Solutions to Problem Set #3

Every problem in this set deals with a situation where a variable is a function of one or more random variables and is therefore itself a random variable. The problems all use the procedure given in the notes for finding the probability density of the new random variable when we know the probability densities for the variables that it depends upon.

Problem 1: Energy in an Ideal Gas

In this problem the kinetic energy $E$ is a function of three random variables $(v_x, v_y, v_z)$ in three dimensions, two random variables $(v_x, v_y)$ in two dimensions, and just one random variable, $v_x$, in one dimension.

(a) In three dimensions

$$E = \frac{1}{2}(v_x^2 + v_y^2 + v_z^2)$$

and the probability density for the three velocity random variables is

$$p(v_x, v_y, v_z) = \frac{1}{(2\pi \sigma^2)^{3/2}} e^{-\left(\frac{v_x^2 + v_y^2 + v_z^2}{2\sigma^2}\right)}.$$  

Following the procedure, we want first to calculate the cumulative probability function $P(E)$, which is the probability that the kinetic energy will have a value which is less than or equal to $E$. To find $P(E)$ we need to integrate $p(v_x, v_y, v_z)$ over all regions of $(v_x, v_y, v_z)$ that correspond to $\frac{1}{2}(v_x^2 + v_y^2 + v_z^2) \leq E$. That region in $(v_x, v_y, v_z)$ space is just a sphere of radius $\sqrt{2E/m}$ centered at the origin.

The integral is most easily done in spherical polar coordinates where

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$$v_x = v \sin \theta \cos \phi$$

$$v_y = v \sin \theta \sin \phi$$

$$v_z = v \cos \theta$$

$$dv_x dv_y dv_z = v^2 dv \sin \theta d\theta d\phi$$
This gives

\[
P(E) = \frac{1}{(2\pi \sigma^2)^{3/2}} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} \, d\phi \int_0^{\sqrt{2E/m}} \sqrt{v^2} \, e^{-v^2/2\sigma^2} \, dv
\]

\[
= \frac{4\pi}{(2\pi \sigma^2)^{3/2}} \int_0^{\sqrt{2E/m}} v^2 \, e^{-v^2/2\sigma^2} \, dv
\]

Thus (substituting \(m \sigma^2 = kT\))

\[
p(E) = \frac{dP(E)}{dE} = \frac{2}{\sqrt{\pi kT}} \sqrt{\frac{E}{kT}} e^{-E/kT} \text{ if } E \geq 0 \quad (0 \text{ if } E < 0)
\]

You can use this to calculate \(<E>=\frac{3}{2}kT\).

(b) In two dimensions

\[E = \frac{1}{2}(v_x^2 + v_y^2)\]

and the probability density for the two velocity random variables is

\[p(v_x, v_y) = \frac{1}{(2\pi \sigma^2)^2} e^{-(v_x^2 + v_y^2)/2\sigma^2}.
\]

To find \(P(E)\) we need to integrate \(p(v_x, v_y)\) over all regions of \((v_x, v_y)\) that correspond to \(\frac{1}{2}(v_x^2 + v_y^2) \leq E\). That region in \((v_x, v_y)\) space is just a circle of radius \(\sqrt{2E/m}\) centered at the origin.

The integral is most easily done in polar coordinates where

\[
v^2 = v_x^2 + v_y^2
\]

\[
v_x = v \cos \theta
\]

\[
v_y = v \sin \theta
\]

\[
dv_x \, dv_y = v \, dv \, d\theta
\]
This gives

\[ P(E) = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\theta \int_0^{\sqrt{2E/m}} \frac{v}{\sigma} e^{-v^2/2\sigma^2} dv = \frac{1}{\sigma^2} \int_0^{\sqrt{2E/m}} e^{-v^2/2\sigma^2} dv \]

Thus (substituting \( m\sigma^2 = kT \))

\[ p(E) = \frac{dP(E)}{dE} = \frac{1}{kT} e^{-E/kT} \text{ if } E \geq 0 \text{ (and } 0 \text{ if } E < 0) \]

You can use this to calculate \( <E> = kT \).

(c) In one dimension (not required)

\[ E = \frac{1}{2}v_x^2 \]

and the probability density for the single velocity random variable is

\[ p(v_x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v_x^2/2\sigma^2}. \]

To find \( P(E) \) we need to integrate \( p(v_x) \) over all regions of \( v_x \) that correspond to \( \frac{1}{2}v_x^2 \leq E \). That region in \( v_x \) space is just a line from \(-\sqrt{2E/m}\) to \( \sqrt{2E/m}\).

\[ P(E) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\sqrt{2E/m}}^{\sqrt{2E/m}} v e^{-v^2/2\sigma^2} dv \]

Thus (substituting \( m\sigma^2 = kT \))

\[ p(E) = \frac{dP(E)}{dE} = \frac{1}{\sqrt{\pi kT E}} e^{-E/kT} \text{ if } E \geq 0 \text{ (0 if } E < 0) \]

You can use this to calculate \( <E> = \frac{1}{2}kT \).
Problem 2: Distance to the Center of the Galaxy

(a) \[ R_0 = \frac{v_{\parallel}}{\sigma_{\theta,\perp}} = \frac{20 \text{ km/sec}}{9.1 \times 10^{-17} \text{ radians/sec}} = 2.2 \times 10^{17} \text{ km} \approx 7.1 \text{ kiloparsecs} \]

(b) Here the random variable \( v_{\parallel} \) (the velocity along the line from the center of the galaxy to an observer on the earth) is a function of the three random variables \( v_x, v_y, v_z \) which are the components of the velocities of the water vapor masers. The first thing to note is that all masers have the same speed \( \dot{r} \) and that the randomness is only in the direction of their motion. If we choose spherical polar coordinates

\[
\begin{align*}
v_x &= v \sin \theta \cos \phi = \dot{r} \sin \theta \cos \phi \\
v_y &= v \sin \theta \sin \phi = \dot{r} \sin \theta \sin \phi \\
v_z &= v \cos \theta = \dot{r} \cos \theta
\end{align*}
\]

there will be only two random variables \( \theta \) and \( \phi \) because the magnitude \( v \) is fixed.

Then it is convenient to choose the \( z \) axis to be the direction from the center of the galaxy to the earth, because this makes \( v_{\parallel} = v_z \). Solving the problem requires us to find \( p(v_{\parallel}) = p(v_z) \) using the appropriate \( p(v_x, v_y, v_z) \) for a fixed magnitude and random orientation of \( \vec{v} \). Since \( v_{\parallel} = \dot{r} \cos \theta \), the possible values of \( v_{\parallel} \) will be from \( -\dot{r} \) when \( \theta = \pi \) to \( \dot{r} \) when \( \theta = 0 \). We first want to find \( P(v_{\parallel}) \), the probability that \( v_z \leq v_{\parallel} \); since \( v_{\parallel} = \dot{r} \cos \theta \) this is equal to the probability that \( \cos \theta \leq v_{\parallel}/\dot{r} \) or \( \theta \geq \cos^{-1}(v_{\parallel}/\dot{r}) \).

\[
P(v_{\parallel}) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{\cos^{-1}(v_{\parallel}/\dot{r})}^\pi \sin \theta d\theta
\]

\[
= \frac{1}{2} \left[ -\cos \theta \right]_{\cos^{-1}(v_{\parallel}/\dot{r})}^\pi = \begin{cases} 
0 & \text{if } v_{\parallel} < \dot{r} \\
\frac{1}{2} \left( 1 + \frac{v_{\parallel}}{\dot{r}} \right) & \text{if } -\dot{r} \leq v_{\parallel} \leq \dot{r} \\
1 & \text{if } v_{\parallel} > \dot{r}
\end{cases}
\]

\[
p(v_{\parallel}) = \frac{dP(v_{\parallel})}{dv_{\parallel}} = \begin{cases} 
(2\dot{r})^{-1} & \text{if } -\dot{r} \leq v_{\parallel} \leq \dot{r} \\
0 & \text{otherwise}
\end{cases}
\]
Problem 3: Acceleration of a Star

In this problem we want to find the probability density function for the acceleration random variable $a$ when $a = GM/r^2$, where $r$ is the distance to a nearest neighbor star of mass $M$.

![Graph showing the relationship between $a(r)$ and $r$.

(a) $P(a) = \int_{\sqrt{GM/a_0}}^{\infty} p_r(r) \, dr$

$p(a) = \frac{dP(a)}{da} = \frac{1}{2a} \sqrt{\frac{GM}{a}} \, p_r\left(\sqrt{\frac{GM}{a}}\right)$

Distant neighbors will produce small forces and accelerations, so their effect on $p(a)$ will be greatest when $a$ is small.

(b) We use the $p_r(r)$ found in problem 4 of problem set 2:

$p_r(r) = 4\pi \rho \, r^2 \, e^{-4\pi \rho r^3/3}$

to calculate

$p(a) = \frac{2\pi \rho}{GM} \left(\frac{GM}{a}\right)^{5/2} \, e^{-(4\pi \rho/3)(GM/a)^{3/2}}$
If there are binary stars and other complex units at close distances, these will have the greatest effect when $r$ is small or when $a$ is large.

(c) The model considered here assumes all neighbor stars have the same mass $M$. To improve the model, one should consider a distribution of $M$ values. One could even include the binary stars and other complex units, from part (b), with a suitable distribution.

Problem 4: Atomic Velocity Profile

\[
\begin{align*}
\text{In this problem we have two random variables related by } s &= A/v_x^2. \text{ We are given } p_s(\zeta) \text{ and are to find } p(v_x). \\
&\text{To find the probability } P(v) \text{ that } v_x \leq v \text{ we must calculate the probability that } s \geq A/v^2. \text{ This is (the only possible values of } v_x \text{ are } \geq 0)\\
&\quad P(v_x) = \frac{A^{3/2}}{\sigma^2 \sqrt{2\pi}\sigma^2} \int_{A/v^2}^{\infty} \frac{1}{\zeta^{5/2}} e^{-A/(2\sigma^2\zeta)} \, d\zeta\\
&\quad p(v_x) = \frac{dP(v_x)}{dv_x} = -\frac{A^{3/2}}{\sigma^2 \sqrt{2\pi}\sigma^2} \left( \frac{v_x^2}{A} \right)^{5/2} e^{-v_x^2/2\sigma^2} \frac{A}{d(v_x^2/v_x^2)} = \frac{2v_x^2}{\sigma^2 \sqrt{2\pi}\sigma^2} e^{-v_x^2/2\sigma^2}
\end{align*}
\]
Problem 5: Planetary Nebulae

In this problem we are looking at matter distributed with equal probability over a spherical shell of radius $R$. When we look at it from a distance it appears as a ring because we are looking edgewise through the shell (and see much more matter) near the outer edge of the shell. If we choose coordinates so that the $z$ axis points from the shell to us as observers on the earth, then $r_\perp = \sqrt{x^2 + y^2} = R \sin \theta$ will be the radius of the ring that we observe. The task for this problem is to find a probability distribution function $p(r_\perp)$ for the amount of matter that seems to us to be in a ring of radius $r_\perp$. This is a function of the two random variables $\theta$ and $\phi$, which are the angular coordinates locating a point on the shell of the planetary nebula. Since the nebulae are spherically symmetric shells, we know that $p(\theta, \phi) = (4\pi)^{-1}$.

We want first to find $P(r_\perp)$, the probability that the perpendicular distance from the line of sight is less than or equal to $r_\perp$. Since $r_\perp = R \sin \theta$, that is equal to the probability that $\sin \theta \leq r_\perp / R$. This corresponds to the two polar patches on the nebula: $0 \leq \theta \leq \sin^{-1}(r_\perp / R)$ in the “top” hemisphere and $\sin^{-1}(r_\perp / R) \geq \theta \geq \pi$ in the “bottom” hemisphere.

\[
P(r_\perp) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \left( \int_0^{\sin^{-1}(r_\perp / R)} \sin \theta d\theta + \int_{\sin^{-1}(r_\perp / R)}^\pi \sin \theta d\theta \right) = 1 - \sqrt{1 - \frac{r_\perp^2}{R^2}}
\]

\[
p(r_\perp) = \frac{dP(r_\perp)}{dr_\perp} = \frac{r_\perp}{R \sqrt{R^2 - r_\perp^2}} \quad \text{for} \quad 0 \leq r_\perp \leq R \quad \text{(and 0 otherwise)}
\]

Note: $\cos (\sin^{-1}(r_\perp / R)) = \sqrt{1 - \frac{r_\perp^2}{R^2}}$. 

\[
p(r_\perp)
\]

\[
0 \quad 1 \quad \frac{r_\perp}{R}
\]