Lecture 15: Contour Integration and Applications
(Text 154-161)

Remarks on Lecture 15

In parts 4 and 5 (p. 154-160) some clarification of the use of the logarithm are called for.

Example 4 p.159

The relation

\((-z)^{2\alpha} = e^{2\pi i\alpha}z^{2\alpha}\)

which is crucial for proof deserves an explanation.

![Fig. 15-1](image)

We consider the function

\[\log_\theta z = \log|z| + i\arg_\theta z\]

in the region \(\mathbb{C} - l_\theta\) (the plane with the ray \(l_\theta\) removed) where the angle is fixed by

\[\theta < \arg_\theta z < \theta + 2\pi.\]

In the problem of computing

\[\int_0^\infty x^\alpha R(x) \, dx\]

we consider

\[\log_{-\frac{\pi}{2}}(z)\]
in the plane \( \mathbb{C} \) with the negative imaginary axis removed and us the Residue theorem on the contour in Fig. 4.13. As in the text we arrive at the integral

\[
\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) \, dz = \int_{0}^{\infty} \left( z^{2\alpha+1} + (-z)^{2\alpha+1} \right) R(z^2) \, dz.
\]

On the right \( z \) belongs to \((0, \infty)\) and

\[
\log_{-\frac{\pi}{2}}(z) = \log |z| + \left( -\frac{\pi}{2} + \frac{\pi}{2} \right) i, \quad -\frac{\pi}{2} < \arg_{-\frac{\pi}{2}} z < \frac{3\pi}{2},
\]

\[
\log_{-\frac{\pi}{2}}(-z) = \log |z| + \left( -\frac{\pi}{2} + \frac{3\pi}{2} \right) i
\]

\[= \log_{-\frac{\pi}{2}}(z) + i\pi, \quad z > 0.\]

Thus for \( z > 0 \),

\[
(-z)^{2\alpha+1} = e^{(2\alpha+1) \log_{-\frac{\pi}{2}}(-z)}
\]

\[= e^{(2\alpha+1) \log_{-\frac{\pi}{2}}(z+i\pi)}
\]

\[= -e^{2\alpha i\pi} z^{2\alpha+1},\]

so the last integrals combine to

\[
(1 - e^{2\alpha i\pi}) \int_{0}^{\infty} z^{2\alpha+1} R(z^2) \, dz.
\]

For \( z > 0 \) we have from the above

\[
\log_{-\frac{\pi}{2}}(z) = \log |z|,
\]

so

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) \, dz = \frac{1}{2\pi i} (1 - e^{2\alpha i\pi}) \int_{0}^{\infty} x^{2\alpha+1} R(x^2) \, dx
\]

\[= -\frac{1}{\pi} e^{\alpha i\pi} \sin \pi \alpha \int_{0}^{\infty} x^{2\alpha+1} R(x^2) \, dx. \quad (1)
\]

The left hand side of (1) is the sum of the residues of

\[
z^{2\alpha+1} R(z^2) = f(z)
\]

in the upper half plane. If

\[
R(z^2) = \frac{g(z)}{h(z)},
\]

where \( g \) and \( h \) are holomorphic, \( g(a) \neq 0 \), and \( h \) has a simple zero at \( a \), then

\[
\text{Res}_{z=a} f(z) = z^{(2\alpha+1)}(a) \frac{g(a)}{h'(a)}, \quad (2)
\]

where

\[
z^{2\alpha+1} = e^{(2\alpha+1) \log_{-\frac{\pi}{2}}(z)}.
\]
Example: Exercise 3(g) p.161

To calculate
\[ \int_0^\infty \frac{x^{1/3}}{1 + x^2} \, dx, \]
we use \( x = t^2 \) and arrive at
\[ \int_{-\infty}^{\infty} \frac{z^{1/3}}{1 + z^4} \, dz \]
in (1). The poles in the upper half plane are
\[ z = e^{i/4} \quad \text{and} \quad z = e^{i(\frac{3}{4} + \frac{\pi}{2})}. \]

We use (2) to calculate the residues:

\[
\text{Res}_{z=e^{i/4}} \left( z^{1/3} \frac{1}{1 + z^4} \right) = z^{1/3} \left( e^{i/4} \right) \frac{1}{4(e^{i/4})^3} = \frac{1}{4} e^{-i/4},
\]
and
\[
\text{Res}_{z=e^{i(3/4)}} \left( z^{1/3} \frac{1}{1 + z^4} \right) = z^{1/3} \left( e^{i(3/4)} \right) \frac{1}{4(e^{i(3/4)})^3} = \frac{1}{4} e^{-i\pi}.
\]

Thus (1) gives
\[
\frac{1}{4} e^{-i\pi} + \frac{1}{4} e^{-i\pi} = \frac{1}{\pi} e^{\pi i} \sin \frac{\pi}{3} \int_0^{\infty} \frac{x^{1/3}}{1 + x^3} \, dx,
\]
so
\[
\int_0^{\infty} \frac{x^{1/3}}{1 + x^3} \, dx = \frac{\pi}{2\sqrt{3}}.
\]
Example 5 p.160

The last four lines on the page are a bit misleading because the specific logarithm has already been chosen. So here is a completion of the proof after the equation

$$\int_0^\pi \log(-2ie^{ix} \sin x) \, dx = 0.$$

We know (Lecture 2) that

$$\log(z_1z_2) = \log z_1 + \log z_2, \quad \text{if} \quad -\pi < \text{Arg} z_1 + \text{Arg} z_2 < \pi. \quad (3)$$

Using this for $z = 2 \sin x$ we get

$$\int_0^\pi \log(2 \sin x) \, dx + \int_0^\pi \log(-ie^{ix}) \, dx = 0. \quad (4)$$

But

$$\log(-i) = -\frac{\pi i}{2}, \quad \log e^{ix} = ix \quad (0 < x < \pi),$$

so since $-\frac{\pi}{2} + x$ is in $(-\pi, \pi)$, (3) implies

$$\log(-ie^{ix}) = -\frac{\pi i}{2} + ix.$$

Now (4) implies the result

$$\int_0^\pi \log \sin \theta \, d\theta = -\pi \log 2.$$