Massachusetts Institute of Technology
Department of Physics

Course: 8.09 Classical Mechanics
Term: Fall 2003

Quiz 2
November 19, 2000

Instructions

• Do not start until you are told to do so.
• Solve all problems.
• Put your name and recitation section number on the covers of all notebooks you are using.
• Show all work neatly in the blue book, label the problem you are working on.
• Mark the final answers.
• Books and notes are not to be used. Calculators are unnecessary.
Useful Formulae

Newton and Basic Kinematics:
\[
\vec{F} = m\ddot{\vec{r}} = m\ddot{\vec{a}} \quad \text{for} \quad v \ll c \\
\vec{v} = \vec{v}_0 + \int_{t'=0}^{t} dt' \vec{a} \\
\vec{r} = \vec{r}_0 + \vec{v}_0 t + \int_{t'=0}^{t} \int_{t''=0}^{t'} dt'' \vec{F}(t'')/m
\]

Gravitational Law:
\[
\vec{F} = -\frac{Gm_1m_2}{r_{12}^2} \hat{r}_{12}
\]

Lagrangian and Hamiltonian:
\[
L(q, \dot{q}) = T - U; \quad H(p, q) = T + U = p \frac{\partial q}{\partial t} - L
\]

Hamilton Equation of Motion:
\[
\frac{\partial H}{\partial \dot{q}} = -\dot{p}; \quad \frac{\partial H}{\partial p} = \dot{q}
\]

Euler-Lagrange (without and with constraints):
\[
\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0; \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial q}{\partial x} = 0
\]

Polar Coordinates:
\[
x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta
\]

Orbit Equation:
\[
u'' + u = -\frac{\mu}{\ell^2 u^2} F(u) \quad \text{with} \quad u = \frac{1}{r}
\]

Effective Potential:
\[
V(r) = U(r) + \frac{\ell^2}{2\mu r^2}
\]

Keplerian Orbits:
\[
U(r) = -\frac{k}{r}; \quad \frac{\alpha}{r} = \varepsilon \cos \theta + 1 \\
\alpha = \frac{\ell^2}{\mu k}; \quad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}; \quad \tau^2 = \frac{4\pi^2\mu}{k} a^3
\]
\[
r_{\min} = a(1 - \varepsilon) = \frac{\alpha}{1 + \varepsilon}; \quad r_{\max} = a(1 + \varepsilon) = \frac{\alpha}{1 - \varepsilon}
\]
Polar Coordinates:

\[ x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta \]

Inelastic Scattering (coefficient of restitution):

\[ \epsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|} \]

Scattering:

\[ \sigma(\theta) = \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| \]

\[ \tan \psi = \frac{\sin \theta}{\cos \theta + (m_1/m_2)}; \quad \zeta = \frac{\pi}{2} - \frac{\theta}{2} \]

Acceleration in accelerated frame:

\[ \ddot{\alpha} = \ddot{g} - \dot{V} - (\ddot{\omega} \times \ddot{r}) - 2(\ddot{\omega} \times \dot{r}) - (\ddot{\omega} \times (\ddot{\omega} \times r)) \]
Problem 1: Orbit (25 points)

A particle of mass, $m$, moves under the influence of an effective potential, $V(r)$, given by:

$$V(r) = \frac{l^2}{2mr^2} + \frac{a}{3r^3} + \frac{b^2}{4r^4},$$

where $a$ and $b$ are real numbers.

a) Calculate the force $\vec{F}(r)$ which corresponds to the effective potential (do not include fictitious forces).

b) Investigate for which exact range of values of $a$ and $b$, stable orbits are possible. Therefore first consider the general behavior of the effective potential for very small and very large values of $r$. Then identify the total of six different cases for values of $a$ and $b$ and give the position of the stable orbit if there is a stable orbit.

c) Sketch the qualitative behaviour for three representative cases from part b). Make sure you indicate the behavior for small and large distances $r$ properly. Three separate diagrams, please.

d) Choose a case from part c), where stable orbits are possible. Draw a separate diagram with the effective potential. Choose a total energy for the motion which allows bound orbits which are not stable. Indicate in the sketch your chosen total energy, the point of closest approach and the furthest point. Finally indicate the point where the motion has the largest kinetic energy and indicate the size of the kinetic energy in the diagram.
Problem 2: Scattering by an Ideal Lens (25 points)

Consider the path of light rays that are bent by an optical lens of diameter $D = 2R$, where $R$ is defined in the sketch. As is well known from geometrical optics, rays of light that enter the lens parallel to the optical axis are deviated in such a way that they all pass through a common focal point at a distance $f$ from the lens. The Figure 1 illustrates the path of a typical light ray as it passes through the lens. We are going to analyze this situation as if it were a classical mechanics scattering problem.

![Figure 1: “Scattering” by a lens.](image)

**a)** The scattering angle $\theta$ of the light is the angle between the original direction of the light and the light after it has passed through the lens. Using the figure, derive a relation between the scattering angle $\theta$ and the impact parameter $b$.

**b)** What are the maximum and minimum scattering angles, $\theta_{\text{min}}$ and $\theta_{\text{max}}$, and what values of $b$ do they correspond to? Also express $\cos \theta_{\text{max}}$ in terms of $f$ and $R$; this will be needed later in the integration!

**c)** Based on your result from part a) derive the differential cross section $d\sigma(\theta)/d\Omega$.

**d)** Without explicitly doing any calculation, what do you expect the total cross section to be? Briefly explain your answer.

**e)** Verify your result by directly calculating the total scattering cross section: For the integration you might find it useful to replace $\cos \theta$ with $x$. When setting up the integration limits you can refer to your result in part b).
Problem 3: Inelastic Collisions (20 points)

A steal ball is dropped from a height, \( h_0 \), onto a flat and heavy steal plate. The collision is inelastic and has a coefficient of restitution, \( \epsilon \).

a) When you do the following calculations you have to make one important assumption. What is this assumption?

b) Calculate the maximum height, \( h_1 \), the steal ball reaches after the first bounce. Express the result in terms of the initial height, \( h_0 \), and the coefficient of restitution, \( \epsilon \).

c) Calculate the time, \( t_1 \), it takes the ball to bounce back to height \( h_1 \). To be precise: The time starts when the ball hits the steal plate for the first time and stops when is reaches height \( h_1 \).

d) Will the ball ever stop bouncing? If yes, what is the time it takes for the ball to stop and how far has the ball traveled in the meanwhile?

\[ \sum_{i=0}^{n-1} x^i = \frac{1 - x^n}{1 - x} \]

*Hint: A series is a sequence of numbers. The sum of a geometric series is given by:*
Problem 4:  Non-Inertial Frames (30 points)

A projectile is fired due east from a point on the Earth surface on the northern hemisphere with latitude, $\lambda$. The initial velocity has a magnitude of $v_0$ and an angle of $\alpha$ with respect to the horizontal. The Earth rotation frequency is given by $\omega$.

In this problem we are going to derive the change of range of the projectile due to the Earth rotation. The curvature of the Earth can be neglected in this problem. Follow the steps below.

a) Find the range, $R$ of the projectile assuming the Earth does not rotate ($\omega = 0$).

b) Write the three vector components of the two relevant terms of the acceleration in this problem (gravitation and Coriolis force). Therefore choose your coordinate system as follows: The $z$ axis points upwards with respect to the Earth’s surface; the projectile is fired in direction east which is the positive $x$ direction.

c) From the acceleration in part b) derive the changes in range due to the acceleration in $x$ direction.

d) From the acceleration in part b) derive the change in $z$, $\Delta z$, due to the acceleration in $z$ direction. Convert $\Delta z$ into a change in the $x$ direction. Therefore assume that the trajectory is symmetric.

e) Add both components to obtain the change in range:

$$\Delta R = \sqrt{\frac{2R^3}{g}} \omega \cos \lambda (\sqrt{\cot \alpha} - \frac{1}{3} \sqrt{\tan^3 \alpha})$$