You must answer ALL questions in order to fulfill the course requirement. For each student, the answers will be ordered by score and the top four answers will then comprise the final problem set grade.

Question 1:
Let $X_1, X_2, \ldots, X_n$ be a random sample where $X_i \sim \text{exponential}(\beta)$, that is: $f_{X_i}(x_i) = \frac{1}{\beta} e^{-x_i/\beta}$, $x_i > 0$ and 0 elsewhere.

a. Derive the MLE for $\beta$.

b. Find the MLE for $\sqrt{\beta}$.

c. Is your MLE in part a. unbiased? Formally prove or disprove.

d. Is your MLE from part a. consistent?

Question 2:
Assume a sample of continuous random variables: $X_1, X_2, \ldots, X_n$, where $E[X_i] = \mu, \text{Var}[X_i] = \sigma^2 > 0$. Consider the following estimators: $\hat{\mu}_{1,n} = X_n, \hat{\mu}_{2,n} = \frac{1}{n+1} \sum_{i=1}^{n} X_i$.

a. Are $\hat{\mu}_{1,n}$ and $\hat{\mu}_{2,n}$ unbiased?

b. Are $\hat{\mu}_{1,n}$ and $\hat{\mu}_{2,n}$ consistent?

c. What do you conclude about the relation between unbiased and consistent estimators?

Question 3: Consider a sample of random variables: $X_1, X_2, \ldots, X_n$, where $n > 10, E[X_i] = \mu, \text{Var}[X_i] = \sigma^2 > 0$ and the estimator: $\hat{\mu}_n = \frac{1}{n-10} \sum_{i=11}^{n} X_i$. [Hint: note that the sum is over $(n - 10)$ random variables].

a. Calculate the bias of $\hat{\mu}_n$

b. Calculate the variance of $\hat{\mu}_n$.

c. Calculate the MSE of $\hat{\mu}_n$.

d. Is $\hat{\mu}_n$ efficient in a finite sample?

e. Can you think of a scenario where you might want to use $\hat{\mu}_n$?
Question 4:
Suppose all MIT undergraduates have one of the four following living arrangements: (i) fraternity/sorority, (ii) independent living group, (iii) a dormitory, (iv) an off-campus apartment. Assume this list is exhaustive and mutually exclusive. You are interested in estimating the probability that an incoming freshman will choose each of the four options. Assume that there is no individual heterogeneity, so each freshman chooses any option with the same (independent) probability as all other freshmen. Assume also that the probabilities do not differ by year. You conduct a survey of students to determine the probability of each alternative.

a. Using the results from your survey as your i.i.d. random sample \((X_1, X_2, \ldots, X_n)\) derive the MLE for each of the four probabilities.

b. What happens if one of the categories in your survey had zero respondents?

Question 5:
You observe a random sample \((X_1, X_2, \ldots, X_n)\) from a distribution \(f_X(x) = \begin{cases} e^{-(x-\theta)} & \text{if } x > \theta \\ 0 & \text{otherwise} \end{cases} \)
You do not know \(\theta\) and you wish to estimate it.

a. Calculate a method of moments (MM) estimator for \(\theta\).

b. Is the MM estimator unbiased?

c. Is the MM estimator consistent?

Question 6:

a. Find a Method of Moments estimator for \(\theta = (\theta_1, \theta_2)\) given a random sample from the uniform distribution \(U[\theta_1, \theta_2]\).

b. Assume \((X_1, X_2, \ldots, X_n)\) are a random sample from the distribution with a cdf: \(F_X(x) = \begin{cases} 1 - (\theta_1/x)^{\theta_2} & \text{if } x \geq \theta_1 \\ 0 & \text{otherwise} \end{cases} \), where \(\theta_1 > 0, \theta_2 > 0\). Find the Maximum Likelihood estimator of \(\theta = (\theta_1, \theta_2)\).