Problem 1: Two Quantum Particles

a)

\[ p(x_1, x_2) = |\psi(x_1, x_2)|^2 \]
\[ = \frac{1}{\pi x_0^2} \left( \frac{x_1 - x_2}{x_0} \right)^2 \exp\left[-\frac{x_1^2 + x_2^2}{x_0^2}\right] \]

The figure on the left shows in a simple way the location of the maxima and minima of this probability density. On the right is a plot generated by a computer application, in this case Mathematica.

b)

\[ p(x_1) = \int_{-\infty}^{\infty} p(x_1, x_2) \, dx_2 \]
\[ = \frac{1}{\pi x_0^2} \left( \frac{x_1}{x_0} \right)^2 \exp\left[-\frac{x_1^2}{x_0^2}\right] \int_{-\infty}^{\infty} \left(x_1^2 - 2x_1x_2 + x_2^2\right) \exp\left[-\frac{x_2^2}{x_0^2}\right] \, dx_2 \]
\[ = \frac{1}{\pi x_0^2} \left( \frac{x_1}{x_0} \right)^2 \left[ x_1^2 \left( \sqrt{\frac{x_1^2}{x_0^2}} \right) - 2x_1 \times 0 + \frac{x_0^2}{2} \left( \sqrt{\frac{x_1^2}{x_0^2}} \right) \right] \]
\[ = \frac{1}{\sqrt{\pi x_0^2}} \left( \frac{x_1^2}{x_0^2} + 1/2 \right) \exp\left[-\frac{x_1^2}{x_0^2}\right] \]
By symmetry, the result for $p(x_2)$ has the same functional form.

$$p(x_2) = \frac{1}{\sqrt{\pi x_0^2}} \left( \frac{x_2^2}{x_0^2} + 1/2 \right) \exp[-x_2^2/x_0^2]$$

By inspection of these two results one sees that $p(x_1, x_2) \neq p(x_1)p(x_2)$, therefore $x_1$ and $x_2$ are not statistically independent.

c)

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$= \frac{\sqrt{\pi x_0^2}}{\pi x_0^2} \frac{(x_1 - x_2)^2 \exp[-(x_1^2 + x_3^2)/x_0^2]}{x_0^2 + 1/2 x_0^2} \exp[-x_2^2/x_0^2]$$

$$= \frac{2}{\sqrt{\pi x_0^2}} \frac{1}{(1 + 2(x_2/x_0)^2)} \left( \frac{x_1 - x_2}{x_0} \right)^2 \exp[-x_1^2/x_0^2]$$

It appears that these particles are anti-social: they avoid each other.

For those who have had some quantum mechanics, the $\psi(x_1, x_2)$ given here corresponds to two non-interacting spinless\(^1\) Fermi particles (particles which obey Fermi-Dirac statistics) in a harmonic oscillator potential. The ground and first excited single

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\(^1\)Why spinless? If the particles have spin, there is a spin part to the wavefunction. Under these circumstances the spin part of the wavefunction could carry the antisymmetry (assuming that the spatial and spin parts factor) and the spatial part of Fermionic wavefunction would have to be symmetric.
Particle states are used to construct the two-particle wavefunction. The wavefunction is antisymmetric in that it changes sign when the two particles are exchanged: \( \psi(x_2, x_1) = -\psi(x_1, x_2) \). Note that this antisymmetric property precludes putting both particles in the same single particle state, for example both in the single particle ground state.

For spinless particles obeying Bose-Einstein statistics (Bosons) the wavefunction must be symmetric under interchange of the two particles: \( \psi(x_2, x_1) = \psi(x_1, x_2) \). We can make such a wavefunction by replacing the term \( x_1 - x_2 \) in the current wavefunction by \( x_1 + x_2 \). Under these circumstances \( p(x_1|x_2) \) could be substantial near \( x_1 = x_2 \).

**Problem 2: A Joint Density of Limited Extent**

a) The tricky part here is getting correct limits on the integral that must be done to eliminate \( y \) from the probability density. It is clear that the integral must start at \( y = 0 \), but one must also be careful to get the correct upper limit. A simple sketch such as that at the right is helpful.

\[
p(x) = \int_{-\infty}^{\infty} p(x, y) \, dy = 6 \int_{0}^{1-x} (1 - x - y) \, dy
\]

\[
= 6 \left\{ \left( (1 - x)y - \frac{y^2}{2} \right) \right\}_{0}^{1-x} = 6 \{ (1 - x)^2 - \frac{1}{2} (1 - x)^2 \}
\]

\[
= 3(1 - x)^2 \quad 0 \leq x \leq 1
\]

b) \[
p(y|x) = \frac{p(x, y)}{p(x)} = \frac{6(1 - x - y)}{3(1 - x)^2}
\]

\[
= \frac{2}{(1 - x)^2} (1 - x - y) \quad 0 \leq y \leq 1 - x
\]

Note that this is simply a linear function of \( y \).
Problem 3: A Discrete Joint Density

a) \[ p(n) = \sum_l p(n, l) = \sum_{l=-n}^n c \exp[-an] \]
\[ = c(2n+1)\exp[-an] \]

b) \[ p(l|n) = \frac{p(l, n)}{p(n)} = \frac{c \exp[-an]}{c(2n+1)\exp[-an]} \]
\[ = \frac{1}{2n+1} \quad |l| \leq n \]
\[ = 0 \quad \text{otherwise} \]
Problem 4: Distance to the Nearest Star

First consider the quantity \( p(\text{no stars in a sphere of radius } r) \). Since the stars are distributed at random with a mean density \( \rho \) one can treat the problem as a Poisson process in three dimensions with the mean number of stars in the volume \( V \) given by:

\[
< n > = \rho V = \frac{4}{3} \pi \rho r^3.
\]

Thus

\[
p(\text{no stars in a sphere of radius } r) = \exp\left[ -\frac{4}{3} \pi \rho r^3 \right]
\]
Next consider the quantity $p$ (at least one star in a shell between $r$ and $r + dr$). When the differential volume element involved is so small that the expected number of stars within it is much less than one, this quantity can be replaced by $p$ (exactly one star in a shell between $r$ and $r + dr$). Now the volume element is that of the shell and $< n > = \rho \Delta V = 4\pi \rho r^2 dr$. Thus

$$p(\text{at least one star in a shell between } r \text{ and } r + dr) = p(n = 1)$$

$$= \frac{1}{1!} < n >^{(1)} e^{-<n>} = 1$$

$$\approx < n > = 4\pi \rho r^2 dr$$

Now $p(r)$ is defined as the probability density for the event “the first star occurs between $r$ and $r + dr$”. Since the positions of the stars are (in this model) statistically independent, this can be written as the product of the two separate probabilities found above.

$$p(r)dr = p(\text{no star out to } r) \times p(1 \text{ star between } r \text{ and } r + dr)$$

$$= 4\pi \rho r^2 \exp[-4/3 \pi \rho r^3] dr$$

Dividing out the differential $dr$ and being careful about the range of applicability leaves us with

$$p(r) = 4\pi \rho r^2 \exp[-4/3 \pi \rho r^3] \quad r \geq 0$$

$$= 0 \quad r < 0$$
Problem 5: Shot Noise

a) This is a Poisson process with a rate equal to $r_s$ counts per second and an interval of $T$ seconds. The probability of getting $n$ counts in the interval is given by the Poisson expression for $p(n)$ with a mean $<n> = r_sT$. Since we have defined the mean as the signal in this case, $S = r_sT$.

b) One tries to use a record such as the one above to determine $I$ through the relation $I = r_s(\eta A/\hbar \nu)^{-1} = <n> (\eta AT/\hbar \nu)^{-1}$. The problem is to determine $<n>$ since all the other factors are known constants. Using a single measurement taken in an interval $T$ to determine $<n>$ could be in error by an amount of the order of $\sigma = (\text{Variance})^{1/2} = \sqrt{<n>}$ which we define as the noise, $N$, for the measurement. A histogram of the measured results in an interval $T$ would peak near $<n>$ and would have a width $\sim \sqrt{<n>}$.

c) Now consider the signal to noise ratio, $S/N$.

$$S/N = <n> / \sqrt{<n>} = \sqrt{<n>}$$

$$= \sqrt{r_sT} = \left(\frac{\eta A}{\hbar \nu IT}\right)^{1/2}$$

This shows that the signal to noise ratio grows as the square root of both the light intensity and the counting time.

Note that if one had the entire record shown in the above figure available, about $36T$, one could do better at estimating $I$ by using the number of counts detected in the expanded interval $T' = 36T$. The signal to noise ratio would be increased by a factor of $\sqrt{T'/T} = \sqrt{36} = 6$. 
d) When $I$ is on, one has a Poisson process with $<n> = (r_s + r_d)T$ and $\sigma = \sqrt{(r_s + r_d)T}$. When $I$ is off, one has a Poisson process with $<n> = r_dT$ and $\sigma = \sqrt{r_dT}$. The signal is now defined as $S \equiv <n>_{\text{on}} - <n>_{\text{off}} = r_sT$. One tries to estimate this quantity from $n_{\text{on}} - n_{\text{off}}$ for a single measurement. The noise is associated with the uncertainty in both measurements, but when $r_d >> r_s$ the contribution from $r_s$ is small and $N \approx \sqrt{r_dT}$. Thus in the small signal limit

$$\frac{S}{N} = \frac{r_sT}{\sqrt{r_dT}} = \frac{\eta A}{h\nu\sqrt{r_d}}I\sqrt{T}.\,$$

Now the signal to noise ratio is linearly proportional to the intensity of the signal but it still only grows as the square root of the counting interval.

In the example shown below, $r_d = 100$ counts per second and $r_s = 20$ counts per second. The signal is cycled on for 25 counting intervals, each of length one second, and off for an equal number of counting intervals.