Lecture 13: The General Cauchy Theorem

(Replacing Text 137-148)

Here we shall give a brief proof of the general form of Cauchy’s Theorem.

Definition 1 A closed curve $\gamma$ in an open set $\Omega$ is homologous to 0 (written $\gamma \sim 0$) with respect to $\Omega$ if

$$n(\gamma, a) = 0 \quad \text{for all } a \notin \Omega.$$  

Definition 2 A region is simply connected if its complement with respect to the extended plane is connected.

Remark: If $\Omega$ is simply connected and $\gamma \subset \Omega$ a closed curve, then $\gamma \sim 0$ with respect to $\Omega$. In fact, $n(\gamma, z)$ is constant in each component of $\mathbb{C} - \gamma$, hence constant in $\mathbb{C} - \Omega$ and is 0 for $z$ sufficiently large.

Theorem 1 (Cauchy’s Theorem) If $f$ is analytic in an open set $\Omega$, then

$$\int_{\gamma} f(z) \, dz = 0$$

for every closed curve $\gamma \subset \Omega$ such that $\gamma \sim 0$.

We shall first prove

Theorem 2 (Cauchy’s Integral Formula) Let $f$ be holomorphic in an open set $\Omega$. Then

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad (1)$$

where $\gamma \sim 0$ with respect to $\Omega$.  

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Proof: The prove is based on the following three claims.

Define \( g(z, \zeta) \) on \( \Omega \times \Omega \) by

\[
g(z, \zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{for } z \neq \zeta, \\ f'(z) & \text{for } z = \zeta. \end{cases}
\]

Claim 1: \( g \) is continuous on \( \Omega \times \Omega \) and holomorphic in each variable.

Clearly \( g \) is continuous outside the diagonal in \( \Omega \times \Omega \). Let \((z_0, z_0)\) be a point on the diagonal and \( D \subset \Omega \) a disk with center \( z_0 \). Let \( z \neq \zeta \) in \( D \) and

\[
\delta : w(t) = z + t(\zeta - z) \quad (0 \leq t \leq 1)
\]

the segment jointing them. Then

\[
\int_{\delta} f'(w) \, dw = \int_0^1 \frac{d}{dt} f(w(t)) \frac{1}{\zeta - z} w'(t) \, dt
\]

\[
= f(\zeta) - f(z),
\]

and

\[
g(z, \zeta) - g(z_0, z_0) = (\zeta - z)^{-1} \int_{\delta} (f'(w) - f'(z_0)) \, dw.
\]

So the continuous at \((z_0, z_0)\) is obvious.

For the holomorphy statement, it is clear that for each \( \zeta_0 \in \Omega \) the function

\[ z \mapsto g(z, \zeta_0) \]

is holomorphic on \( \Omega - \zeta_0 \). Since

\[
\lim_{z \to \zeta_0} g(z, \zeta_0)(z - \zeta_0) = 0
\]

the point \( \zeta_0 \) is a removable singularity (Theorem 7, p.124), so

\[ z \mapsto g(z, \zeta_0) \]

is indeed holomorphic on \( \Omega \). This proves Claim 1.
Let
\[ \Omega' = \{ z \in \mathbb{C} : n(\gamma, z) = 0 \}. \]

Define function \( h \) on \( \mathbb{C} \) by
\[ h(z) = \frac{1}{2\pi i} \int_\gamma g(z, \zeta) \, d\zeta, \quad z \in \Omega; \quad (2) \]
\[ h(z) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in \Omega'. \quad (3) \]

Since both expressions agree on \( \Omega \cap \Omega' \) and since \( \Omega \cup \Omega' = \mathbb{C} \), this is a valid definition.

**Claim 2:** \( h \) is holomorphic.

This is obvious on the open sets \( \Omega' \) and \( \Omega - \gamma \). To show holomorphy at \( z_0 \in \gamma \), consider a disk \( D \subset \Omega \) with center \( z_0 \). Let \( \delta \) be any closed curve in \( D \). Then
\[
\int_\delta h(z) \, dz = \frac{1}{2\pi i} \int_\delta \left( \int_\gamma g(z, \zeta) \, d\zeta \right) dz \\
= \frac{1}{2\pi i} \int_\gamma \left( \int_\delta g(z, \zeta) \, dz \right) d\zeta.
\]

For each \( \zeta \),
\[ z \mapsto g(z, \zeta) \]
is holomorphic on \( D \) (even \( \Omega \)). So by the Cauchy’s theorem for disks,
\[ \int_\delta g(z, \zeta) \, dz = 0. \]

Now the Morera’s Theorem implies \( h \) is holomorphic.

Now we can prove:

**Claim 3:** \( h \equiv 0 \), so (1) holds.

We have \( z \in \Omega' \) for \( |z| \) sufficiently large. So by (3),
\[ \lim_{z \to \infty} h(z) = 0. \]

By Liouville’s Theorem, \( h \equiv 0 \). Q.E.D.
Proof of Theorem 1: To derive Cauchy’s theorem, let $z_0 \in \Omega - \gamma$ and put

$$F(z) = (z - z_0)f(z).$$

By (1),

$$\frac{1}{2\pi i} \int_\gamma f(z) \, dz = \frac{1}{2\pi i} \int_\gamma \frac{F(z)}{z - z_0} \, dz$$

$$= n(\gamma, z_0)F(z_0)$$

$$= 0.$$

Q.E.D.

Note finally that Corollary 2 on p.142 is an immediately consequence of Cauchy’s Theorem.