\section{Tensors}

Let $V$ be an $n$-dimensional vector space and let $V^k$ be the set of all $k$-tuples, $(v_1, \ldots, v_k)$, $v_i \in V$. A function
\[ T : V^k \rightarrow \mathbb{R} \]
is said to be linear in its $i$\textsuperscript{th} variable if, when we fix vectors, $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$, the map
\[ v \in V \rightarrow T(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k) \]
is linear in $V$. If $T$ is linear in its $i$\textsuperscript{th} variable for $i = 1, \ldots, k$ it is said to be $k$-\textit{linear}, or alternatively is said to be a $k$-\textit{tensor}. We denote the set of all $k$-tensors by $\mathcal{L}^k(V)$.

Let $T_1$ and $T_2$ be functions on $V^k$. It is clear from (2.1) that if $T_1$ and $T_2$ are $k$-linear, so is $T_1 + T_2$. Similarly if $T$ is $k$-linear and $\lambda$ is a real number, $\lambda T$ is $k$-linear. Hence $\mathcal{L}^k(V)$ is a vector space. Note that for $k = 1$, “$k$-linear” just means “linear”, so $\mathcal{L}^1(V) = V^\ast$.

We will next prove that this vector space is finite dimensional. Let
\[ I = (i_1, \ldots, i_k) \]
be a sequence of integers with $1 \leq i_r \leq n$, $r = 1, \ldots, k$. We will call such a sequence a \textit{multi-index} of length $k$. For instance the multi-indices of length 2 are the square array of pairs of integers
\[(i, j), \ 1 \leq i, j \leq n\]
and there are exactly $n^2$ of them.

\begin{exercise}

Show that there are exactly $n^k$ multi-indices of length $k$.

Now fix a basis, $e_1, \ldots, e_n$, of $V$ and for $T \in \mathcal{L}^k(V)$ let
\[ T_I = T(e_{i_1}, \ldots, e_{i_k}) \]
for every multi-index of length $k$, $I$.

\begin{proposition}

The $T_I$’s determine $T$, i.e., if $T$ and $T'$ are $k$-tensors and $T_I = T'_I$ for all $I$, then $T = T'$.

\begin{proof}

By induction on $n$. For $n = 1$ we proved this result in §1. Let’s prove that if this assertion is true for $n - 1$, it’s true for $n$. For each $e_i$ let $T_i$ be the $(k - 1)$-tensor
\[(v_1, \ldots, v_{n-1}) \rightarrow T(v_1, \ldots, v_{n-1}, e_i).\]
Then for $v = c_1 e_1 + \cdots c_n e_n$
\[ T(v_1, \ldots, v_{n-1}, v) = \sum c_i T_i(v_1, \ldots, v_{n-1}), \]
so the $T_I$’s determine $T$. Now apply induction.
\end{proof}
\end{proposition}
The tensor product operation

If $T_1$ is a $k$-tensor and $T_2$ is an $\ell$-tensor, one can define a $k + \ell$-tensor, $T_1 \otimes T_2$, by setting

$$(T_1 \otimes T_2)(v_1, \ldots, v_{k+\ell}) = T_1(v_1, \ldots, v_k)T_2(v_{k+1}, \ldots, v_{k+\ell}).$$

This tensor is called the tensor product of $T_1$ and $T_2$. Similarly, given a $k$-tensor, $T_1$, an $\ell$-tensor, $T_2$ and an $m$-tensor, $T_3$, one can define a $(k + \ell + m)$-tensor, $T_1 \otimes T_2 \otimes T_3$ by setting

$$(2.3) \quad T_1 \otimes T_2 \otimes T_3(v_1, \ldots, v_{k+\ell})$$
$$= T_1(v_1, \ldots, v_k)T_2(v_{k+1}, \ldots, v_{k+\ell})T_3(v_{k+\ell+1}, \ldots, v_{k+\ell+m}).$$

Alternatively, one can define (2.3) by defining it to be the tensor product of $T_1 \otimes T_2$ and $T_3$ or the tensor product of $T_1$ and $T_2 \otimes T_3$. It’s easy to see that both these tensor products are identical with (2.3):

$$(2.4) \quad (T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3) = T_1 \otimes T_2 \otimes T_3.$$ 

We leave for you to check that if $\lambda$ is a real number

$$(2.5) \quad \lambda(T_1 \otimes T_2) = (\lambda T_1) \otimes T_2 = T_1 \otimes (\lambda T_2)$$

and that the left and right distributive laws are valid: For $k_1 = k_2$,

$$(2.6) \quad (T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$

and for $k_2 = k_3$

$$(2.7) \quad T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3.$$ 

A particularly interesting tensor product is the following. For $i = 1, \ldots, k$ let $\ell_i \in V^*$ and let

$$(2.8) \quad T = \ell_1 \otimes \cdots \otimes \ell_k.$$ 

Thus, by definition,

$$(2.9) \quad T(v_1, \ldots, v_k) = \ell_1(v_1) \cdots \ell_k(v_k).$$

A tensor of the form (2.9) is called a decomposable $k$-tensor. These tensors, as we will see, play an important role in multilinear algebra. In particular, let $e_1, \ldots, e_n$ be a basis of $V$ and $e_1^*, \ldots, e_n^*$ the dual basis of $V^*$. For every multi-index, $I$, of length $k$ let

$$e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*.$$ 

Then if $J$ is another multi-index of length $k$, 

$$(2.10) \quad e_I^*(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}$$

by (1.6), (2.8) and (2.9). From (2.10) it’s easy to conclude
Theorem 2.2. The $e_i^*$’s are a basis of $L^k(V)$.

Proof. Given $T \in L^k(V)$, let

$$ T' = \sum T_i e_i^* $$

where the $T_i$’s are defined by (2.2). Then

$$(2.11) \quad T'(e_{j_1}, \ldots, e_{j_k}) = \sum T_i e_i^*(e_{j_1}, \ldots, e_{j_k}) = T_J $$

by (2.10); however, by Proposition 2.1 the $T_J$’s determine $T$, so $T' = T$. This proves that the $e_i^*$’s are a spanning set of vectors for $L^k(V)$. To prove they’re a basis, suppose

$$ \sum C_i e_i^* = 0 $$

for constants, $C_i \in \mathbb{R}$. Then by (2.11) with $T = 0$, $C_J = 0$, so the $e_i^*$’s are linearly independent.

Corollary. $\dim L^k(V) = n^k$.

The pull-back operation

Let $V$ and $W$ be finite dimensional vector spaces and let $A : V \to W$ be a linear mapping. If $T \in L^k(W)$, we define

$$ A^* T : V^k \to \mathbb{R} $$

to be the function

$$(2.12) \quad A^* T(v_1, \ldots, v_k) = T(Av_1, \ldots, Av_k).$$

It’s clear from the linearity of $A$ that this function is linear in its $i$th variable for all $i$, and hence is a $k$-tensor. We will call $A^* T$ the pull-back of $T$ by the map, $A$.

Proposition 2.3. The map

$$(2.13) \quad A^* : L^k(W) \to L^k(V), \quad T \to A^* T,$$

is a linear mapping.

We leave this as an exercise. We also leave as an exercise the identity

$$(2.14) \quad A^*(T_1 \otimes T_2) = A^* T_1 \otimes A^* T_2$$

for $T_1 \in L^k(W)$ and $T_2 \in L^m(W)$. Also, if $U$ is a vector space and $B : U \to V$ a linear mapping, we leave for you to check that

$$(2.15) \quad (AB)^* T = B^*(A^* T)$$

for all $T \in L^k(W)$. 

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Exercises.

1. Verify that there are exactly $n^k$ multi-indices of length $k$.

2. Prove Proposition 2.3.


4. Verify (2.15).