Massachusetts Institute of Technology
Department of Physics

Course: 8.09 Classical Mechanics
Term: Fall 2004

Final Examination
December 17, 2004

Instructions

• Do not start until you are told to do so.
• Solve all problems.
• Put your name and recitation section number on the covers of all notebooks you are using.
• Show all work neatly in the blue book, label the problem you are working on.
• Mark the final answers.
• Books and notes are not to be used. Calculators are unnecessary.
Useful Formulae

Gravitational Law:
\[ \vec{F} = -\frac{G m_1 m_2}{r_{12}^2} \hat{r}_{12} \]

Lagrangian and Hamiltonian:
\[ L(q, \dot{q}) = T - U; \quad p = \frac{\partial L}{\partial \dot{q}} \quad H(p, q) = p \frac{\partial q}{\partial t} - L \]

Hamilton Equation of Motion:
\[ \frac{\partial H}{\partial q} = -\dot{p}; \quad \frac{\partial H}{\partial p} = \dot{q} \]

Poisson Brackets:
\[ [g, f] = \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} - \frac{\partial g}{\partial p} \frac{\partial f}{\partial q} \]

Euler-Lagrange (without and with constraints):
\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0; \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial g}{\partial x} = 0 \]

Polar Coordinates:
\[ x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta \]

Coordinate system rotating with angular velocity \( \vec{\omega} \):
\[ \left( \frac{d\vec{a}}{dt} \right)_{\text{fixed}} = \left( \frac{d\vec{a}}{dt} \right)_{\text{rotating}} + \vec{\omega} \times \vec{a}; \]

Orbit Equation:
\[ u'' + u = -\frac{\mu}{\ell^2 u^2} F(u) \quad \text{with} \quad u = \frac{1}{r} \]

Effective Potential:
\[ V(r) = U(r) + \frac{\ell^2}{2\mu r^2} \]

Keplerian Orbits:
\[ U(r) = -\frac{k}{r}; \quad \frac{\alpha}{r} = \varepsilon \cos \theta + 1 \]
\[ \alpha = \frac{\ell^2}{\mu k}; \quad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}; \quad \tau = \frac{4\pi^2 \mu}{k} a^3 \]
\[ r_{\text{min}} = a(1 - \varepsilon) = \frac{\alpha}{1 + \varepsilon}; \quad r_{\text{max}} = a(1 + \varepsilon) = \frac{\alpha}{1 - \varepsilon} \]
Polar Coordinates:
\[ x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta \]

Inelastic Scattering (coefficient of restitution):
\[ \epsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|} \]

Scattering:
\[ \sigma(\theta) = \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \frac{db}{d\theta} \]
\[ \tan \psi = \frac{\sin \theta}{\cos \theta + (m_1/m_2)}; \quad \zeta = \frac{\pi}{2} - \frac{\theta}{2}; \]

Acceleration in accelerated frame:
\[ \vec{a} = \vec{g} - \vec{V} - (\vec{\omega} \times \vec{r}) - 2(\vec{\omega} \times \vec{v}) - (\vec{\omega} \times (\vec{\omega} \times \vec{r})) ; \]

Electromagnetic Fields
\[ \vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}; \]
\[ \vec{B} = \vec{\nabla} \times \vec{A}; \]

Summation Convention
\[ \epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}; \]
\[ (\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k; \]

Lagrangian Density
\[ L(\eta, \eta', \dot{\eta}, x, t); \]
\[ \frac{\partial L}{\partial \dot{\eta}} - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \eta'} \right) - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\eta}} \right) = 0; \]
Problem 1: Pendulum (15 points)

Consider a simple plane pendulum consisting of mass $m$ attached to a string of length $\ell$. After the pendulum is set into motion the length of the string is being shortened such that its length $\ell(t)$ is time-dependent where $\ell(t) \leq 0$. The suspension point remains fixed.

a) Write the Lagrangian function $L(\theta, \dot{\theta}, t)$ and find the Euler-Lagrange equations of motion.

b) Write the Hamiltonian function $H(\theta, p_\theta, t)$ and find the Hamilton equations of motion.

c) Write the mechanical energy of the system. Is the energy conserved? Is Hamiltonian equal to the total mechanical energy of the pendulum?
Problem 2: Two masses (20 points)

Consider masses $m_1$ and $m_2$ connected with a string of fixed length $L$. Mass $m_1$ can move without friction on a horizontal table. Mass $m_2$ is hanging from the string below the table (see Figure). Assume that the mass $m_2$ is constrained to move along vertical direction only. The gravitational acceleration is $g$.

![Diagram of two masses connected by a string]

a) Write the Langrangian function for the system as a function of $r$, $\phi$ and $z$.

b) Write the constraint function that indicates that the two masses are connected by a fixed length string.

c) Write Euler-Lagrange equations of motion for each of the generalized coordinates. Include the effect of the constraint by introducing Lagrange multipliers.

d) What is the physical interpretation of the forces corresponding to the Lagrange multipliers?

e) Describe the motion of the system such that the mass $m_2$ remains stationary. Solve the equations of motion for the mass $m_1$ in that case.
Problem 3: Scattering (15 points)

Consider the scattering of very small particles of mass $m$ from a fixed hard sphere of radius $R$. The collisions are elastic in the direction normal to the surface but the surface of the sphere is so rough such that the component of the velocity tangential to the surface of the sphere after the collision is zero. The energy of the incoming particles is $E_0$.

\[ \begin{array}{c}
\theta \\
\hline
\end{array} \]

b) What is the differential cross section of the scattering from this sphere $\sigma(\theta)$?

c) What is the energy of the scattered particles as a function of the scattering angle $E(\theta)$?
Problem 4: Rolling Cylinder (15 points)

Consider a solid cylinder of mass $m$, radius $R$ and with non-uniform mass density distribution. The center of mass of the cylinder is displaced by a distance $a$ from the geometrical symmetry axis (see Figure). The rotational inertia about the axis that includes the center of mass and that is parallel to the geometrical symmetry axis is $I_0$. The cylinder can roll without slipping on a horizontal surface. Use angle $\theta$ as a generalized coordinate. The gravitational acceleration is $g$.

![Diagram of a cylinder with center of mass displaced by a distance a from the geometrical symmetry axis]

a) What is the gravitational potential energy of the cylinder?

b) What is the kinetic energy of the cylinder?

c) Write Lagrangian for the system and the equations of motion for the cylinder.
Problem 5: Hamiltonian in rotating coordinate system (20 points)

A particle of mass $m$ is moving in a spherically symmetrical potential $U(r)$. Consider
the motion of the particle as observed from a coordinate system that has a center
at $r = 0$ and it is rotating with respect to the inertial system at a constant angular
velocity $\vec{\Omega}$. Note: Use vector algebra as much as possible. For the equations of motion
use cartesian coordinate system. Do not use spherical coordinate system.

a) Rewrite the Lagrangian in terms of coordinates and velocities measured in the
rotating system.

b) Calculate the Hamiltonian. Is the Hamiltonian a constant of motion?

c) Write the Hamilton equations of motion for the particle in the rotating system.

d) The expression for Hamiltonian includes the standard formula $\frac{mv^2}{2} + U$ and an
additional term. Can you give a physical interpretation of that term?
Problem 6: Electromagnetic Lagrangian Density (15 points)

The Lagrangian density that describes electromagnetic fields and their interaction with the charges and currents contains information about Maxwell equations. Use the Lagrangian density below to show that the Euler-Lagrange equations correspond to Gauss and Ampere’s Law. The density of charges is \( \rho \) and the current is \( \vec{j} \). Electromagnetic field potentials are \( \phi \) and \( \vec{A} \).

The Lagrangian density is:

\[
L = L_{\text{particles}} + L_{\text{interaction}} + L_{\text{EM}}
\]

where

\[
L_{\text{particles}} = -mc^2 \sqrt{1 - \frac{1}{c^2e^2\eta^2j^2}}
\]

\[
L_{\text{interaction}} = -\rho \phi + \frac{1}{c} \vec{j} \cdot \vec{A}
\]

and

\[
L_{\text{EM}} = \frac{1}{8\pi} \left[ \left( \frac{\partial \phi}{c} + \frac{1}{c} \frac{\partial A_i}{\partial t} \right)^2 - (\partial_j A_k) (\partial_j A_k) + (\partial_j A_k) (\partial_k A_j) \right]
\]

a) Use electric potential \( \phi \) as a field variable and derive Gauss’ Law

\[
\vec{n} \cdot \vec{E} = 4\pi\rho
\]

b) Use components of the vector potential field \( A_i \) as field variables to derive Ampere’s Law:

\[
\vec{n} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\]

Hint: You may want to make use of the formula: \( \vec{n} \times (\vec{n} \times \vec{a}) = \vec{n}(\vec{n} \cdot \vec{a}) - \nabla^2 \vec{a} \)