Lecture 2: Exponential function & Logarithm
for a complex argument
(Replacing Text p.10 - 20)

For $b > 1$, $x \in \mathbb{R}$, we defined in 18.100B,

$$b^x = \sup_{t \in \mathbb{Q}, t \leq x} b^t$$

(where $b^t$ was easy to define for $t \in \mathbb{Q}$). Then the formula

$$b^{x+y} = b^x b^y$$

was hard to prove directly. We shall obtain another expression for $b^x$ making proof easy.

Let

$$L(x) = \int_1^x \frac{dt}{t}, \quad x > 0.$$  

Then

$$L(xy) = L(x) + L(y)$$

and

$$L'(x) = \frac{1}{x} > 0.$$  

So $L(x)$ has an inverse $E(x)$ satisfying

$$E(L(x)) = x.$$  

By 18.100B,

$$E'(L(x))L'(x) = 1,$$

so

$$E'(L(x)) = x.$$  

If $y = L(x)$, so $x = E(y)$, we thus have

$$E'(y) = E(y),$$

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It is easy to see $E(0) = 1$, so by uniqueness,

$$E(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \quad \text{and} \quad E(1) = e.$$  

**Theorem 1** $b^x = E(xL(b))$, $\forall x \in \mathbb{R}$.

**Proof:** Let $u = L(x)$, $v = L(y)$, then

$$E(u + v) = E(L(x) + L(y)) = E(L(xy)) = xy = E(u)E(v),$$

$$E(n) = E(1)^n = e^n,$$

and if $t = \frac{n}{m}$,

$$E(t)^m = E(mt) = E(n) = e^n.$$

so

$$E(t) = e^t, \quad t \in \mathbb{Q}, \quad t > 0.$$

Since

$$E(t)E(-t) = 1,$$

So

$$E(t) = e^t, \quad t \in \mathbb{Q}.$$  

Now

$$b^n = E(nL(b))$$

and

$$b^\frac{1}{m} = E\left(\frac{1}{m}L(b)\right)$$

since both have same $m^{th}$ power.

$$\left(b^\frac{1}{m}\right)^n = b^\frac{n}{m} = E\left(\frac{1}{m}L(b)\right)^n = E\left(\frac{n}{m}L(b)\right),$$

so

$$b^t = E(tL(b)), \quad t \in \mathbb{Q}.$$  

Now for $x \in \mathbb{R}$,

$$b^x = \sup_{t \leq x, \ t \in \mathbb{Q}} (b^t) = \sup_{t \leq x, \ t \in \mathbb{Q}} E(tL(b)) = E(xL(b))$$

since $E(x)$ is continuous.

Q.E.D.

**Corollary 1** For any $b > 0$, $x, y \in \mathbb{R}$, we have $b^{x+y} = b^x b^y$.  

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In particular $e^x = E(x)$, so we have the amazing formula

$$
\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \right)^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots .
$$

The formula for $e^x$ suggests defining $e^z$ for $z \in \mathbb{C}$ by

$$
e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots.
$$

the convergence being obvious.

**Proposition 1** $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

**Proof:** Look at the functions

$$
f(t) = e^{tz+w}, \quad g(t) = e^{t z} e^w
$$

for $t \in \mathbb{R}$. Differentiating the series for $e^{tz+w}$ and $e^{t z}$ with respect to $t$, term-by-term, we see that

$$
\frac{df}{dt} = z f(t), \quad \frac{dg}{dt} = z g(t)
$$

and

$$
f(0) = e^w, \quad g(0) = e^w .
$$

By the uniqueness for these equations, we deduce $f \equiv g$. Thus $f(1) = g(1)$. **Q.E.D.**

Note that if $t \in \mathbb{R}$,

$$
e^{it}e^{-it} = 1, \quad \text{and} \quad (e^{it})^{-1} = e^{-it}.
$$

Thus

$$
|e^{it}| = 1.
$$

So $e^{it}$ lies on the unit circle.

Put

$$
\cos t = \frac{e^{it} + e^{-it}}{2} = 1 - \frac{t^2}{2} + \cdots ,
$$

$$
\sin t = \frac{e^{it} - e^{-it}}{2i} = t - \frac{t^3}{3!} + \cdots .
$$

Thus we verify the old geometric meaning $e^{it} = \cos t + i \sin t$. Note that the $e^{it}(t \in \mathbb{R})$ fill up the unit circle. In fact by the intermediate value theorem, $\{\cos t \mid t \in \mathbb{R}\}$ fills up the interval $[-1, 1]$, so $e^{it} = \cos t + i \sin t$ is for a suitable $t$ an arbitrary point on the circle.
Note that \(z \mapsto e^z\) takes all values \(w \in \mathbb{C}\) except 0. For this note
\[
e^z = e^x \cdot e^{iy}, \quad z = x + iy.
\]
Choose \(x\) with
\[
e^x = |w|
\]
and then \(y\) so that
\[
e^{iy} = \frac{w}{|w|},
\]
then \(e^z = w\).

If
\[
z = |z| e^{i\varphi}, \quad w = |w| e^{i\psi},
\]
then
\[
zw = |z||w| e^{i(\varphi + \psi)} = |z||w|(\cos(\varphi + \psi) + i\sin(\varphi + \psi)),
\]
which gives a geometric interpretation of the multiplication.

From this we also have the following very useful formula
\[
(\cos \varphi + i \sin \varphi)^n = e^{in\varphi} = \cos n\varphi + i\sin n\varphi.
\]
Thus

**Theorem 2** The roots of \(z^n = 1\) are \(1, \omega, \omega^2, \ldots, \omega^{n-1}\), where
\[
\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.
\]

Geometric meanings for some useful complex number sets:
\[
\begin{align*}
|z - a| &= r \quad \longleftrightarrow \quad \text{circle} \\
|z - a| + |z - b| &= r, \quad (|a - b| < r) \quad \longleftrightarrow \quad \text{ellipse} \\
|z - a| &= |z - b| \quad \longleftrightarrow \quad \text{perpendicular bisector} \\
\{z \mid z = a + tb, \, t \in \mathbb{R}\} \quad \longleftrightarrow \quad \text{line} \\
\{z \mid \text{Im}z < 0\} \quad \longleftrightarrow \quad \text{lower half plane} \\
\{z \mid \text{Im} \left(\frac{z - a}{b}\right) < 0\} \quad \longleftrightarrow \quad \text{general half plane}
\end{align*}
\]
For $x$ real, $x \mapsto e^x$ has an inverse. This is NOT the case for $z \mapsto e^z$, because

$$e^{z+2\pi i} = e^z,$$

thus $e^z$ does not have an inverse. Moreover, for $w \neq 0$,

$$e^z = w$$

has infinitely many solutions:

$$e^x = |w|, \quad e^{iy} = \frac{w}{|w|} \implies x = \log |w|, \quad y = \arg(w).$$

So

$$\log w = \log |w| + i\arg(w)$$

takes infinitely many values, thus not a function.

Define

$$\text{Arg}(w) \triangleq \text{principal argument of } w \text{ in interval } -\pi < \text{Arg}(w) < \pi$$

and define the principal value of logarithm to be

$$\text{Log}(w) \triangleq \log |w| + i\text{Arg}(w),$$

which is defined in slit plane (removing the negative real axis).

We still have

$$\log z_1 z_2 = \log z_1 + \log z_2$$

in the sense that both sides take the same infinitely many values. We can be more specific:

**Theorem 3** In slit plane,

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2) + n \cdot 2\pi i, \quad n = 0 \text{ or } \pm 1$$

and $n = 0$ if

$$-\pi < \text{Arg}(z_1) + \text{Arg}(z_2) < \pi.$$

In particular, $n = 0$ if $z_1 > 0$.

**Proof:** In fact, Arg($z_1$), Arg($z_2$) and Arg($z_1 z_2$) are all in ($-\pi, \pi$), thus

$$-\pi - \pi - \pi < \text{Arg}(z_1) + \text{Arg}(z_2) - \text{Arg}(z_1 z_2) < \pi + \pi + \pi,$$

but

$$\text{Arg}(z_1) + \text{Arg}(z_2) - \text{Arg}(z_1 z_2) = n \cdot 2\pi i,$$
thus \[ |n| \leq 1. \]

If \[ |\text{Arg}(z_1 + \text{Arg}(z_2)| < \pi, \]

since \[ |\text{Arg}(z_1z_2)| < \pi, \]

they must agree since difference is a multiple of \(2\pi\). \hspace{1cm} \text{Q.E.D.}