8.251 Homework 7

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Problem 1. (20 points) Current of a charged point particle

Consider a point particle with charge $e$ moving in a $(d+1)$ dimensional spacetime as described by functions $x^\mu(\tau) = \{x^0(\tau), \vec{x}(\tau)\}$ where $\tau$ is the proper time parameter. Recall that the EM current $j^\mu$ is defined as $j^\mu = (\rho, \vec{j})$ where $\rho$ is the charge density (charge per unit volume) and $\vec{j}$ is the current density (current per unit area).

(a) Use delta functions to write expressions for $j^0(\vec{x}, t)$ and $j^i(\vec{x}, t)$ describing the electromagnetic current associated to the point particle.

(b) Show that the expressions you wrote above can be obtained from the following integral representation (with $(d+1)$ spacetime dimensions)

$$ j^\mu(\vec{x}, t) = ec \int d\tau \delta^{d+1}(x^\mu - x^\mu(\tau)) \frac{dx^\mu(\tau)}{d\tau} $$

Problem 2. (20 points) Point particle coupled to EM field.

To study how a charged point particle moves in an EM field we must write an action that includes the point particle action and a term coupling the particle to the gauge potential:

$$ S = -mc^2 \int_\gamma d\tau + \frac{e}{c} \int_\gamma A_\mu(x) dx^\mu $$

Here the integrals are over the world-line $\gamma$ of the particle, $\tau$ is proper time, and $e$ is the electric charge. More explicitly, the second integral can be written as $\frac{e}{c} \int_\gamma A_\mu(x(\tau)) \frac{dx^\mu(\tau)}{d\tau} d\tau$.

Now consider the variation of the action $S$ under a change $\delta x^\mu$ of the particle trajectory. The first term in the action was varied in the notes. Vary also the second term and obtain the equation of motion for the point particle in the presence of an EM field. Your result should be the equation considered in homework 1 problem 4 (see the solution for the relativistic form).

Problem 3. (20 points) EM field dynamics with charged particle.

The action for the dynamics of both a charged point particle and the EM field is given by

$$ S' = -mc^2 \int_\gamma d\tau + \frac{e}{c} \int_\gamma A_\mu(x) dx^\mu - \frac{1}{16\pi c} \int d^{d+1}x \left( F_{\mu\nu} F^{\mu\nu} \right) $$

Notice that the total action $S'$ is a hybrid. The last term is an integral over spacetime and the first two terms are integrals over the particle worldline. While included for completeness, the first term will play no role here. Vary the action $S'$ under a fluctuation $\delta A_\mu(x)$ and obtain the equation of motion for the electromagnetic field in the presence of the charged point particle. The answer should be the equation you had in hwk 2, prob. 2(b), where the current is the one calculated in prob. 1 above. [Hint: to vary $A_\mu(x)$ in the worldline action it is useful to first turn this term into a full spacetime integral with the help of delta functions].
Problem 4. (30 points) Kalb-Ramond field $B_{\mu\nu}$.

The purpose of this problem is to develop the light cone formulation of the field theory of a massless antisymmetric tensor gauge field $B_{\mu\nu} = -B_{\nu\mu}$. This gauge field is a tensor analogue of the Maxwell gauge field, which only has one index. Just as for $A_\mu$, where we defined a field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, for the present case we define a field strength $H_{\mu\nu\rho}$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{1}$$

(a) Show that $H_{\mu\nu\rho}$ is totally antisymmetric. Prove that $H_{\mu\nu\rho}$ is invariant under

$$\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \tag{2}$$

(b) Consider the spacetime action principle

$$S \sim \int d^Dx \left( -\frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \tag{3}$$

By variation of this action write down the field equation satisfied by $B_{\mu\nu}$. Write the field equation and the gauge transformations in momentum space.

(c) Consider the equations for $B^{\mu\nu}(p)$ for $p^2 \neq 0$ and for $p^2 = 0$. Show that there are no degrees of freedom when $p^2 \neq 0$ by proving that $p_\mu B^{\mu\nu}$ can be gauged away (Hint: try using a Lorentz frame where a vector $p^\mu$ satisfying $p^2 \neq 0$ has the just one component). Similarly, discuss the $p^2 = 0$ case and find what components of $B^{\mu\nu}$ can be gauged away, and which represent truly independent degrees of freedom.

Problem 5. (20 points) Lorentz generators and Lorentz algebra

For a Lorentz invariant classical mechanics system with dynamical variables $x^\mu(\tau)$ and associated momenta $p^\mu = \frac{\partial L}{\partial \dot{x}^\mu}$ the objects

$$M^{\mu\nu} = x^\mu(\tau)p^\nu(\tau) - x^\nu(\tau)p^\mu(\tau)$$

will be constants of the motion, that is, $\tau$ independent. Consider now the case when we declare $x^\mu(\tau)$ and $p^\nu(\tau)$ to be quantum Heisenberg operators with the commutation relations

$$[x^\mu(\tau), p^\nu(\tau)] = i \eta^{\mu\nu}$$

(a) Calculate $[M^{\mu\nu}, x^\alpha(\tau)]$. Show that the $M^{\mu\nu}$ generate infinitesimal Lorentz transformations: any such transformation $\delta x^\alpha = \omega^{\alpha\beta} x_\beta$ ($\omega^{\alpha\beta} = -\omega^{\beta\alpha}$) can be obtained as a commutator with $M$

$$\delta x^\alpha(\tau) = \left[ -\frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}, x^\alpha(\tau) \right]$$

(b) Calculate the commutator $[M^{\mu\nu}, M^{\rho\sigma}]$. Your answer should have four terms of the form $\eta M$. The fact that the commutator of two $M$'s gives a series of $M$'s is the statement that the $M$'s define a Lie algebra. In the present case this Lie algebra is the Lorentz algebra.

(c) Consider the Lorentz algebra in light cone coordinates. Calculate the commutators

$$[M^{\pm I}, M^{JK}], \quad [M^{\pm I}, M^{\mp J}], \quad [M^{+-}, M^{\pm I}], \quad [M^{\pm I}, M^{\pm J}]$$
Problem 6. (20 points) Virasoro operators

Consider the Virasoro combinations setting $\alpha' = 1/2$

$$L_n = \frac{1}{2} \sum_{p=-\infty}^{\infty} \alpha_p^\mu \alpha_{n-p,\mu}$$

In quantum string theory these are Virasoro operators. Moreover, the oscillators satisfy the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m \delta_{m+n,0} \eta^{\mu\nu}$$

Note that only the Virasoro operator $L_0$ has an ordering ambiguity. This ambiguity plays no role in the present problem.

(a) Consider two Virasoro operators $L_m, L_n$, with $m + n \neq 0$. Show that

$$[L_m, L_n] = (m - n)L_{m+n}, \quad m + n \neq 0$$

This commutator, assumed valid even for $m = -n$ defines the Virasoro algebra without central extension. It is an infinite dimensional Lie algebra (it has an infinite number of generators). Given a set of generators $x, y, z, \ldots$ and a Lie bracket $[\cdot, \cdot]$, we have a Lie algebra if

(i) Antisymmetry: $[x, y] = -[y, x]$ for all generators $x, y, and$

(ii) Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = 0$ for all generators $x, y, z$.

(b) Show that the commutators in (A) assumed valid for all $m$ and $n$ define a Lie algebra.

Problem 7. (20 points) Virasoro anomaly

A true calculation of the algebra in (A) would show that there is an extra term – a constant one, appearing when $m = -n$. So we must have

$$[L_m, L_n] = (m - n)L_{m+n} + A(m)\delta_{m+n,0}, \quad (B)$$

where $A(m)$ is a function of $m$ that will be calculated directly in lecture. This is a Virasoro algebra with a central extension. The central extension is the object $A(m)$. The name central comes about because we assume that the $A(m)$’s commute with everything.

(i) What does the antisymmetry requirement on a Lie algebra tell you about $A(m)$? What is $A(0)$?

(ii) Consider now the Jacobi identity for generators $L_m, L_n, L_k$ with $m + n + k = 0$. Show that

$$(m - n)A(k) + (n - k)A(m) + (k - m)A(n) = 0 \quad (C)$$

(iii) Use equation (C) to show that $A(m) = \alpha m$ and $A(m) = \beta m^3$ for constants $\alpha$ and $\beta$ yield consistent central extensions.

(iii) Consider equation (C) with $k = 1$. Show that $A(1)$ and $A(2)$ determine all $A(n)$’s.