\[ \vec{L} = mrv \]

\[ E = \frac{1}{2} m v^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r} = \frac{1}{2} m v^2 + \frac{MBv^2}{2R^2} - \frac{GMm}{r} \]

\[ = E_0 = \frac{1}{2} mv^2 \]

At closest, is \( r = R \) allowed?

Requires \( \left( \frac{b^2}{R^2} - 1 \right) \leq \frac{2GM}{R} \)

\[ \Rightarrow \quad \frac{b^2}{R^2} \leq 1 + \frac{2GM}{R} \]
This is \( b = R \) as \( \nu \to \infty \), but is enhanced over the "geometrical" capture for finite \( \nu \).

Angular Momentum in Systems:

\[ m_i \frac{d\vec{r}_i}{dt^2} = \vec{F}_i + \sum_j \vec{F}_{ij} \]

\[ \sum_i m_i \vec{r}_i \times \frac{d\vec{r}_i}{dt^2} = \sum_i \vec{r}_i \times \vec{F}_i + \sum_j \sum_i \vec{r}_i \times \vec{F}_{ij} \]

\[ \sum_j \sum_i \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2} \sum_j \left( \sum_i \vec{r}_i \times \vec{F}_{ij} + \sum_j \vec{r}_j \times \vec{F}_{ji} \right) \]

\[ = \frac{1}{2} \sum_j \vec{F}_{ij} \times \left( \vec{r}_i - \vec{r}_j \right) \]

\[ \text{3rd law} \quad \frac{1}{2} \sum_j \vec{r}_j \times \vec{F}_{ij} \]

Now if \( \vec{F}_{ij} \propto \vec{r}_i - \vec{r}_j \) (central force), this = 0.

Then

\[ \sum_i m_i \vec{r}_i \times \frac{d\vec{r}_i}{dt^2} = \sum_i \vec{r}_i \times \vec{F}_i \]

\[ \sum_i \frac{d}{dt} \sum_i \vec{r}_i \times \vec{R} \]

\[ \Rightarrow \sum_i \frac{d}{dt} \sum_i \vec{r}_i \times \vec{r}_i = \vec{0} \]

\[ \Rightarrow \frac{d}{dt} \sum_i \vec{r}_i = \vec{0} \]

\[ \Rightarrow \sum_i \vec{r}_i = \vec{L} \]

\[ \Rightarrow \frac{d}{dt} \sum_i \vec{r}_i = \vec{L} \quad \text{with} \quad \vec{L} = \sum_i \vec{L}_i \]

\[ \vec{L} = \sum_i \vec{r}_i \]
The result is more general than this derivation.

Example: rod-pendulum

\[ L = \int_0^1 \sum m \cdot \left( \ddot{\theta} r \right) 
\]

\[ \lambda = \text{mass/length} \]

\[ \ddot{\theta} = \frac{1}{\lambda} \int_0^1 \sum m \ddot{r} \cdot \dddot{\theta} \]

\[ \text{into here} \]

General theorem for near-Earth gravity:

\[ \ddot{r} = \int_0^1 \sum F \cdot \ddot{r} \]

\[ \dddot{r} = \text{d} \cdot \text{d} F \]

\[ \Rightarrow \text{The torque may be calculated as if the force is acting at the CM} \]

\[ \text{NB: Near-Earth gravity only}!! \]

In present example:

\[ \ddot{r} = \text{Mg} \frac{r}{2} \sin \theta \hat{z} \quad \text{(evaluating x product)} \]

\[ \frac{d^2 \theta}{dt^2} = \ddot{\theta} \Rightarrow \frac{d}{dt} \left( \frac{1}{2} \dot{\theta}^2 \right) = M \frac{r}{2} \sin \theta \quad \text{a} \quad \ddot{\theta} = \frac{-3g}{2r} \sin \theta \]

Small \( \theta \):

\[ \ddot{\theta} = \frac{-3g}{2r} \theta \quad \omega = \frac{3g}{2r} \text{ oscillation} \]
Statics

If a rigid body is not moving (no internal motion):

i) $\Sigma F_{\text{ext.}} = 0$

ii) $\Sigma \vec{F}_{\text{ext.}} = 0$

since $\frac{dx}{dt} = 0$, $\frac{d^2x}{dt^2} = 0$. And if $\Sigma F_{\text{ext.}} \neq 0$ or $\Sigma \vec{F}_{\text{ext.}} \neq 0$,
in a rigid body, you will have \underline{visible motion}. So

i) and ii) are the fundamental equations \underline{of statics}. This is
a vast subject in engineering (you want buildings, bridges, etc. to
be static!)

example: lever

\[ F = m_1 g + m_2 g \]

around fulcrum: $l_2 m_2 g - l_1 m_1 g = 0$

\[ \frac{l_1}{l_2} = \frac{m_2}{m_1} \]

example: motorcycle lift-off

\[ \text{torques around } O: \quad h Ma - L Mg + 2LN = 0 \]

\[ \text{with } N > 0 \]

\[ \text{possible only for } a < \frac{L}{h} g \]

\[ \text{with bigger } a, \text{ "lift off"} \]
Note high CM and limited wheelbase are conducive to lift-off.

Converse? If $\Sigma F = 0$, $\Sigma I = 0$ we have $\vec{P} = \vec{P}_0$, $\vec{I} = \vec{I}_0$. The first is removed "trivially" by going to CM frame. The 2nd ($\vec{I}_0 \neq 0$) is much more interesting. It does not correspond to a constant rotation, which in general. We'll treat it down the road.
Rigid Body Dynamics: General Setting

The position and orientation of a rigid body can be specified using 6 quantities. Specifically, we can specify:

i) the position of a reference point A (e.g., the CM) — 3 coordinates

ii) having fixed A, we need to specify the position of a second point B. This lies on a definite sphere around A, since distances are fixed (the body is rigid). This brings in 2 more coordinates

iii) having fixed A and B, we can still make a rotation around the axis through them. We use this to match the actual position of a third point C (fixed in the body). This brings in 1 more angle.

With three (collinear) points fixed, nothing we can "triangulate" to other points based on the distances to A,B,C — the position and orientation of the rigid body are completely specified. It took 6 parameters — three distances, three angles.
We have 6 equations for how things change in time:

\[ \frac{d\vec{x}}{dt} = \vec{F}, \quad \frac{d\vec{v}}{dt} = \vec{a} \]

This is just enough so that given initial values of position, orientation, and their rate of change (so we can get \( \vec{x}(0), \vec{v}(0) \)), we can solve for later values!

**6.1 Rotation Around a Fixed Axis**

The phenomena of rigid body motion can be intricate and counter-intuitive, so we'll build up from special cases, that are of importance in themselves.

Here: pure rotation around a fixed axis

For conciseness choose the rotation axis to be the z-axis. To get an equation for the rate of rotation, we analyze the z-component of \( \frac{d\vec{L}}{dt} = \vec{\omega} \).

\[
\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i
\]

\[
\vec{v}_i = \vec{\omega} \times \vec{r}_i, \quad \text{with} \quad \vec{\omega} = (0, 0, \omega), \quad \vec{r}_i = (x_i, y_i, z_i)
\]

So \( \vec{\omega} \times \vec{r}_i = (-\omega y_i, \omega x_i, 0) \)

\[
\vec{r}_i \times \vec{v}_i = \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = (\ast, \ast, \omega (x_i^2 + y_i^2))
\]

\[
L_z = \sum m_i (x_i^2 + y_i^2) \omega = I_{zz} \omega \quad \text{(or just } I \omega)\]
with \( I = \sum m_i (x_i^2 + y_i^2) = \sum m_i r_i^2 \rightarrow \int \rho(x)r^2 \) 

\( \text{distance}^2 \) from axis.

The value of \( I \) is a property of the body (and the axis!) that does not change with time, as long as these don't. Specifically, it does not change during rotation around the axis. Thus our dynamical equation becomes

\[
\frac{d(I\omega)}{dt} = r_z
\]

\[\text{I} \frac{d\omega}{dt} = I \frac{d^2\theta}{dt^2} \]

Since \( \omega = \frac{d\theta}{dt} \)

Example: rotating mass arrangement

![Diagram of mass arrangement]

Large \( I \)  

Small \( I \)