8.251 Homework 11

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Problem 1. (20 points) Massive vector.

The purpose of this problem is to understand what kind of action principle and equations of motion are associated to massive Maxwell fields. We will also see that in a $D$ dimensional spacetime, while a massless gauge field has $D-2$ degrees of freedom, the massive vector has $D-1$ degrees of freedom.

Consider the action $S = \int d^Dx \mathcal{L}$ with

$$
\mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\nu A_\mu \partial^\mu A^\nu - \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - (\partial \cdot A)m\phi
$$

The first two terms in the above equation are the familiar ones for the Maxwell field. The third looks like a mass term for the Maxwell field, but alone would not suffice. The additional terms show a scalar, the one that is eaten to give the gauge field a mass, as we will see.

(a) Show that the action $S$ is invariant under the gauge transformations

$$
\delta A_\mu = \partial_\mu \epsilon, \quad \delta \phi = \beta m \epsilon,
$$

where $\beta$ is a constant you will determine. Note that the gauge field is transforming under the usual Maxwell gauge transformation, while the scalar field has an unusual transformation law.

(b) Vary the action and write down the field equations for $A_\mu$ and for $\phi$.

(c) Argue that the gauge transformations allow us to set $\phi = 0$. Since the field $\phi$ dissappears from sight, we say it was eaten to give the gauge field a mass, as we will see.

(d) Write the simplified equations in momentum space and show that for $p^2 \neq -m^2$ there are no nontrivial solutions, while for $p^2 = -m^2$ the solution implies that there are $D-1$ degrees of freedom (It may be useful to use a Lorentz trasformation to represent the vector $p^\mu$ satisfying $p^2 = -m^2$ as a vector having a component only in one direction).

Problem 2. (40 points) Polyakov action

The Polyakov action for the string is given as

$$
S = -\frac{1}{4\pi \alpha'} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}
$$

Here $h \equiv \det(h_{\alpha\beta})$ and $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$, namely

$$
h^{\alpha\beta}h_{\beta\gamma} = \delta^\alpha_\gamma, \quad h^{\alpha\beta}h_{\alpha\beta} = 2.
$$

The indices $\alpha, \beta$ run over two values, say 1 and 2 and $\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$ with $\xi^\alpha = (\tau, \sigma)$. The object $h_{\alpha\beta}$ represents a metric on the mathematical two-dimensional world-sheet.

(a) Find the equation of motion for $X^\mu$ by variation of the action in (1). (Do not expand out the derivatives!!).
(b) In order to vary the metric $h_{\alpha\beta}$ we first need a preliminary result. Let $A$ be a two by two matrix and $\delta A$ denote its variation

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \delta A = \begin{pmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{pmatrix}$$

(3)

Show that the variation of the determinant of $A$ can be written as

$$\delta \det A = \det A \text{Tr}(A^{-1} \delta A),$$

(4)

where $A^{-1}$ denotes the inverse of $A$, and Tr stands for trace. This identity actually holds for matrices of arbitrary dimension.

(c) Find the equations of motion that result from the action (1) under a metric variation $\delta h_{\alpha\beta}$. Note that $\delta h^{\alpha\beta}$ and $\delta h_{\alpha\beta}$ can be related to each other using equation (2).

(d) Show that the equations of motion derived in (c) imply that

$$h_{\alpha\beta} = f(\xi) \partial_{\alpha}X \cdot \partial_{\beta}X$$

(5)

where $f$ is some scalar function (it has no indices). This means that the equation of motion for the metric $h_{\alpha\beta}$ sets it proportional to the induced metric on surface. Substitute this result back into the action (1) and show that you obtain the Nambu-Goto action!

It is a theorem of two-dimensional geometry that a reparametrisation of coordinates allows any two-dimensional metric $h_{\alpha\beta}$ to be cast in the form

$$h_{\alpha\beta} = \rho(\xi) \eta_{\alpha\beta}$$

(6)

where $\rho$ is a conformal factor and $\eta_{\alpha\beta}$ is the two-dimensional Minkowski Metric. This choice of metric is called the conformal gauge.

(e) Examine the equation of motion for $X^\mu$ under the condition in (6). Show that you get the expected wave equation. Examine the equations of motion for the metric (part (c)) under the condition in (6). Show that you get the two familiar reparametrization constraints.

Problem 3. (40 points) Spectrum of strings stretched between a Dp-brane and a D25 brane.

Consider a Dp-brane located at $x^a = 0$ with $a = p + 1, p + 2, \ldots, d$. Moreover there is also a D25-brane that fills all of space. The purpose of the present problem is to consider the open strings stretching from the Dp-brane to the D25-brane. The string stretching from the D25-brane to the Dp-brane can be treated in exactly the same way (In order to use the light-cone gauge we assume that $p \geq 1$, so that $x^0$ and $x^i$ can be treated using the light cone gauge.) Let $x^i$, with $i = 2, \ldots, p$, denote the (light-cone) transverse coordinates that lie along the Dp-brane.

(a) State the boundary conditions for the coordinates $X^i(\tau, \sigma)$ and $X^a(\tau, \sigma)$ for the class of strings under consideration. Letting N stand for Neumann and D stand for Dirichlet label the coordinates as NN, ND, DN, DD, where the first label refers to boundary condition at $\sigma = 0$ and the second refers to the boundary condition at $\sigma = \pi$.

The quantization of the $X^i$ coordinates is familiar, but the quantization of the $X^a$ coordinates involves an interesting complication.
(b) Consider the usual expansion

\[ X^a(\tau, \sigma) = \frac{1}{2} \left( f^a(\tau + \sigma) + g^a(\tau - \sigma) \right) , \]

and use the boundary conditions to relate \( g^a \) to \( f^a \) and then to constrain \( f^a \).

In fact, you will find that the function \( f^a(\tau) \) is antiperiodic when translated by \( 2\pi \). This implies that \( f^a(\tau) \) is periodic with period \( 4\pi \). Use this information to find a suitable mode expansion.

In defining useful \( \alpha_n^a \) oscillators note that in all the examples considered thus far the model label \( n \) of oscillators \( \alpha_n \) is matched with an exponential \( e^{-in\tau} \) in the coordinate expansion. In the present case this will lead to fractional mode labels!

Show that you can write

\[ X^a(\tau, \sigma) = \sum_{n \text{ odd}} \alpha_n^a e^{-in\frac{\tau}{2}} \sin\left( \frac{n\sigma}{2} \right) \]

(c) Examine the commutator \([X^a, P^b]\) and derive the commutation relations for the new fractional oscillators.

(d) Calculate the contribution of the \( X^a \) coordinates to the Virasoro operators \( L_\perp \). Note that while the modes of \( X^a \) are half-integers, we only get integrally moded Virasoro operators.

(e) Attempt to order properly the Virasoro operator \( L_0^\perp \). That is, find the constant \( \alpha(0) \) to be used in

\[ P^- = \frac{1}{P^+}(L_0^\perp - \alpha(0)) \]

where \( L_0^\perp \) is defined as an ordered operator with no additional constants. For this purpose, recall that each of the 24 light cone coordinates in the D25 case contributed

\[ \frac{1}{2} \left( 1 + 2 + 3 + 4 + \cdots \right) = \frac{1}{2} \left( -\frac{1}{12} \right) = -\frac{1}{24} \]

How much does each coordinate \( X^a \) contribute to the normal ordering constant? What is the resulting value of \( \alpha(0) \)?

(f) Give the new mass-squared formula. Give a list of the states appearing at the two lowest possible mass levels.