Truss Design and Convex Optimization

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Outline

- Physical Laws of a Truss System
- The Truss Design Problem
- Second-Order Cone Optimization
- Truss Design and Second-Order Cone Optimization
- Some Computational Results
- Extensions of the Truss Design Problem
Outline

- Semi-Definite Optimization
- Truss Design and Semi-Definite Optimization
- Truss Design and Linear Optimization
A truss is a structure in $d = 2$ or $d = 3$ dimensions, formed by $n$ nodes and $m$ bars joining these nodes.

Example of a truss in $d = 2$ dimensions, with $n = 6$ nodes and $m = 13$ bars.

Examples of trusses include bridges, cranes, and the Eiffel Tower.
The data used to describe a truss is:

- a set of nodes (given in physical space)
- a set of bars joining pairs of nodes with associated data for each bar:
  - the length $L_k$ of bar $k$
  - the Young’s modulus $E_k$ of bar $k$
  - the volume $t_k$ of bar $k$
- an external force vector $F$ on the nodes
Physical Laws of a Truss System

Data Description of a Truss

A truss problem.

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Nodes can be:

- *static* nodes (fixed in place), or
- *free* nodes (movable when the truss is stressed).

A truss problem.
The allowable movements of the nodes defines the degrees of freedom (dof) of the truss. We say that the truss has $N$ degrees of freedom.

$$N \leq nd.$$
Movements of nodes in the truss will be represented by a vector $u$ of displacements $u \in \mathbb{R}^N$.

The external force on the truss is given by a vector $F \in \mathbb{R}^N$.

Displacements of the nodes in the truss cause internal forces of compression and/or expansion to appear in the bars in the truss.

Let $f_k$ denote the internal force of bar $k$. The vector $f \in \mathbb{R}^m$ is the vector of forces of the bars.
Physical Laws of a Truss System

Example of a Truss Problem

There is a single external force applied to node 3 in the direction indicated.

This will result in internal forces $f$ along the bars in the truss and will simultaneously cause small displacements $u$ in all of the nodes.

Nodes 1 and 5 are fixed, and the other nodes are free.

A truss problem.
Physical Laws of a Truss System

Example of a Truss Problem

A truss problem.

We can label the 13 different bars by the nodes they link. The bars will be: 12, 13, \ldots, 56.
A truss problem.

The internal force $f_k$ of bar $k$ can be positive or negative.
If $f_k \geq 0$, bar $k$ has been expanded and its internal force counteracts the expansion with compression.
Physical Laws of a Truss System

Forces on the Bars

Compression

\[ \Delta < 0 \]

A bar under compression.

If \( f_k \leq 0 \), bar \( k \) has been compressed and its internal force counteracts the compression with expansion.
In a static truss the internal forces will balance the external forces in every degree of freedom. This is a law of conservation of forces.

A truss problem.

Consider the balance of forces on node 3:

\[
\begin{align*}
\text{x coordinate:} & \quad -f_{13} - f_{23} \cos(\pi/4) + f_{35} + f_{36} \cos(\pi/4) = -F_{3x} \\
\text{y coordinate:} & \quad +f_{23} \sin(\pi/4) + f_{34} + f_{36} \sin(\pi/4) = -F_{3y}
\end{align*}
\]
For the entire truss we write $N$ linear equations that represent the balance of forces in each degree of freedom.

In matrix notation this is:

$$Af = -F$$

$A$ is an $N \times m$ matrix.

Each column of $A$, denoted as $a_k$, is the projection of the bar onto the degrees of freedom of the nodes that bar $k$ meets.
Physical Laws of a Truss System

...Force Balance Equations...

A truss problem.
In our example, we have:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 1 & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 - x \\
-1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 - y \\
0 & -1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 3 - x \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 3 - y \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 1 & 0 & 0 & 0 & 0 & 4 - x \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 4 - y \\
0 & 0 & 0 & -\frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -1 & 0 & 0 & 0 & 6 - x \\
0 & 0 & 0 & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -1 & 0 & 6 - y
\end{bmatrix}
\]
If bar $k$ has length $L_k$ and cross-sectional area $A_k^c$, Young’s modulus $E_k$, and its length is changed by $\Delta_k$, then the internal force $f_k$ is given by:

$$f_k = E_k \frac{A_k^c \Delta_k}{L_k}.$$ 

$$t_k = A_k^c L_k.$$ 

We write:

$$f_k = \frac{E_k}{L_k^2} t_k \Delta_k.$$
Displacements in the nodes will cause the lengths of the bars to change. Suppose that $L_{12} = L_{13} = L_{35} = 1.0$, with all other bars measured proportionately.

Consider a displacement of: $u = (-\epsilon, -\epsilon, 0, 0, 0, 0, 0, 0, 0)$.

Example of a small displacement in a node.
Physical Laws of a Truss System

Physical Laws

...Distortion and Displacements...

\[ u = (-\epsilon, -\epsilon, 0, 0, 0, 0, 0, 0, 0) \ . \]

Example of a small displacement in a node.
### Physical Laws of a Truss System

Changes in the lengths of the bars under nodal displacement.

<table>
<thead>
<tr>
<th>Bar $k$</th>
<th>New Length</th>
<th>Linearized Length</th>
<th>Linearized Change $\Delta_k$</th>
<th>$(a_k)^T u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$\sqrt{1 + 2\epsilon(\epsilon - 1)}$</td>
<td>$1 - \epsilon$</td>
<td>$-\epsilon$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>$\sqrt{5}$</td>
<td>$\sqrt{5}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td>$\sqrt{2 + 2\epsilon^2}$</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>24</td>
<td>$\sqrt{1 + 2\epsilon(\epsilon + 1)}$</td>
<td>$1 + \epsilon$</td>
<td>$\epsilon$</td>
<td>$-\epsilon$</td>
</tr>
<tr>
<td>25</td>
<td>$\sqrt{5 + 2\epsilon(\epsilon + 1)}$</td>
<td>$\sqrt{5} + \frac{1}{\sqrt{5}}\epsilon$</td>
<td>$\frac{1}{\sqrt{5}}\epsilon$</td>
<td>$-\frac{1}{\sqrt{5}}\epsilon$</td>
</tr>
<tr>
<td>34</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>35</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>36</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>45</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>46</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>56</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

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If $\epsilon$ is small, then the distortion of bar $k$ is nicely approximated by the linear expression:

$$
\Delta_k = -(a_k)^T u .
$$

We approximate the internal force on bar $k$ due to a feasible displacement $u$ as:

$$
f_k = \frac{E_k}{L_k^2} t_k \Delta_k = -\frac{E_k}{L_k^2} t_k (a_k)^T u
$$
Define:

\[
B = \begin{pmatrix}
\frac{L_1^2}{E_1 t_1} & 0 & \cdots & 0 \\
0 & \frac{L_2^2}{E_2 t_2} & \cdots & 0 \\
0 & 0 & \cdots & \frac{L_m^2}{E_m t_m}
\end{pmatrix}, \quad B^{-1} = \begin{pmatrix}
\frac{E_1 t_1}{L_1^2} & 0 & \cdots & 0 \\
0 & \frac{E_2 t_2}{L_2^2} & \cdots & 0 \\
0 & 0 & \cdots & \frac{E_m t_m}{L_m^2}
\end{pmatrix}.
\]

\[
f_k = -\frac{E_k}{L_k^2} t_k (a_k)^T u
\]

can be written as: \[f = -B^{-1} A^T u,
\]
which is
\[Bf + A^T u = 0.\]
We write the physical laws of the truss system as:

\[
Af = -F \quad \text{(conservation of forces)}
\]

\[
Bf + A^T u = 0 \quad \text{(forces, distortions, and displacements)}
\]
Physical Laws of a Truss System

Physical Laws

...Equilibrium Conditions

\[ Af = -F \]  
\[ Bf + A^T u = 0 \]

(conservation of forces)  
(forces, distortions, and displacements)

Combining these yields:

\[ F = -Af \]
\[ = AB^{-1} A^T u \]
\[ = Gu \]

where:

\[ G = AB^{-1} A^T . \]

The matrix \( G \) is called the stiffness matrix of the truss.

Note that \( G \) is an SPSD matrix.
Physical Laws of a Truss System

Physical Laws

Solving the Equations

\[ A f = -F \quad \text{(conservation of forces)} \]
\[ B f + A^T u = 0 \quad \text{(forces, distortions, and displacements)} \]

Form:

\[ G = AB^{-1}A^T . \]

Solve for \( u \):

\[ Gu = F \]

Then compute:

\[ f = -B^{-1}A^T u . \]

This last expression is simply:

\[ f_k = -\frac{E_k}{L_k^2} t_k (a_k)^T u . \]

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The compliance of the truss is the work (or energy) performed by the truss.

The compliance is the sum of the forces times displacements:

\[ F^T u = u^T G u \geq 0. \]

The compliance will generally be a positive quantity.
Consider the following optimization problem:

\[
\text{OP : minimize} \quad \tilde{f} \sum_{k=1}^{m} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} \tilde{f}_k^2
\]

\[
\text{s.t.} \quad \sum_{k=1}^{m} a_k \tilde{f}_k = -F.
\]

The constraints can be written:

\[
-F = \sum_{k=1}^{m} a_k \tilde{f}_k = A \tilde{f}
\]
Physical Laws of a Truss System

...A View from Optimization...

\[
\text{OP : minimize} \quad \tilde{f} \sum_{k=1}^{m} \frac{1}{2} t_k \frac{L_k^2}{E_k} \tilde{f}_k^2
\]

\[
\text{s.t.} \quad \sum_{k=1}^{m} a_k \tilde{f}_k = -F.
\]

This is a convex quadratic problem. The optimality conditions are:

\[
\sum_{k=1}^{m} a_k f_k = -F
\]

\[
\frac{L_k^2}{t_k E_k} f_k + a_k^T u = 0 \quad \text{for} \quad k = 1, \ldots, m.
\]

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Optimality conditions:

$$\sum_{k=1}^{m} a_k f_k = -F$$

$$\frac{L_k^2}{t_k E_k} f_k + a_k^T u = 0 \quad \text{for} \quad k = 1, \ldots, m.$$ 

Re-write as:

$$Af = -F$$

$$Bf + A^T u = 0.$$ 

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Optimality conditions of (OP):

\[ Af = -F \]

\[ Bf + A^T u = 0. \]

These are the original equilibrium conditions of the truss.

The truss system is nature’s solution to an optimization problem.
Physical Laws of a Truss System

...A View from Optimization

$$\text{OP : minimize } \sum_{k=1}^{m} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} \tilde{f}_k^2$$

s.t. $$\sum_{k=1}^{m} a_k \tilde{f}_k = -F.$$  

The optimal objective function value is:

$$\sum_{k=1}^{m} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} f_k^2 = \frac{1}{2} f^T B f = -\frac{1}{2} f^T A^T u = -\frac{1}{2} u^T A f = \frac{1}{2} F^T u$$

The optimal objective value of (OP) is the compliance of the truss system.
The Truss Design Problem

We now consider the volumes \( t_k \) of the bars \( k \) to be design parameters that we wish to determine.

\[
F = G u = A B^{-1} A^T u = \sum_{k=1}^{m} (a_k) (B^{-1} A^T u)_k = \sum_{k=1}^{m} -(a_k) f_k = \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k) (a_k)^T u = G(t) u
\]

where

\[
G(t) = \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k) (a_k)^T .
\]
The Truss Design Problem

\[ G(t) = \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \]

\[ G = G(t) \] is a weighted sum of rank-one matrices (weighted by the volumes \( t_k \)).

\[ G = G(t) \] is a weighted sum of outer-product matrices.

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In designing the truss, we choose the volumes $t_k$ of the bars subject to linear constraints:

$$Mt \leq d \quad t \geq 0.$$  

Typically, these constraints include upper and lower bounds on the volumes of certain bars, as well as an overall cost or volume constraint:

$$\sum_{k=1}^{m} t_k \leq V.$$  

The criteria in truss design is to choose the volumes $t_k$ of the bars so as to minimize the compliance of the truss, namely:

$$F^T u.$$  

Such a truss will be the most resistant to the external force $F$. 
The Truss Design Problem

(TDP): \( \text{minimize}_{t,u} \ F^T u \)

s.t. \( G(t)u = F \)
\( Mt \leq d \)
\( t \geq 0 \)
\( u \in \mathbb{R}^N, \ t \in \mathbb{R}^m \).
The Truss Design Problem

(TDP): minimize \( t, u \) \( F^T u \)

s.t. \[
\begin{bmatrix}
\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k) (a_k)^T
\end{bmatrix} u = F
\]

\[ Mt \leq d \]

\[ t \geq 0 \]

\[ u \in \mathbb{R}^N, \ t \in \mathbb{R}^m. \]

The decision variables are \( u \) and \( t \).

The real decision variables are \( t \) only.

Once \( t \) is chosen, \( u \) will be determined by the solution to the system of equations:

\[ G(t)u = F. \]
The Truss Design Problem

(TDP): minimize_{t,u} \ F^T u
s.t. \ \left[ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] u = F
\Mt \leq d
\t \geq 0
\u \in \mathbb{R}^N, \ \t \in \mathbb{R}^m.

Note that as written, the truss design problem TDP is not a convex problem.
Recall problem (OP):

\[
\text{OP : minimize } f \sum_{k=1}^{m} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} \tilde{f}_k^2
\]

\[
\text{s.t. } \sum_{k=1}^{m} a_k \tilde{f}_k = -F.
\]

Convert \( t = (t_1, \ldots, t_k) \) to be decision variables and add constraints

\[
Mt \leq d
\]

\[
t \geq 0.
\]
Write this as:

$$(TDP_2) : \text{minimize}_{f,t} \sum_{t_k > 0} \frac{1}{2} \frac{L_k^2}{E_k} f_k^2$$

s.t. $$\sum_{t_k > 0} a_k f_k = -F$$

$$Mt \leq d$$

$$t \geq 0$$

$$f \in \mathbb{R}^m, t \in \mathbb{R}^m.$$
The Truss Design Problem

A Convex Version of TDP

\[ \text{OP}: \minimize_f \sum_{k=1}^{m} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} \tilde{f}_k^2 \]

s.t. \[ \sum_{k=1}^{m} a_k \tilde{f}_k = -F \]

\[(\text{TDP}_2): \minimize_{f,t} \sum_{t_k > 0} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} f_k^2 \]

s.t. \[ \sum_{t_k > 0} a_k f_k = -F \]
\[ Mt \leq d \]
\[ t \geq 0 \]
\[ f \in \mathbb{R}^m, t \in \mathbb{R}^m. \]
The Truss Design Problem

A Convex Version of TDP

Two modifications:

1. bars with zero volume \( t_k = 0 \) are no longer counted
2. \( f \) instead of \( \tilde{f} \)
\[(TDP_2): \text{minimize}_{f, t} \sum_{t_k > 0} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} f_k^2\]

s.t. \[\sum_{t_k > 0} a_k f_k = -F\]

\[Mt \leq d\]

\[t \geq 0\]

\[f \in \mathbb{R}^m, t \in \mathbb{R}^m.\]
The Truss Design Problem

A Convex Version of TDP

Re-write as:

\[(TDP_2): \text{minimize}_{f,t,s} \frac{1}{2} \sum_{k=1}^{m} s_k\]

\[\text{s.t.} \quad \sum_{k=1}^{m} a_k f_k = -F\]

\[Mt \leq d\]

\[\frac{L_k^2}{E_k} f_k^2 \leq t_k s_k \text{ for } k = 1, \ldots m\]

\[t \geq 0, \quad s \geq 0\]

\[f, t, s \in \mathbb{R}^m.\]
(TDP₂): minimize \( f, t, s \) \( \frac{1}{2} \sum_{k=1}^{m} s_k \)

s.t. \( \sum_{k=1}^{m} a_k f_k = -F \)

\( Mt \leq d \)

\( \frac{L_k^2}{E_k} f_k^2 \leq t_k s_k \) for \( k = 1, \ldots, m \)

\( t \geq 0, \ s \geq 0 \)

\( f, t, s \in \mathbb{R}^m \).
Further re-write as:

\[(TDP_2) : \text{minimize}_{f, t, s} \frac{1}{2} \sum_{k=1}^{m} s_k\]

s.t. \[Af = -F\]

\[Mt \leq d\]

\[\frac{L_k^2}{E_k} f_k^2 \leq t_k s_k \text{ for } k = 1, \ldots, m\]

\[t \geq 0, \ s \geq 0\]

\[f, t, s \in \mathbb{R}^m.\]
(TDP\(_2\)) : minimize\(_{f,t,s} \frac{1}{2} \sum_{k=1}^{m} s_k \)

s.t. \(Af = -F\)

\(Mt \leq d\)

\(\frac{L_k^2}{E_k} f_k^2 \leq t_k s_k\) for \(k = 1, \ldots m\)

\(t \geq 0, s \geq 0\)

\(f, t, s \in \mathbb{R}^m\).
We have a linear objective function and almost all constraints are linear.

It is pretty easy to show that the constraints:

\[
\frac{L_k^2}{E_k} f_k^2 \leq t_k s_k, \quad t_k \geq 0, \quad s_k \geq 0
\]

describe a convex region.

This is a convex optimization problem.
A second-order cone optimization problem (SOCP) is an optimization problem of the form:

\[
\begin{align*}
\text{SOCP: } \quad & \min_x c^T x \\
\text{s.t. } & \quad Ax = b \\
& \quad \|Q_i x + d_i\| \leq (g_i^T x + h_i), \quad i = 1, \ldots, k.
\end{align*}
\]

In this problem, the norm \(\|v\|\) is the standard Euclidean norm:

\[
\|v\| := \sqrt{v^T v}.
\]

The norm constraints in SOCP are called 
“second-order cone” constraints.
Second-Order Cone Optimization

Linear Optimization is an SOCP

SOCP: \[ \min_x c^T x \]
\[ \text{s.t. } Ax = b \]
\[ \|Q_i x + d_i\| \leq (g_i^T x + h_i), \quad i = 1, \ldots, k. \]

SOCP is a convex problem, because
\[ \|Q_i x + d_i\| - (g_i^T x + h_i) \] is a convex function.

Linear optimization is a special case of SOCP: just set \( Q_i = 0, d_i = 0, h_i = 0, \) and \( g_i \) to be the \( i^{th} \) unit vector, \( i = 1, \ldots, n. \)
Second-Order Cone Optimization

**SOCP:** \( \min_x c^T x \)

s.t. \( Ax = b \)

\[ \| Q_i x + d_i \| \leq (g_i^T x + h_i) , \ i = 1, \ldots, k . \]

Suppose we have the constraint: \( \frac{1}{2} x^T Q x + q^T x + r \leq 0 . \)

Factor \( Q = M^T M \)

\[ \left\| \left( \frac{1}{\sqrt{2}} M x , \ \frac{q^T x + r + 1}{2} \right) \right\| \leq \frac{-q^T x - r + 1}{2} . \]

Square both sides and collect terms.
Second-Order Cone Optimization

\[
\text{SOCP: } \min_x c^T x \\
\text{s.t. } Ax = b \\
\|Q_i x + d_i\| \leq (g_i^T x + h_i), \ i = 1, \ldots, k.
\]

Suppose we have a convex quadratic objective:
\[
f(x) := \frac{1}{2} x^T Q x + q^T x
\]

Create a new variable \(x_{n+1}\) and write:
\[
\min_{x, x_{n+1}} x_{n+1} \\
\text{s.t. } \\
\frac{1}{2} x^T Q x + q^T x \leq x_{n+1},
\]

This can be further re-written as an SOCP.
Recall:

$$(TDP_2) : \text{minimize}_{f,t,s} \frac{1}{2} \sum_{k=1}^{m} s_k$$

s.t. 

$$A f = -F$$

$$M t \leq d$$

$$\frac{L_k^2}{E_k} f_k^2 \leq t_k s_k \text{ for } k = 1, \ldots, m$$

$$t \geq 0, \ s \geq 0$$

$$f, t, s \in \mathbb{R}^m.$$
Make the simple change of variables:

\[ w_k = \frac{1}{2} t_k + \frac{1}{2} s_k \]

\[ y_k = -\frac{1}{2} t_k + \frac{1}{2} s_k \]

Then \( s_k = w_k + y_k \) and \( t_k = w_k - y_k \), and so:

\[ t_k s_k = (w_k - y_k)(w_k + y_k) = w_k^2 - y_k^2, \]
The constraint

\[ \frac{L_k^2}{E_k} f_k^2 \leq t_k s_k \]

becomes:

\[ \frac{L_k^2}{E_k} f_k^2 \leq w_k^2 - y_k^2 . \]

Rearrange this and take square roots:

\[ \sqrt{\frac{L_k^2}{E_k} f_k^2 + y_k^2} \leq w_k , \]

and re-write as:

\[ \left\| \left( y_k, \frac{L_k}{\sqrt{E_k}} f_k \right) \right\| \leq w_k . \]
(TDP$_2$) : minimize$_{f,t,s} \frac{1}{2} \sum_{k=1}^{m} s_k$

s.t. \[ A f = -F \]

\[ Mt \leq d \]

\[ \frac{L_k^2}{E_k} f_k^2 \leq t_k s_k \text{ for } k = 1, \ldots, m \]

\[ t \geq 0, \ s \geq 0 \]

\[ f, t, s \in \mathbb{R}^m. \]
can be re-written as:

\[(\text{CTDP}): \text{minimize}_{f, w, y} \frac{1}{2} e^T (w + y)\]

s.t. \[Af = -F\]

\[\left\| \left( y_k, \frac{L_k}{\sqrt{E_k}} f_k \right) \right\| \leq w_k, \quad k = 1, \ldots, m\]

\[M(w - y) \leq d\]

\[w, y, f \in \mathbb{R}^m.\]
(CTDP): minimize \( f, w, y \)

\[
\frac{1}{2} e^T (w + y)
\]

s.t. \( A f = -F \)

\[
\left\| \left( y_k, \frac{L_k}{\sqrt{E_k}} f_k \right) \right\| \leq w_k, \quad k = 1, \ldots, m
\]

\[
M (w - y) \leq d
\]

\( w, y, f \in \mathbb{R}^m \).
CTDP is a second-order cone problem. The LHS is the norm of a 2-dimensional vector:

\[ \| (v_1, v_2) \| := \left\| \begin{pmatrix} y_k & \left( \frac{L_k}{\sqrt{E_k}} \right) f_k \end{pmatrix} \right\|, \]

and the RHS is the linear expression:

\[ w_k. \]

**Proposition** Suppose that \((f, w, y)\) is a feasible or optimal solution of CTDP. Let:

\[ t = w - y \text{ and } s = w + y. \]

Then \((f, t, s)\) is the corresponding feasible or optimal solution of TDP_2.
We have seen TDP$_2$, which is a convex problem.
We have also seen CTDP, which is a second-order cone problem.
We can also formulate the TDP as a “semidefinite optimization” (SDO for short) problem.

In the special case when the constraints on the volume variables $t_1, \ldots, t_m$ are:

$$t \geq 0 \ , \quad \sum_{k=1}^{m} t_k \leq V \ ,$$

TDP can actually be solved by linear optimization.
The forces and nodes for the basic bridge design model.
Some Computational Results

Three Truss Design Problems

Possible Bars for the Bridge

The set of possible bars for the basic bridge design problem.
Some Computational Results

Hanging Sign Design Problem

The forces and nodes for the hanging sign design model.

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Some Computational Results

Crane Design Problem

The forces and nodes for the crane design model.
Optimal solution to the basic bridge design problem.
Optimal solution to the hanging sign design problem.
Some Computational Results

Solution of the Crane Design Problem

Optimal solution to the crane design problem.
Some Computational Results

Solution of the Enhanced Bridge Design Problem

Lower Bounds on “Road-Surface” Bars

Optimal solution of the bridge design problem, with lower bounds on the “road surface” bar volumes.
### Some Computational Results

#### Details of Problems Solved

Dimensions of the Truss Design Problems

<table>
<thead>
<tr>
<th>Name</th>
<th>Size of Grid</th>
<th>Nodes</th>
<th>Arcs</th>
<th>Degrees of Freedom</th>
<th>Maximum Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridge-16×7</td>
<td>16×7</td>
<td>136</td>
<td>1,547</td>
<td>268</td>
<td>1,600</td>
</tr>
<tr>
<td>Bridge-20×10</td>
<td>20×10</td>
<td>231</td>
<td>2,878</td>
<td>458</td>
<td>2,000</td>
</tr>
<tr>
<td>Bridge-30×10</td>
<td>30×10</td>
<td>341</td>
<td>4,368</td>
<td>678</td>
<td>4,000</td>
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<tr>
<td>Sign-10×20</td>
<td>10×20</td>
<td>231</td>
<td>2,878</td>
<td>444</td>
<td>1,000</td>
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<tr>
<td>Sign-20×30</td>
<td>20×30</td>
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<td>Sign-30×40</td>
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<tr>
<td>Crane-20×40</td>
<td>20×40</td>
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<td>3,188</td>
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<tr>
<td>Crane-30×40</td>
<td>30×40</td>
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<td>4,678</td>
<td>798</td>
<td>10,000</td>
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</table>
### Number of variables and inequalities in the truss design problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>Problem Size</th>
<th>Variables $2m$</th>
<th>Inequalities $2m$</th>
<th>Variables $3m$</th>
<th>Inequalities $m + 1 + LBs$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridge-16×7</td>
<td></td>
<td>†</td>
<td>†</td>
<td>4,641</td>
<td>1,564</td>
</tr>
<tr>
<td>Bridge-20×10</td>
<td></td>
<td>†</td>
<td>†</td>
<td>8,634</td>
<td>2,899</td>
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<tr>
<td>Bridge-30×10</td>
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<td>†</td>
<td>†</td>
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<td>4,399</td>
</tr>
<tr>
<td>Sign-10×20</td>
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<td>5,756</td>
<td>5,756</td>
<td>8,634</td>
<td>2,879</td>
</tr>
<tr>
<td>Sign-20×30</td>
<td></td>
<td>18,116</td>
<td>18,116</td>
<td>27,174</td>
<td>9,059</td>
</tr>
<tr>
<td>Sign-30×40</td>
<td></td>
<td>36,876</td>
<td>36,876</td>
<td>55,314</td>
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<tr>
<td>Crane-10×20</td>
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<td>1,636</td>
<td>1,636</td>
<td>2,454</td>
<td>819</td>
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<tr>
<td>Crane-20×40</td>
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<td>6,376</td>
<td>6,376</td>
<td>9,564</td>
<td>3,189</td>
</tr>
<tr>
<td>Crane-30×40</td>
<td></td>
<td>9,356</td>
<td>9,356</td>
<td>14,034</td>
<td>4,679</td>
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</tbody>
</table>

† The linear optimization model cannot be used for the modified bridge design model.
### Some Computational Results

#### Details of Problems Solved

#### Variables and Equations

<table>
<thead>
<tr>
<th>Problem</th>
<th>LP Model</th>
<th>SOCP Model</th>
<th>SDO Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Variables</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2m$</td>
<td>$3m$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Equations</td>
<td>Constraints</td>
<td>Matrix Dimension</td>
</tr>
<tr>
<td></td>
<td>$N$</td>
<td>$N + m + 1 + LBs$</td>
<td>$N + m + 2$</td>
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<td>1,832</td>
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<tr>
<td>Bridge-20×10</td>
<td>†</td>
<td>8,634</td>
<td>3,357</td>
</tr>
<tr>
<td>Bridge-30×10</td>
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<tr>
<td>Sign-10×20</td>
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<td>8,634</td>
<td>3,323</td>
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<tr>
<td>Sign-20×30</td>
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<td>27,174</td>
<td>10,339</td>
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<tr>
<td>Sign-30×40</td>
<td>36,876</td>
<td>55,314</td>
<td>20,951</td>
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<tr>
<td>Crane-10×20</td>
<td>1,636</td>
<td>2,454</td>
<td>1,007</td>
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<tr>
<td>Crane-20×40</td>
<td>6,376</td>
<td>9,564</td>
<td>3,767</td>
</tr>
<tr>
<td>Crane-30×40</td>
<td>9,356</td>
<td>14,034</td>
<td>5,477</td>
</tr>
</tbody>
</table>

† The linear optimization model cannot be used for the modified bridge design model.
### Compliance of the optimized truss design for the truss design problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>LP</th>
<th>SOCP</th>
<th>SDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridge-16×7</td>
<td>†</td>
<td>27.43365</td>
<td>27.15510</td>
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<td>Bridge-20×10</td>
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<td>52.58314</td>
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<tr>
<td>Bridge-30×10</td>
<td>†</td>
<td>132.46038</td>
<td>‡</td>
</tr>
<tr>
<td>Sign-10×20</td>
<td>0.77279</td>
<td>0.77293</td>
<td>0.77279</td>
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<tr>
<td>Sign-20×30</td>
<td>2.52003</td>
<td>2.52056</td>
<td>‡</td>
</tr>
<tr>
<td>Sign-30×40</td>
<td>4.08573</td>
<td>4.08681</td>
<td>‡</td>
</tr>
<tr>
<td>Crane-10×20</td>
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<td>Crane-20×40</td>
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<td>286.24954</td>
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<td>Crane-30×40</td>
<td>596.73177</td>
<td>596.73324</td>
<td>596.63740</td>
</tr>
</tbody>
</table>

† The linear optimization model cannot be used for the modified bridge design model.
‡ The data to run the SDO model could not be prepared for this instance due to memory restrictions.
Some Computational Results

Details of Problems Solved

Iterations of Interior-Point Method

LOQO was used to solve the LP and SOCP models.

SDPa was used to solve the SDO models.

<table>
<thead>
<tr>
<th>Problem</th>
<th>LP</th>
<th>SOCP</th>
<th>SDO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridge-16×7</td>
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<td>36</td>
</tr>
<tr>
<td>Bridge-20×10</td>
<td>†</td>
<td>55</td>
<td>43</td>
</tr>
<tr>
<td>Bridge-30×10</td>
<td>†</td>
<td>47</td>
<td>‡</td>
</tr>
<tr>
<td>Sign-10×20</td>
<td>18</td>
<td>61</td>
<td>37</td>
</tr>
<tr>
<td>Sign-20×30</td>
<td>21</td>
<td>47</td>
<td>‡</td>
</tr>
<tr>
<td>Sign-30×40</td>
<td>24</td>
<td>53</td>
<td>‡</td>
</tr>
<tr>
<td>Crane-10×20</td>
<td>17</td>
<td>52</td>
<td>34</td>
</tr>
<tr>
<td>Crane-20×40</td>
<td>31</td>
<td>158</td>
<td>58</td>
</tr>
<tr>
<td>Crane-30×40</td>
<td>31</td>
<td>144</td>
<td>60</td>
</tr>
</tbody>
</table>

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### Details of Problems Solved

#### Running Times (in seconds)

Running time (in seconds) of the interior-point algorithm to solve the truss design problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>LP</th>
<th>SOCP</th>
<th>SDO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridge-16×7</td>
<td>†</td>
<td>93.90</td>
<td>257.01</td>
</tr>
<tr>
<td>Bridge-20×10</td>
<td>†</td>
<td>460.96</td>
<td>1,088.09</td>
</tr>
<tr>
<td>Bridge-30×10</td>
<td>†</td>
<td>904.86</td>
<td>†</td>
</tr>
<tr>
<td>Sign-10×20</td>
<td>3.33</td>
<td>513.37</td>
<td>1,081.54</td>
</tr>
<tr>
<td>Sign-20×30</td>
<td>32.08</td>
<td>4,254.08</td>
<td>†</td>
</tr>
<tr>
<td>Sign-30×40</td>
<td>111.03</td>
<td>26,552.74</td>
<td>†</td>
</tr>
<tr>
<td>Crane-10×20</td>
<td>0.52</td>
<td>41.12</td>
<td>446.98</td>
</tr>
<tr>
<td>Crane-20×40</td>
<td>6.46</td>
<td>1,299.12</td>
<td>33,926.20</td>
</tr>
<tr>
<td>Crane-30×40</td>
<td>10.67</td>
<td>1,599.85</td>
<td>95,131.49</td>
</tr>
</tbody>
</table>

† The linear optimization model cannot be used for the modified bridge design model.
‡ The data to run the SDO model could not be prepared for this instance due to memory restrictions.
If $S$ is a $k \times k$ matrix, then $S$ is a symmetric positive semi-definite (SPSD) matrix if $S$ is symmetric:

$$S_{ij} = S_{ji} \quad \text{for any} \quad i, j = 1, \ldots, k$$

and

$$v^T S v \geq 0 \quad \text{for any} \quad v \in \mathbb{R}^k.$$

If $S$ is a $k \times k$ matrix, then $S$ is a symmetric positive definite (SPD) matrix if $S$ is symmetric and

$$v^T S v > 0 \quad \text{for any} \quad v \in \mathbb{R}^k, v \neq 0.$$
Let $S^k$ denote the set of symmetric $k \times k$ matrices.

Let $S^k_+$ denote the set of symmetric positive semi-definite (SPSD) $k \times k$ matrices.

Let $S^k_{++}$ denote the set of symmetric positive definite (SPD) $k \times k$ matrices.
Let $S$ and $X$ be any symmetric matrices.

We write “$S \succeq 0$” to denote that $S$ is symmetric and positive semi-definite.

We write “$S \succ 0$” to denote that $S$ is symmetric and positive definite.
Remark 1 \( S^k_+ = \{ S \in S^k \mid S \succeq 0 \} \) is a convex set in \( \mathbb{R}^{k^2} \).

Proof: Suppose that \( S, X \in S^k_+ \). Pick any scalars \( \alpha, \beta \geq 0 \) for which \( \alpha + \beta = 1 \). For any \( v \in \mathbb{R}^k \), we have:

\[
v^T(\alpha S + \beta X)v = \alpha v^T Sv + \beta v^T Xv \geq 0,
\]

whereby \( \alpha S + \beta X \in S^k_+ \). This shows that \( S^k_+ \) is a convex set. \( \text{q.e.d.} \)
SDO: minimize $y \ b^T \ y$

s.t. $C + \sum_{i=1}^{m} y_i A_i \geq 0$

$My \geq g$.

The matrices $C, A_1, \ldots, A_m$ are symmetric matrices.
Semi-Definite Optimization

Interpretation of SDO

SDO: minimize \( y \) \( \mathbf{b}^T \mathbf{y} \)

s.t. \[ \mathbf{C} + \sum_{i=1}^{m} y_i \mathbf{A}_i \geq 0 \]

\[ \mathbf{M} \mathbf{y} \geq \mathbf{g} . \]

The objective is to minimize the linear function:

\[ \sum_{i=1}^{m} b_i y_i \]

of the \( m \) scalar variables \( \mathbf{y} = (y_1, \ldots, y_m) \).
Semi-Definite Optimization

Interpretation of SDO

\[
\text{SDO: minimize}_{y} \ b^T y \\
\text{s.t.} \quad C + \sum_{i=1}^{m} y_i A_i \succeq 0 \\
\quad My \succeq g.
\]

\( y = (y_1, \ldots, y_m) \) must satisfy \( My \succeq g. \)
Semi-Definite Optimization

Interpretation of SDO

SDO: minimize \( y^T b \)
\[
\text{s.t.} \quad C + \sum_{i=1}^{m} y_i A_i \succeq 0 \\
M y \geq g.
\]

\( y = (y_1, \ldots, y_m) \) must satisfy the condition that the matrix \( S \), defined by:
\[
S := C + \sum_{i=1}^{m} y_i A_i,
\]
must be positive semi-definite. That is,
\[
S := C + \sum_{i=1}^{m} y_i A_i \succeq 0.
\]
Semi-Definite Optimization

Illustration of SDO

SDO: minimize \( y_1, y_2 \) \( 11y_1 + 19y_2 \)

\[
\begin{align*}
\text{s.t. } & \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix} + y_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix} = S \succeq 0 \\
3y_1 + 7y_2 & \leq 12 \\
2y_1 + y_2 & \leq 6.
\end{align*}
\]

SDO: minimize

\[
\begin{align*}
\text{s.t. } & \quad 11y_1 + 19y_2 \\
& \quad \begin{pmatrix} 1 + 1y_1 + 0y_2 & 2 + 0y_1 + 2y_2 & 3 + 1y_1 + 8y_2 \\ 2 + 0y_1 + 2y_2 & 9 + 3y_1 + 6y_2 & 0 + 7y_1 + 0y_2 \\ 3 + 1y_1 + 8y_2 & 0 + 7y_1 + 0y_2 & 7 + 5y_1 + 4y_2 \end{pmatrix} \succeq 0 \\
3y_1 + 7y_2 & \leq 12 \\
2y_1 + y_2 & \leq 6.
\end{align*}
\]

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Semi-Definite Optimization

SDO: minimize \( y \) \( b^T y \)

s.t. \( C + \sum_{i=1}^{m} y_i A_i \geq 0 \)

\( My \geq g \).

Remark 2. SDO is a convex minimization problem.
Semi-definite optimization is a unifying model that includes:
- linear optimization;
- quadratic optimization;
- second-order cone optimization;
- certain other convex optimization problems.

Semi-definite optimization has applications that span convex optimization, discrete optimization, and control theory.

Semi-definite optimization may very well become the canonical way that optimizers will think about optimization in the next decade.
Truss Design and Semi-Definite Optimization

(STDP): minimize $t, \theta$ $\theta$

s.t. $\begin{pmatrix} \theta \\ F \\ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \end{pmatrix} \succeq 0$

$Mt \leq d$

$t \geq 0$

$\theta \in \mathbb{R}, \ t \in \mathbb{R}^m$.

Notice that STDP is a semi-definite optimization problem.
Truss Design and Semi-Definite Optimization

(TDP): minimize\(_{t,u}\) \(F^T u\)

s.t. \[ \left[ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] u = F \]

\[ Mt \leq d \]
\[ t \geq 0 \]
\[ u \in \mathbb{R}^N, \ t \in \mathbb{R}^m. \]

(STD|P): minimize\(_{t,\theta}\) \(\theta\)

s.t. \[ \left( \frac{\theta}{F} \left[ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] \right) \geq 0 \]

\[ Mt \leq d \]
\[ t \geq 0 \]
\[ \theta \in \mathbb{R}, \ t \in \mathbb{R}^m. \]

**Proposition 3.** Suppose that \((t,u)\) is a feasible solution of TDP. Let:

\[ \theta = F^T u. \]

Then \((t,\theta)\) is a feasible solution of STD|P, and \(\theta = F^T u.\)
Proposition 4. Suppose that \((t, \theta)\) is a feasible solution of STDP. Then there exists a vector \(u\) for which \((t, u)\) is feasible for TDP, and in fact:

\[
F^T u \leq \theta.
\]
(TDP): minimize \( t, u \) \( F^T u \)

\[
\begin{align*}
\text{s.t.} \quad & \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T u = F \\
& Mt \leq d \\
& t \geq 0 \\
& u \in \mathbb{R}^N, \ t \in \mathbb{R}^m.
\end{align*}
\]

(StDP): minimize \( t, \theta \) \( \theta \)

\[
\begin{align*}
\text{s.t.} \quad & \left( F^T \left[ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] \right) \geq 0 \\
& Mt \leq d \\
& t \geq 0 \\
& \theta \in \mathbb{R}, \ t \in \mathbb{R}^m.
\end{align*}
\]

Solve StDP for \( (t^*, \theta^*) \) and then solve the following linear equation system for \( u^* \):

\[
\left[ \sum_{k=1}^{m} \frac{t_k E_k}{L_k^2} a_k a_k^T \right] u^* = -F.
\]
(TDP): minimize\( t, u \) \( F^T u \)

\[
\text{s.t.} \quad \left[ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] u = F
\]

\[
\sum_{k=1}^{m} t_k \leq V
\]

\[
t \geq 0
\]

\[
u \in \mathbb{R}^N, t \in \mathbb{R}^m.
\]

The only constraint on the volumes of the bars \( t_k \) is a volume constraint limiting the total volume of the bars to not exceed the given value \( V \).

This is a special case of TDP, but it is not too unusual.
(TDP): minimize\(_{t,u}\) \(F^T u\)

\[
\text{s.t.} \quad \left[ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] u = F \\
\sum_{k=1}^{m} t_k \leq V \\
t \geq 0 \\
u \in \mathbb{R}^N, \ t \in \mathbb{R}^m .
\]

\(G(t) := \left[ \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] , \quad \text{(TDP): minimize}_{t,u} \ F^T u \)

\[
\text{s.t.} \quad G(t)u = F \\
\sum_{k=1}^{m} t_k \leq V \\
t \geq 0 \\
u \in \mathbb{R}^N, \ t \in \mathbb{R}^m .
\]
(TDP): minimize $t,u \quad F^T u$
\[
\text{s.t.} \quad G(t)u = F
\]
\[
\sum_{k=1}^{m} t_k \leq V
\]
\[
t \geq 0
\]
\[
u \in \mathbb{R}^N, \quad t \in \mathbb{R}^m.
\]

(DTDP): maximize $v,z \quad -2F^T v - V z$
\[
\text{s.t.} \quad \left(a_k^Tv\right)^2 \leq \frac{L_k^2}{E_k} z, \quad k = 1, \ldots, m
\]
\[
y \in \mathbb{R}^N.
\]
Proposition 5. Suppose that \((t, u)\) is feasible for TDP and that \((v, z)\) is feasible for DTDP. Then:

\[
F^T u \geq -2F^T v - Vz.
\]
Consider the following pair of primal and dual linear optimization models:

(LP): \( \text{minimize}_{f^+, f^-} \sum_{k=1}^{m} \frac{L_k}{\sqrt{E_k}} (f^+_k + f^-_k) \)

\[ \text{s.t.} \quad A(f^+ - f^-) = -F \]
\[ f^+ \geq 0, \ f^- \geq 0 \]
\[ f^+, f^- \in \mathbb{R}^m. \]

(LD): \( \text{maximize}_y -F^Ty \)

\[ \text{s.t.} \quad -\frac{L_k}{\sqrt{E_k}} \leq a_k^Ty \leq \frac{L_k}{\sqrt{E_k}}, \quad k = 1, \ldots, m \]
\[ y \in \mathbb{R}^N. \]
Truss Design and Linear Optimization

Linear Models and Original Models

Dual Problems

(LP): minimize \( f^+ + f^- \sum_{k=1}^{m} \frac{L_k}{\sqrt{E_k}} (f_k^+ + f_k^-) \)

s.t. \( A(f^+ - f^-) = -F \)

\( f^+ \geq 0, \ f^- \geq 0 \)

\( f^+, f^- \in \mathbb{R}^m \).

(LD): maximize \( y - F^T y \)

s.t. \( -\frac{L_k}{\sqrt{E_k}} \leq a_k^T y \leq \frac{L_k}{\sqrt{E_k}}, \ k = 1, \ldots, m \)

\( y \in \mathbb{R}^N \).

(TDP): minimize \( t, u \) \( F^T u \)

s.t. \( G(t)u = F \)

\( \sum_{k=1}^{m} t_k \leq V \)

\( t \geq 0 \)

\( u \in \mathbb{R}^N, \ t \in \mathbb{R}^m \).

(DTDP): maximize \( v, z \) \( -2F^T v - Vz \)

s.t. \( (a_k^T v)^2 \leq \frac{L_k^2}{E_k} z, \ k = 1, \ldots, m \)

\( y \in \mathbb{R}^N \).
Proposition 6. Suppose that \((\bar{f}^+, \bar{f}^-)\) is an optimal solution of LP and that \(\bar{y}\) is an optimal solution of LD, and consider the following assignment of variables:

\[
R = \sum_{k=1}^{m} \frac{L_k}{\sqrt{E_k}} (f_k^+ + f_k^-)
\]

\[
\bar{t}_k = \frac{V}{R} \frac{L_k}{\sqrt{E_k}} (f_k^+ + f_k^-) \quad k = 1, \ldots, m,
\]

\[
\bar{u} = -\frac{R}{V} \bar{y}
\]

\[
\bar{v} = \frac{R}{V} \bar{y}
\]

\[
\bar{z} = \frac{R^2}{V^2}
\]

Then \((\bar{t}, \bar{u})\) solves TDP and \((\bar{v}, \bar{z})\) solves DTDP.
Truss Design and Linear Optimization

\[ \text{(LP): minimize}_{f^+, f^-} \sum_{k=1}^{m} \frac{L_k}{\sqrt{E_k}} (f_k^+ + f_k^-) \]
\[ \text{s.t.} \quad A(f^+ - f^-) = -F \]
\[ f^+ \geq 0, \quad f^- \geq 0 \]
\[ f^+, f^- \in \mathbb{R}^m. \]

\[ \text{(LD): maximize}_{y} \quad -F^T y \]
\[ \text{s.t.} \quad -\frac{L_k}{\sqrt{E_k}} \leq a_k^T y \leq \frac{L_k}{\sqrt{E_k}}, \quad k = 1, \ldots, m \]
\[ y \in \mathbb{R}^N. \]

\[ \text{(TDP): minimize}_{t, u} \quad F^T u \]
\[ \text{s.t.} \quad G(t)u = F \]
\[ \sum_{k=1}^{m} t_k \leq V \]
\[ t \geq 0 \]
\[ u \in \mathbb{R}^N, \quad t \in \mathbb{R}^m. \]

\[ \text{(DTDP): maximize}_{v, z} \quad -2F^T v - V z \]
\[ \text{s.t.} \quad (a_k^T v)^2 \leq \frac{L_k^2}{E_k} z, \quad k = 1, \ldots, m \]
\[ y \in \mathbb{R}^N. \]
We consider a finite set of external loads on the truss:

\[ \mathcal{F} = \{ F_1, F_2, \ldots, F_J \} \subset \mathbb{R}^N. \]

A conservative strategy would be to minimize the maximum compliance:

\[
\begin{align*}
\text{minimize}_t & \quad \max_{j=1,\ldots,J} \{ F_j^T u_j \} \\
Mt & \leq d, \\
t & \geq 0 \\
\text{s.t.} & \quad G(t) u_j = F_j.
\end{align*}
\]
Extensions of the Truss Design Problem

Multiple Loads

Averaging Model

We consider a finite set of external loads on the truss:

\[ \mathcal{F} = \{F_1, F_2, \ldots, F_J\} \subset \mathbb{R}^N. \]

Let \( \lambda_j \) denote the relative frequency or importance associated with the truss structure undergoing the external force \( F_j \).

\[ \lambda_1, \ldots, \lambda_J \geq 0. \]

\[ \sum_{j=1}^{J} \lambda_j = 1.0. \]

\[ \text{minimize}_t \quad \sum_{j=1}^{J} \lambda_j F_j^T u_j \]

\[ Mt \leq d, \quad t \geq 0 \]

s.t. \( G(t)u_j = F_j \).
Extensions of the Truss Design Problem

We have assumed that the truss structure itself is not affected by its own weight.

Add an external force corresponding to the gravitational force associated with bar $k$, and linearly proportional to $t_k$ for $k = 1, \ldots, m$.

Let $g_k \in \mathbb{R}^N$ denote the vector that projects the gravitational force of bar $k$ onto the appropriate nodes.

\[
\text{(TDP): minimize}_{t,u} \left( F + \sum_{k=1}^{m} t_k g_k \right)^T u
\]
\[
\text{s.t.} \quad G(t) u = \left( F + \sum_{k=1}^{m} t_k g_k \right)
\]
\[
Mt \leq d
\]
\[
t \geq 0
\]
\[
u \in \mathbb{R}^N, \quad t \in \mathbb{R}^m.
\]
Extensions of the Truss Design Problem

Reinforcement

We are given an existing truss structure. We must determine how to strengthen it. We add lower bounds on the volumes of the existing bars equal to the current volumes of the bars:

\[(TDP): \text{minimize}_{t,u} \ F^T u\]

\[\text{s.t.} \quad G(t)u = F\]
\[Mt \leq d\]
\[t \geq 0\]
\[t_k \geq \bar{t}_k, \ k = 1, \ldots, m\]
\[u \in \mathbb{R}^N, \ t \in \mathbb{R}^m.\]
The optimal solution of the TDP might yield a truss design that is not rigid for a different external force than the one used.

To obtain a robust solution, we must consider multiple external loads that will contain the possible loads that the truss will be subject to.
Extensions of the Truss Design Problem

Buckling Constraints

We have ignored the fact that if a bar is under great compression it might actually collapse instead of counteracting that external force with its internal force.

For this we have to add lower bounds on the allowable internal forces in terms of the design variables $t_k$ and the geometry of the bars.

These constraints cause the resulting problem to lose its convex structure.