18.03 Recitation Problems 14  
March 30, 2004  
Laplace Transform; Poles

Laplace transform:

\[ f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st} \, dt. \]

s-shift law: \( e^{at}f(t) \mapsto F(s-a). \)

\[
1 \mapsto \frac{1}{s}, \quad e^{at} \mapsto \frac{1}{s-a}, \quad \cos(\omega t) \mapsto \frac{s}{s^2 + \omega^2}, \quad \sin(\omega t) \mapsto \frac{\omega}{s^2 + \omega^2}.
\]

A “pole” of a complex function \( F(s) \) is a complex number \( z \) at which the function value becomes infinite.

1. Sketch a graph of the “window” or “bump” function \( f(t) = u(t-a) - u(t-b), \) \( 0 \leq a < b, \) and compute its Laplace transform \( F(s) \) using the integral definition.

2. Using fact that the Laplace transform is linear to deduce from 1. what the Laplace transform of \( g(t) = (1/b)(u(t) - u(t-b)) \) is. The “limit” of these functions \( g(t) \) as \( b \to 0 \) is the delta function. What is the limit of their Laplace transforms? (Hint: use l’Hôpital’s rule, or, better, the definition of the derivative.)

3. Using the fact that \( e^{wt} \mapsto 1/(s-w) \) for any complex number \( w, \) together with the expressions

\[
\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i},
\]

to compute the Laplace transforms of \( e^{at}\cos(\omega t) \) and \( e^{at}\sin(\omega t). \)

Where are the poles of these functions of \( s? \)

Sketch a graph of \( e^{-t}\cos(t) \) and of the pole diagram of its Laplace transform. Do the same for \( e^{-t}\sin(t) \) and \( e^t\sin(2t). \)