6 The degree of a differentiable mapping

Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and $f : U \to V$ a proper $C^2$ mapping. In this section we will show how to compute the degree of $f$ and, in particular, show that it is always an integer. From this fact we will be able to conclude that the degree of $f$ is a topological invariant of $f$: if we deform $f$ smoothly, its degree doesn’t change.

**Definition 6.1.** Let $C_f$ be the set of critical points of $f$. A point, $q \in V$, is a regular value of $f$ if it is not in the image, $f(C_f)$, of $C_f$.

By Sard’s theorem “almost all” points, $q$ in $V$ are regular values of $f$; i.e., the set of points which are not regular values of $f$ is a set of measure zero. Notice, by the way, that a point, $q$, can qualify as a regular value of $f$ by not being in the image of $f$. For instance, for the constant map, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(p) = c$, the points, $q \in \mathbb{R}^n - \{c\}$ are all regular values of $f$.

Picking a regular value, $q$, of $f$ we will prove:

**Theorem 6.2.** The set, $f^{-1}(q)$ is a finite set. Moreover, if $f^{-1}(q) = \{p_1,\ldots, p_n\}$ there exist connected open neighborhoods, $U_i$, of $p_i$ in $Y$ and an open neighborhood, $W$, of $q$ in $V$ such that:

(i) for $i \neq j$ $U_i$ and $U_j$ are disjoint;

(ii) $f^{-1}(W) = \bigcup U_i$,

(iii) $f$ maps $U_i$ diffeomorphically onto $W$.

**Proof.** If $p \in f^{-1}(q)$ then, since $q$ is a regular value, $p \notin C_f$; so

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^n$$

is bijective. Hence by the inverse function theorem, $f$ maps a neighborhood, $U_p$, of $p$ diffeomorphically onto a neighborhood of $q$. The open sets

$$\{U_p, \quad p \in f^{-1}(q)\}$$

are a covering of $f^{-1}(q)$; and, since $f$ is proper, $f^{-1}(q)$ is compact; so we can extract a finite subcovering

$$\{U_{p_i}, \quad i = 1,\ldots, N\}$$

and since $p_i$ is the only point in $U_{p_i}$ which maps onto $q$, $f^{-1}(q) = \{p_1,\ldots, p_N\}$.

Without loss of generality we can assume that the $U_{p_i}$’s are disjoint from each other; for, if not, we can replace them by smaller neighborhoods of the $p_i$’s which have this property. By Theorem 4.3 there exists a connected open neighborhood, $W$, of $q$ in $V$ for which

$$f^{-1}(W) \subset \bigcup U_{p_i}.$$ 

To conclude the proof let $U_i = f^{-1}(W) \cap U_{p_i}$. 

$\square$
The main result of this section is a recipe for computing the degree of \( f \) by counting the number of \( p_i \)’s above, keeping track of orientation.

**Theorem 6.3.** For each \( p_i \in f^{-1}(q) \) let \( \sigma_{p_i} = +1 \) if \( f : U_i \to W \) is orientation preserving and \( -1 \) if \( f : U_i \to W \) is orientation reversing. Then

\[
\deg(f) = \sum_{i=1}^{N} \sigma_{p_i}.
\]  

(6.1)

**Proof.** Let \( \omega \) be a compactly supported \( n \)-form on \( W \) of class \( C^1 \) whose integral is one. Then

\[
\deg(f) = \int_U f^* \omega = \sum_{i=1}^{N} \int_{U_i} f^* \omega.
\]

Since \( f : U_i \to W \) is a diffeomorphism

\[
\int_{U_i} f^* \omega = \pm \int_W \omega = +1 \text{ or } -1
\]

depending on whether \( f : U_i \to W \) is orientation preserving or not. Thus \( \deg(f) \) is equal to the sum (6.1).

\[\Box\]

As we pointed out above, a point, \( q \in V \) can qualify as a regular value of \( f \) “by default”, i.e., by not being in the image of \( f \). In this case the recipe (6.1) for computing the degree gives “by default” the answer zero. Let’s corroborate this directly.

**Theorem 6.4.** If \( f : U \to V \) isn’t onto, \( \deg(f) = 0 \).

**Proof.** By exercise 3 of §4, \( V - f(U) \) is open; so if it is non-empty, there exists a compactly supported \( n \)-form, \( \omega \), of class \( C^\infty \) with support in \( V - f(U) \) and with integral equal to one. Since \( \omega = 0 \) on the image of \( f \), \( f^* \omega = 0 \); so

\[
0 = \int_U f^* \omega = \deg(f) \int_V \omega = \deg(f).
\]

\[\Box\]

**Remark:** In applications the contrapositive of this theorem is much more useful than the theorem itself.

**Theorem 6.5.** If \( \deg(f) \neq 0 \) \( f \) maps \( U \) onto \( V \).

In other words if \( \deg(f) \neq 0 \) the equation

\[
f(x) = y
\]

has a solution, \( x \in U \) for every \( y \in V \).

We will now show that the degree of \( f \) is a topological invariant of \( f \): if we deform \( f \) by a “homotopy” we don’t change its degree. To prove this we will need a slight generalization of the notion of “proper mapping”.

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**Definition 6.6.** Let \( X \) be a subset of \( \mathbb{R}^m \) and \( Y \) a subset of \( \mathbb{R}^m \). A continuous map \( f : X \to Y \) is proper if for every compact subset, \( A \), of \( Y \), \( f^{-1}(A) \) is compact.

In particular, let \( U \) and \( V \) be open subsets of \( \mathbb{R}^n \) and let \( a \) be a positive real number. Suppose that

\[
g : [0, a] \times U \to V
\]

is a proper \( C^1 \) mapping. For \( t \in [0, a] \) let

\[
f_t : U \to V
\]

be the mapping, \( f_t(p) = g(p, t) \). Then, if \( A \) is a compact subset of \( V \), \( f_t^{-1}(A) \) is the intersection of the compact set

\[
\{(s, p) \in [0, a] \times U \mid g(s, p) \in A\}
\]

with the set: \( s = t \), and hence is compact. Therefore, for every \( t \in [0, a] \), \( f_t : U \to V \) is a proper \( C^1 \) mapping. If \( f_0 = f \), we will call the family of mappings

\[
f_t, \quad 0 \leq t \leq a
\]

a “deformation” or “homotopy” of \( f \).

**Theorem 6.7.** For all \( t \in [0, a] \), the degree of \( f_t \) is equal to the degree of \( f \).

**Proof.** Let

\[
\omega = \phi(y)dy_1 \wedge \cdots \wedge dy_n
\]

be a compactly supported \( n \)-form of class \( C^1 \) on \( U \) with integral equal to one. Then the degree of \( f_t \) is equal to the integral of \( f_t^*\omega \) over \( U \):

\[
\int_U \phi(g_1(x, t), \ldots, g_n(x, t)) \det D_x g(x, t) \, dx.
\]

(6.4)

The integrand in (6.4) is continuous and is supported on a compact subset of \([0, a] \times U\); hence (6.4) is continuous as a function of \( t \). However, \( \deg(f_t) \) is integer valued, so (6.4) is an (integer-valued) constant, not depending on \( t \).

There are many other nice applications of Theorem 6.3. We’ll content ourselves with two relatively simple and prosaic ones:

**Application 1.** The Brouwer fixed point theorem

Let \( B^n \) be the closed unit ball in \( \mathbb{R}^n \):

\[
\{x \in \mathbb{R}^n, \|x\| \leq 1\}.
\]
Theorem 6.8. If \( f : B^n \rightarrow B^n \) is a \( C^2 \) mapping then \( f \) has a fixed point, i.e., maps some point, \( x_0 \in B^n \) onto itself.

The idea of the proof will be to assume that there isn’t a fixed point and show that this leads to a contradiction. Suppose that for every point, \( x \in B^n \) \( f(x) \neq x \). Consider the ray through \( f(x) \) in the direction of \( x \):

\[
f(x) + s(x - f(x)), \quad 0 \leq s < \infty.
\]

This intersects the boundary, \( S^{n-1} \), of \( B^n \) in a unique point, \( \gamma(x) \), (see figure 1 below); and one of the exercises at the end of this section will be to show that the mapping \( \gamma : B^n \rightarrow S^{n-1}, \quad x \rightarrow \gamma(x) \), is a \( C^2 \) mapping. Also it is clear from figure 1 that \( \gamma(x) = x \) if \( x \in S^{n-1} \).

![Figure 6.1](image)

Let \( B^n(r) \) be the ball, \( \{ x \in \mathbb{R}^n, \| x \| \leq r \} \). Since \( \gamma \) is a \( C^2 \) mapping of \( B^n \) into \( \mathbb{R}^n \), there exists an open set, \( U \), containing \( B^n \) and a \( C^2 \) mapping of \( U \) into \( \mathbb{R}^n \) whose restriction to \( B^n \) is \( \gamma \). (For the sake of economy of notation we’ll continue to call this map \( \gamma \).) Since \( U \) is open and contains \( B^n(1) \) it contains a slightly larger ball, \( B^n(1 + \delta_0) \), \( \delta_0 > 0 \). We claim:

Lemma 6.9. Given \( \epsilon > 0 \) there exists a \( 0 < \delta < \delta_0 \) such that for \( 1 \leq \| x \| \leq 1 + \delta \), \( \| \gamma(x) - x \| \) is less than \( \epsilon \).

Proof. Since \( \gamma \) is uniformly continuous on \( B(1 + \delta_0) \) there exists \( 0 < \delta < \delta_0 \) such that for \( x, y \in B(1 + \delta_0) \) and \( \| x - y \| < \delta \), \( \| \gamma(x) - \gamma(y) \| \) is less than \( \epsilon/2 \). Moreover we can assume \( \delta < \frac{\epsilon}{2} \). Let \( 1 \leq \| x \| \leq 1 + \delta \) and let \( y = x/\| x \| \). Then

\[
x - y = \| x \| y - y = (\| x \| - 1)y
\]
so
\[ \|x - y\| = \|x\| - 1 \leq \delta. \]
Hence \( \|\gamma(x) - \gamma(y)\| \) is less than \( \epsilon/2 \); therefore, since \( \gamma(y) = y \), \( \|\gamma(x) - y\| \) is less than \( \epsilon/2 \). Thus
\[ \|\gamma(x) - x\| \leq \|\gamma(x) - y\| + \|y - x\| \leq \frac{\epsilon}{2} + \delta \leq \epsilon. \]
\[ \square \]

Now let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be a \( C^\infty \) function, with \( 0 \leq \varphi \leq 1 \), which is one on the set, \( \|x\| \leq 1 + \delta/2 \), and zero on the set, \( \|x\| \geq 1 + \delta \), and let
\[ g(x) = \varphi(x)\gamma(x) + (1 - \varphi(x))x. \] (6.5)
The mapping defined by (6.5) is a \( C^2 \) mapping of \( \mathbb{R}^n \) into \( \mathbb{R}^n \) with the properties
\[ g(x) = \gamma(x) \text{ for } \|x\| \leq 1 + \delta/2 \] (6.6)
and
\[ g(x) = (x) \text{ for } \|x\| \geq 1 + \delta \]. (6.7)
We claim that on the set \( 1 \leq \|x\| \leq 1 + \delta \)
\[ \|g(x)\| \geq 1 - \epsilon. \] (6.8)
Indeed, since \( g(x) = \varphi(x)(\gamma(x) - x) + x \)
\[ \|g(x)\| \geq \|x\| - \|\gamma(x) - x\| \]
and by Lemma 6.9, \( \|\gamma(x) - x\| < \epsilon \) if \( 1 \leq \|x\| \leq 1 + \delta \). On the other hand, for \( \|x\| \leq 1 \), \( g(x) = \gamma(x) \) and \( \|\gamma(x)\| = 1 \) so \( \|g(x)\| = 1 \) and for \( \|x\| \geq 1 + \delta \) \( g(x) = x \), so \( \|g(x)\| \geq 1 + \delta \). Hence
\[ \|g(x)\| \geq 1 - \epsilon \] (6.9)
for all \( x \). Moreover, by (6.6), \( g(x) \) is proper. Let’s compute its degree. (6.8) tells us that if \( \epsilon < 1 \), the origin is not in the image of \( g \), so by Theorem 6.4 the degree of \( g \) is zero. On the other hand \( g \) is equal to the identity map on the set \( \|x\| \geq 1 + \delta \), and the degree of the identity map is one, hence so is the degree of \( g \). This gives us the contradiction we were looking for a proves by contradiction that \( f \) has to have a fixed point.

**Application 2. The fundamental theorem of algebra**

Let \( p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) be a polynomial of degree \( n \) with complex coefficients. If we identify the complex plane
\[ \mathbb{C} = \{z = x + iy; \, x, y \in \mathbb{R}\} \]
with \( \mathbb{R}^2 \) via the map, \( (x, y) \in \mathbb{R}^2 \to z = x + iy \), we can think of \( p \) as defining a mapping
\[ p : \mathbb{R}^2 \to \mathbb{R}^2, \, z \to p(z). \]
We will prove
Theorem 6.10. The mapping, \( p \), is proper and \( \deg(p) = n \).

Proof. For \( 0 \leq t \leq 1 \) let

\[
p_t(z) = (1-t)z^n + tp(z)
\]

\[
= z^n + t \sum_{i=0}^{n-1} a_i z^i.
\]

We will show that the mapping

\[
g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2, \ z \to p_t(z)
\]

is a proper mapping. Let

\[
C = \sup\{|a_i|, \ i = 0, \ldots, n-1\}.
\]

Then for \( |z| \geq 1 \)

\[
|a_0 + \cdots + a_{n-1} z^{n-1}| \leq |a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1} \leq C|z|^{n-1},
\]

and hence, for \( |z| > 2C \),

\[
|p_t(z)| \geq |z|^n - C|z|^{n-1} \geq C|z|^{n-1}.
\]

If \( A \) is a compact subset of \( \mathbb{C} \) then for some \( R > 0 \), \( A \) is contained in the disk, \( |w| \leq R \) and hence \( g^{-1}(A) \) is contained in the set

\[
\{(t, z) ; 0 \leq t \leq 1, |p_t(z)| \leq R\};
\]

and hence in the compact set

\[
\{(t, z) ; 0 \leq t \leq 1, C|z|^{n-1} \leq R\};
\]

and this shows that \( g \) is proper. Thus each of the mappings,

\[
p_t : \mathbb{C} \to \mathbb{C},
\]

is proper and \( \deg p_t = \deg p_1 = \deg p = \deg p_0 \). However, \( p_0 : \mathbb{C} \to \mathbb{C} \) is just the mapping, \( z \to z^n \) and an elementary computation (see exercises 5 and 6 below) shows that the degree of this mapping is \( n \).

In particular for \( n > 0 \) the degree of \( p \) is non-zero; so by Theorem 6.4 we conclude that \( p : \mathbb{C} \to \mathbb{C} \) is surjective and hence has zero in its image.

Theorem 6.11. (fundamental theorem of algebra)

Every polynomial,

\[
p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0,
\]

with complex coefficients has a complex root.
Exercises for §6

(1) Let $W$ be a subset of $\mathbb{R}^n$ and let $a(x), b(x)$ and $c(x)$ be real-valued functions on $W$ of class $C^r$. Suppose that for every $x \in W$ the quadratic polynomial

$$a(x)s^2 + b(x)s + c(x)$$

has two distinct real roots, $s_+(x)$ and $s_-(x)$, with $s_+(x) > s_-(x)$. Prove that $s_+$ and $s_-$ are functions of class $C^r$.

*Hint:* What are the roots of the quadratic polynomial: $as^2 + bs + c$?

(2) Show that the function, $\gamma(x)$, defined in figure 1 is a $C^1$ mapping of $B^n$ onto $S^{2n-1}$. *Hint:* $\gamma(x)$ lies on the ray,

$$f(x) + s(x - f(x)), \quad 0 \leq s < \infty$$

and satisfies $\|\gamma(x)\| = 1$; so $\gamma(x)$ is equal to

$$f(x) + s_0(x - f(x))$$

where $s_0$ is a non-negative root of the quadratic polynomial

$$\|f(x) + s(x - f(x))\|^2 - 1.$$ 

Argue from figure 1 that this polynomial has to have two distinct real roots.

(3) Show that the Brouwer fixed point theorem isn’t true if one replaces the closed unit ball by the open unit ball. *Hint:* Let $U$ be the open unit ball (i.e., the interior of $B^n$). Show that the map

$$h : U \to \mathbb{R}^n, \quad h(x) = \frac{x}{1 - \|x\|^2}$$

is a diffeomorphism of $U$ onto $\mathbb{R}^n$, and show that there are lots of mappings of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which don’t have fixed points.

(4) Show that the fixed point in the Brouwer theorem doesn’t have to be an interior point of $B^n$, i.e., show that it can lie on the boundary.

(5) If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the mapping: $(x, y) \to z = x + iy$, we can think of a $\mathbb{C}$-linear mapping of $\mathbb{C}$ into itself, i.e., a mapping of the form

$$z \to cz, \quad c \in \mathbb{C}$$

as being an $\mathbb{R}$-linear mapping of $\mathbb{R}^2$ into itself. Show that the determinant of this mapping is $|c|^2$. 

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(6) (a) Let \( f : \mathbb{C} \to \mathbb{C} \) be the mapping, \( f(z) = z^n \). Show that
\[
Df(z) = nz^{n-1}.
\]

\textit{Hint:} Argue from first principles. Show that for \( h \in \mathbb{C} = \mathbb{R}^2 \)
\[
\frac{(z + h)^n - z^n - n z^{n-1} h}{|h|}
\]
tends to zero as \( |h| \to 0 \).

(b) Conclude from the previous exercise that
\[
\det Df(z) = n^2 |z|^{2n-2}.
\]

(c) Show that at every point \( z \in \mathbb{C} - 0 \), \( f \) is orientation preserving.

(d) Show that every point, \( w \in \mathbb{C} - 0 \) is a regular value of \( f \) and that
\[
f^{-1}(w) = \{z_1, \ldots, z_n\}
\]
with \( \sigma_{z_i} = +1 \).

(e) Conclude that the degree of \( f \) is \( n \).