Quantum Physics III (8.06) Spring 2003
Assignment 7

Readings
Griffiths Chapter 8 on the semiclassical approximation, and the Supplementary Notes on the Connection Formulae.

Problem Set 7

1. The Grover Algorithm (20 points)

Consider the 8 dimensional Hilbert space formed by taking the tensor product of the Hilbert spaces for three spin-one-half particles.

We denote the basis states as follows:

\[
\begin{align*}
|0\rangle & = |0, 0, 0\rangle \\
|1\rangle & = |0, 0, 1\rangle \\
|2\rangle & = |0, 1, 0\rangle \\
|3\rangle & = |0, 1, 1\rangle \\
|4\rangle & = |1, 0, 0\rangle \\
|5\rangle & = |1, 0, 1\rangle \\
|6\rangle & = |1, 1, 0\rangle \\
|7\rangle & = |1, 1, 1\rangle
\end{align*}
\]

where, for example, \(|0, 1, 0\rangle\) means a state in which all three spins are in eigenstates of \(S_z\), with eigenvalues \(+\hbar/2\), \(-\hbar/2\), \(+\hbar/2\).

Throughout this problem you will be constructing a variety of \(8 \times 8\) matrices, working in a basis with basis vectors ordered as above.

(a) The first stage of the Grover algorithm is initialization. Suppose we start with all spins up, namely in state \(|0\rangle\). We want to find a unitary operator \(U_{\text{initialize}}\) such that

\[U_{\text{initialize}}|0\rangle = |s\rangle\]

where the state \(|s\rangle\) is given by

\[
|s\rangle = \frac{1}{\sqrt{8}} \left[ |0\rangle + |1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle \right].
\]
Construct the $8 \times 8$ matrix $U_{\text{initialize}}$ as the product of three $8 \times 8$ unitary matrices each of which acts only within the Hilbert space of one of the three spins.

Note: my guess is that this is the part of this problem that you will find trickiest. Note that you need not do this part of the problem in order to do any of the other parts.

(b) Let's suppose that $f(3) = 1$ and $f(a) = 0$ for $a = 0, 1, 2, 4, 5, 6, 7$. In other words, “3 is the winner”. Define a diagonal unitary matrix called $(-1)^f$ that acts on basis states as follows:

\[
(-1)^f |a\rangle = |a\rangle \quad \text{for} \quad a \neq 3
\]

\[
(-1)^f |3\rangle = -|3\rangle.
\]

Write $(-1)^f$ as an $8 \times 8$ matrix.

Note: this part of the problem is very easy as posed. Too easy, in fact. Doing it this way is a little too much like “looking inside the black box and seeing how f works”. What you should really do is construct this unitary operator by introducing a “work-bit”, introducing an operator $U_f$ which represents a function call via $U_f|a, 0\rangle = |a, f(a)\rangle$ and $U_f|a, 1\rangle = |a, 1 - f(a)\rangle$, introducing the operator $L$ defined in lecture by Prof. Farhi, and then constructing $(-1)^f = U_fLU_f$. I do recommend that you do this explicitly, but adding the work bit means doubling the Hilbert space to $16 \times 16$ so I am not going to ask you to turn this in.

(c) Write the unitary operator $U_s \equiv 2|s\rangle \langle s| - 1$ as an $8 \times 8$ matrix. (You should check that your matrix is unitary, but do not turn this check in.)

(d) Find the state

\[
\left[ U_s (-1)^f \right]^k |s\rangle
\]

for $k = 0, 1, 2, 3$. You should find that for $k = 2$, it is fairly close to the state $|3\rangle$ while for $k = 3$, it has become less close to $|3\rangle$.

Suppose that the state with $k = 2$ “is measured”, meaning that $S_z$ is measured for each of the three spins. What is the probability that the outcome of this measurement will be $+\hbar$, $-\hbar$, $-\hbar$ (which corresponds to the state $|3\rangle$)? That is, what is the probability that upon measurement you get the right answer?

Note: Prof. Farhi proved in lecture that for large $N$, the best choice for $k$ is the integer closest to $\pi\sqrt{N}/4$. For our $N = 8$, which is not even very large, $\pi\sqrt{N}/4 = 2.22$. Now that you have understood the $N = 8$ example explicitly, you should review Prof. Farhi’s proof of the large-$N$ result. [Note: although it is not really necessary, it is fine if you choose to use a program like Mathematica to multiply out matrices.]
2. Using the Semiclassical Approximation on the Ground State (8 points)

Despite the fact that the semiclassical approximation was derived for states of large action, we shall see that it does a pretty good job on the ground states of simple potentials. For example, it gets that of the harmonic oscillator exactly right. Here is another example where the exact solution is known, and you can check how well the semiclassical approximation does. This problem is based on Griffiths 8.12.

Consider a particle of mass \( m \) moving in the potential

\[
V(x) = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax).
\]

(a) Show that \( \psi(x) = A \operatorname{sech}(ax) \) is the wave function for a bound state in this potential. How do you know that this bound state must be the ground state? What is the energy of this state?

(b) Apply the semiclassical approximation to this potential to estimate the lowest energy eigenvalue. Compare your result with the exact result from part (a).

3. Quantum Mechanics of a Bouncing Ball (12 points)

The semiclassical approximation can also be used to estimate the energy eigenvalues and eigenstates for potentials that cannot be treated exactly so easily. This problem is loosely based on Griffiths 8.6. (See Griffiths 8.5 if you’d like to learn how to treat this quantum mechanical problem exactly, using Airy functions.)

Consider the quantum mechanical analogue to the classical problem of a ball of mass \( m \) bouncing elastically on the floor, under the influence of a gravitational potential which gives it a constant acceleration \( g \).

(a) Find the allowed energies \( E_n \) in terms of \( m \), \( g \) and \( \hbar \).

(b) What is the wave function of the \( n \)th state in the semiclassical approximation?

(c) Estimate the zero point energy of a mass of 1 gm “at rest” (ie in the quantum mechanical ground state) on a horizontal surface in the earth’s gravitational field. Express your answer in ergs and in eV. [Ie show that this is a very small energy on both human and atomic scales.]

(d) Now imagine dropping the ball from rest from a height of 1 meter, and letting it bounce. Do the 8.01 “calculation” of the classical energy of the ball. The quantum mechanical state corresponding to a ball following this classical trajectory must be a coherent superposition of energy eigenstates, with mean energy equal to the classical energy. How large is the mean value of the quantum number \( n \) in this state?
4. **Tunnelling and the Stark Effect (20 points)**

When we discussed the Stark effect — the physics of an atom in an electric field — we noticed that turning on an electric field meant that the electron in an atom can tunnel out of the atom, making the atomic bound states unstable. I claimed that this was an extremely small effect, which could be neglected. Let us check this, in a simpler one-dimensional analog problem.

Suppose an electron is trapped in a one-dimensional square well of depth $V_0$ and width $d$:

$$V(x) = \begin{cases} -V_0 & \text{for } |x| < d/2 \\ 0 & \text{for } |x| \geq d/2 \end{cases}. $$

Suppose a weak constant electric field in the $x$-direction with strength $\mathcal{E}$ is turned on. That is $V \rightarrow (V - e\mathcal{E}x)$. Assume throughout this problem that $e\mathcal{E}d \ll \hbar^2/2md^2 \ll V_0$.

(a) Set $\mathcal{E} = 0$ in this part of the problem. Estimate the ground state energy (ie the amount by which the ground state energy is above the bottom of the potential well) by pretending that the well is infinitely deep. (Because $\hbar^2/2md^2 \ll V_0$, this is a good approximation.) Use this estimate of the ground state energy in subsequent parts of the problem.

[Aside: the true ground state energy is lower than what you’ve estimated. (You can show this, but that’s optional.) This means that the tunnelling lifetime you estimate below is an underestimate.]

(b) Sketch the potential with $\mathcal{E} \neq 0$ and explain why the ground state of the $\mathcal{E} = 0$ potential is no longer stable when $\mathcal{E} \neq 0$.

(c) Use the semiclassical approximation to calculate the barrier penetration factor for the ground state. [You should use the fact that $e\mathcal{E}d \ll \hbar^2/2md^2$ to simplify this part of the problem.]

(d) Use classical arguments to convert the barrier penetration factor into an estimate of the lifetime of the bound state.

(e) Now, let’s put in numbers. Calculate the lifetime for $V_0 = 20$ eV, $d = 2 \times 10^{-8}$ cm and an electric field of $7 \times 10^4$ V/cm. Compare the lifetime you estimate to the age of the universe.

(f) Show that the lifetime goes like $\exp -1/\mathcal{E}$ for small $\mathcal{E}$ and explain why this result means that this “instability” could not be obtained in any finite order of perturbation theory.