Readings

The reading assignment for the next two weeks is:

- Supplementary notes on Canonical Quantization and Application to a Charged Particle in a Magnetic Field.
- Griffiths Section 10.2.4 is an excellent treatment of the Aharonov-Bohm effect, but ignore the connection to Berry’s phase for now. We will come back to this later.
- Cohen-Tannoudji Ch. VI Complement E
- Those of you reading Sakurai should read pp. 130-139.

Problem Set 2

1. Deriving the Classical Hamiltonian (5 points)

   In lecture on Feb 10, I verified that

   \[ H = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 + e\phi. \]  

   (1)

   is the classical Hamiltonian for the motion of a charged particle in electric and magnetic fields by checking that it describes motion satisfying the Lorentz force law. In lecture, I made a number of sign errors on the board, some of which I fixed and others of which I did not. I’d like you to check whether I got things correct in the Supplementary Notes. Please check all equations from (2.20) through (2.29). For each equation, if I made any errors (sign errors or other errors) give the corrected form of the equation.
2. Electromagnetic Current Density in Quantum Mechanics (15 points)

The probability flux in the Schrödinger equation can be identified as the electromagnetic current density, provided the proper attention is paid to the effects of the vector potential. This current density will play a role in our discussion of the quantum Hall effect.

Way back in the 8.04 you derived the probability flux in quantum mechanics:

\[ \bar{S}(\vec{x}, t) = \frac{\hbar}{m} \text{Im} \left[ \psi^* \vec{\nabla} \psi \right]. \]

In the presence of electric and magnetic fields, the probability current is modified to

\[ \bar{S}(\vec{x}, t) = \frac{\hbar}{m} \text{Im} \left[ \psi^* \vec{\nabla} \psi \right] - \frac{q}{mc} \psi^* \psi \vec{A} \]  \hspace{0.5cm} \text{(2)}

This probability flux is conserved and when multiplied by \( q \), the particle’s charge, it can be interpreted as the electromagnetic current density, \( \vec{j} \equiv q \bar{S} \).

(a) Consider a system defined by the Hamiltonian

\[ \mathcal{H} = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A}(\vec{x}, t) \right)^2 + e\phi(\vec{x}, t). \] \hspace{0.5cm} \text{(3)}

The corresponding time dependent Schrödinger equation in the presence of (possibly time dependent) electric and magnetic fields is:

\[ i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left( -i\hbar \vec{\nabla} - \frac{q}{c} \vec{A}(\vec{x}, t) \right)^2 \psi(\vec{x}, t) + e\phi(\vec{x}, t)\psi(\vec{x}, t), \] \hspace{0.5cm} \text{(4)}

where the scalar potential \( \phi \) and vector potential \( \vec{A} \) produce electric and magnetic fields,

\[ \vec{E} = -\vec{\nabla} \phi - \frac{e}{c} \vec{A} \]

\[ \vec{B} = \vec{\nabla} \times \vec{A} \]

The term \( \vec{A} = \frac{\partial \vec{A}}{\partial t} \) might surprise you. It’s what is needed to describe an electric field in terms of potentials when the fields are time dependent.

Derive the expression eq. (2) for the probability flux, using the following steps:

- Choose to work in a gauge where \( \vec{\nabla} \cdot \vec{A} = 0. \)\(^1\)

\(^1\)Note: it is always possible to find a gauge transformation that takes a given vector potential \( \vec{A}(\vec{x}) \) and turns it into one with \( \vec{\nabla} \cdot \vec{A} = 0 \). (Optional: show this.) Note that stating that \( \vec{\nabla} \cdot \vec{A} = 0 \) does not fully specify \( \vec{A} \). For example, the magnetic field \( \vec{B} = (0, 0, B_0) \) can be described by \( \vec{A} = (-B_0 y, 0, 0) \) or \( \vec{A} = (0, B_0 x, 0) \), both of which satisfy \( \vec{\nabla} \cdot \vec{A} = 0 \).
• Multiply eq. (4) by $\psi^*$.
• Write down the complex conjugate of eq. (4), multiply by $\psi$, and subtract the two equations.
• The resulting equation can be written in the form:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{S}$$

Show that $\rho = \psi^* \psi$ and that $\vec{S}$ is given by eq. (2).

(b) Assuming that $\psi$ has units $1/f^3/2$ as one would expect from the normalization condition, $\int d^3x \psi^* \psi = 1$, show that $\vec{j} = q\vec{S}$ has units of charge per unit area per unit time, which are the dimensions of current density.

(c) In part (a), you assumed that $\vec{\nabla} \cdot \vec{A} = 0$. Now show that $\vec{S}$ has exactly the same form in any gauge, i.e. show that $\vec{S}$ is gauge invariant. That is, show that if we make the following transformations, then $\vec{S}'$ defined in terms of $\vec{A}'$ and $\psi'$ is identical to $\vec{S}$ defined in terms of $\vec{A}$ and $\psi$.

$$\vec{A}'(\vec{x}, t) = \vec{A}(\vec{x}, t) + \vec{\nabla} f(\vec{x}, t)$$
$$\psi'(\vec{x}, t) = \exp \left( \frac{ie}{\hbar c} f(\vec{x}, t) \right) \psi(\vec{x}, t)$$

where $f$ is any function of $\vec{x}$ and $t$.

3. Translation Invariance in a Uniform Magnetic Field (20 points)

One of the surprising things in our analysis of the quantum mechanics of a particle in a uniform magnetic field is that even though $\vec{B}$ is uniform, and we would therefore expect translation invariance in the $xy$-plane, we find that, in any gauge we choose, the Hamiltonian does not appear to reflect this symmetry. This issue is explored in depth in the supplementary notes. In this problem, you explore it in a different gauge, and in a somewhat different way.

The resolution to this question is that translation operators which do commute with the Hamiltonian can be constructed. We shall see, however, that there is a catch.

Consider a magnetic field $\vec{B} = (0, 0, -B_0)$ and work in the gauge in which $\vec{A} = (B_0 y, 0, 0)$. The time-independent Schrödinger equation (for states in the $xy$-plane) is

$$-\frac{\hbar^2}{2m} \left[ \frac{\partial^2 \psi}{\partial y^2} + \left( \frac{\partial}{\partial x} - \frac{ieB_0}{\hbar c} y \right)^2 \psi \right] = E \psi , \quad (5)$$

and the Hamiltonian is

$$H = \frac{1}{2m} \left[ p_y^2 + \left( p_x - \frac{eB_0}{c} y \right)^2 \right] . \quad (6)$$
(a) The appearance of $y$ destroys (on the face of it) invariance under translation in the $y$ direction. Show, however, that if $\psi(x, y)$ is a solution of (5), then so too is $\tilde{\psi}(x, y)$ defined by

$$\tilde{\psi}(x, y) = \psi(x, y - b) \exp(ieB_0b x / \hbar c).$$  \hspace{1cm} (7)

[Hint: be careful with your notation. Express $(\partial / \partial x - ieB_0 y / \hbar c)\tilde{\psi}$ at the point $(x, y)$ in terms of $\psi$ and $\partial \psi / \partial x$ at the point $(x, y - b)$.

(b) Consider the operator $V_b$ which I define by telling you how it acts on any state $|\psi\rangle$:

$$V_b |\psi\rangle = |\tilde{\psi}\rangle.$$  \hspace{1cm} (8)

This operator clearly has the effect of translating in $y$ by a distance $b$. Show that $V_b$ is unitary, and show that it commutes with the Hamiltonian $H$. [Hint: this part of the problem is easy.]

(c) In parts (c) and (d), I ask you to find an explicit expression for $V_b$. You do not actually need this explicit expression either for part (e) or for Problem 4, but having an explicit expression may make you feel more comfortable with $V_b$. You must first find an operator $Q$ which commutes with $H$ and generates translations in $y$. ie you must find an operator which obeys $[Q, H] = 0$ and $[y, Q] = i\hbar$. Find $Q$. [Hint: $Q$ should be a linear combination of the $p_y$ and $x$ operators.]

(d) Show that $V_b = \exp(-ibQ / \hbar)$. That is, show that this explicit expression for $V_b$ yields $V_b |\psi\rangle = |\tilde{\psi}\rangle$.

(e) In the gauge in which we are working, $x$ does not appear in the Hamiltonian. The translation operator for translation in the $x$-direction is therefore the standard one. Call the operator which translates by $a$ in the $x$-direction $U_a$. That is,

$$\langle x, y | U_a |\psi\rangle = \langle x - a, y |\psi\rangle.$$  \hspace{1cm} (9)

[The explicit expression for $U_a$ is just $U_a = \exp(-iax / \hbar)$ but, again, you will not need this explicit expression.] You now have two translation operators, $U_a$ and $V_b$, both of which commute with $H$ for any values of $a$ and $b$ you like. So, what’s the catch? Calculate $\langle x, y | U_a V_b |\psi\rangle$ and $\langle x, y | V_b U_a |\psi\rangle$ and show that $U_a$ commutes with $V_b$ if and only if $ab$ is an integer multiple of $A_B = 2\pi \hbar c / eB_0$ and hence if and only if $abB_0$ is an integer multiple of $\Phi_0 = \hbar c / e$. 

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4. Counting the States in a Landau Level (20 points)

Consider a charged particle in a magnetic field as in the previous problem. Work in the gauge chosen in the previous problem. The particle is restricted to move in a rectangular region of the xy-plane whose extent is \(0 < x < a\) and \(0 < y < b\). Assume that the boundary conditions on the wave function are periodic. This means that \(\psi(0, y) = \psi(a, y)\) and \(\psi(x, 0) = \psi(x, b)\). If you think about it for a minute, these periodic boundary conditions can equally well be implemented as follows: allow \(\psi\) to be defined throughout the entire xy-plane, but require that \(\psi(x + a, y + b) = \psi(x, y)\) for any choice of \(x, y\). This is a (much) more convenient way of implementing periodic boundary conditions and is the one we shall follow.

[Optional: show that the Hamiltonian (6) is Hermitian on the space of wave functions satisfying the periodic boundary conditions just defined.]

From the result of the previous problem, if we choose \(a\) and \(b\) so that \(abB_0 = N\Phi_0\), with \(N\) some large positive integer, then we should be able to find states \(|\psi\rangle\) which are simultaneous eigenstates of \(H\), \(U_a\) and \(V_b\). In this problem we shall count all the states in the lowest Landau level by counting how many states \(|\psi\rangle\) there are which satisfy \(H|\psi\rangle = E_{\text{LLL}}|\psi\rangle\) and \(U_a|\psi\rangle = |\psi\rangle\) and \(V_b|\psi\rangle = |\psi\rangle\).

Here, \(E_{\text{LLL}} = \hbar eB_0/2mc\) is the energy of the lowest Landau level.

[Optional: In general, the eigenvalues of unitary operators are complex numbers of modulus one. You should therefore be wondering why the eigenvalues of \(U_a\) and \(V_b\) must be 1. You can show that any state which is an eigenstate of \(U_a\) which satisfies the periodic boundary conditions must be an eigenstate of \(U_a\) with eigenvalue 1. Eigenstates of \(U_a\) with other eigenvalues do not satisfy the periodic boundary conditions. Same goes for eigenstates of \(V_b\).]

(a) Show that if \(U_a|\psi\rangle = |\psi\rangle\), \(V_b|\psi\rangle = |\psi\rangle\), and \(\psi(x, y)\) satisfies periodic boundary conditions, then \(\psi(x, y)\) must be of the form

\[
\psi(x, y) = \sum_{n=-\infty}^{\infty} u_n(y) \exp(\frac{2\pi inx}{a})
\]

with \(u_{n+N}(y) = u_n(y + b)\). [Aside: the eigenstates I used when I counted states in lecture were eigenstates of \(U_a\) but not of \(V_b\). The eigenstates (10) will turn out to be linear combinations of those I used in lecture.]

(b) Use the Schrödinger equation \(H|\psi\rangle = E_{\text{LLL}}|\psi\rangle\) (that is, the Schrödinger equation (5)) to show that

\[
u_n(y) = c_n f(y + nb/N)
\]

where the function \(f(y)\) is the solution to the time-independent Schrödinger equation for a particle in the lowest energy state of a simple harmonic oscillator with frequency \(\omega = eB_0/mc\).
(c) It might seem that you have found infinitely many solutions. $\psi$ is specified by an infinite set of constants $c_n$. If these constants can be chosen arbitrarily, then there would indeed be infinitely many linearly independent wave functions satisfying all the conditions. However, show that (a) and (b) imply that $c_{n+N} = c_n$. [Hint: don’t forget that $abB_0 = N\Phi_0$.] This means that only $N$ of the $c_n$’s are independent. You have thus shown that there are exactly $N$ states satisfying all the conditions. Thus, in a system with area $NA_B$ the lowest Landau level contains $N$ states.

[Optional: To complete the argument, you must check that you can find $N$ states which are orthogonal. To do this, construct $N$ states as follows: for each state, choose one out of $c_0 \ldots c_{N-1}$ to be 1, and the others to be zero. For example, the first of these states has $\ldots c_{-2N} = c_{-N} = c_0 = c_N = c_{2N} \ldots = 1$ and all other $c$’s zero. The second has $\ldots c_{-2N+1} = c_{-N+1} = c_1 = c_{N+1} = c_{2N+1} \ldots = 1$ and all other $c$’s zero. Etc. What you have shown above is that any state in the lowest landau level is a linear combination of these $N$ states. All you have to do now is show that these $N$ states are orthogonal. That is easy to do.]