Problem 4.1

As warmup, consider first a simpler system with only one pulley and two masses, \( m_A \) and \( m_B \).

This system has one degree of freedom (we choose \( z \))

\[
L(z, \dot{z}) = T - U
\]

\( T = T_A + T_B \)

\( T_A = \frac{1}{2} m_A \dot{z}^2 \)

\( T_B = \frac{1}{2} m_B \dot{z}^2 \)

\( U = U_A + U_B \)

\( U_A = m_A g \cdot h_A = m_A q \cdot (-z) = -m_A q \dot{z} \)

\( U_B = m_B g \cdot h_B = m_B q \cdot (-2\pi R + z) = m_B q \dot{z} \) plus some constant that is unimportant (so we drop it)

\[ L = \frac{1}{2}(m_A + m_B) \dot{z}^2 + (m_A - m_B)q \dot{z}, \]

Now construct Hamiltonian from Lagrangian

\[ H = \frac{1}{2} p^2 + L = \dot{p} \dot{z} - L \dot{z} \]

\[ p = \frac{\partial L}{\partial \dot{z}} = (m_A + m_B) \dot{z} \]

\[ H(p, z) = (m_A + m_B) \ddot{z} - L = \frac{1}{2}(m_A + m_B) \dot{z}^2 - (m_A - m_B)q \dot{z} = T + U \]

\[ = \frac{p^2}{2(m_A + m_B)} - (m_A - m_B)q \dot{z} \]

Hamilton equations of motion:

\[ p = -\frac{\partial H}{\partial z} = (m_A - m_B)q \dot{z} \Rightarrow p = (m_A - m_B)q \dot{z} \text{ (assuming starting from rest)} \]

\[ \dot{z} = \frac{\partial H}{\partial p} = \frac{p}{m_A + m_B} \Rightarrow \dot{z} = \frac{1}{2} q \left( \frac{m_A - m_B}{m_A + m_B} \right) \dot{z}^2 \]

Now, let's look at the center-of-mass: it falls as

\[ m_A \ddot{z} - m_B \ddot{z} = \frac{1}{2} q \left( \frac{m_A - m_B}{m_A + m_B} \right) \dot{z}^2 \]

Since gravity acts with a force \((m_A + m_B)q \dot{z}\), we can determine what the force of constraint is that keeps the pulley in its place.
4.1. Cont'd.

\[ \text{Force} = F_{\text{grav}} - F_{\text{total}} = (m_A + m_B)q' - q' \left( \frac{(m_A - m_B)^2}{m_A + m_B} \right) = q' \frac{4m_A m_B}{m_A + m_B}. \]

In other words, if you would try to keep the pulley in its place you would feel a mass of \( \frac{4m_A m_B}{m_A + m_B} = \mu_{AB} \) with \( \mu_{AB} \) the reduced mass.

Note that if we had worked with Lagrange undetermined multipliers, we would have obtained the force of constraint directly from the equations of motion.

Now for the actual system of this problem, with two degrees of freedom (x and y).

Next, construct the Hamiltonian

\[ H = \sum_{i} p_i q_i - L = p_x x + p_y y - L \]

with

\[ p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2 + m_3) \dot{x} + (m_3 - m_2) \dot{y} \]

\[ p_y = \frac{\partial L}{\partial \dot{y}} = (m_3 - m_2) \dot{x} + (m_2 + m_3) \dot{y} \]
Notice how we can write \((p_x, p_y) = M(\dot{x}, \dot{y})\)

where \(M\) is a 2x2 (symmetric) matrix

\[
M = \begin{pmatrix} m_1 + m_2 + m_3 & m_3 - m_2 \\ m_3 - m_2 & m_2 + m_3 \end{pmatrix}
\]

so

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = M^{-1} \begin{pmatrix} p_x \\ p_y \end{pmatrix}
\]

and for a 2x2 matrix \((a \ b) \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\text{det}}(d \ -b)
\]

\[
\text{det} = ad - bc
\]

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{\text{det} M} \begin{pmatrix} m_2 + m_3 & m_2 - m_3 \\ m_3 - m_2 & m_2 + m_3 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}
\]

\[
\text{det} M = (m_1 + m_2 + m_3)(m_2 + m_3) - (m_2 - m_3)^2 = m_1 (m_2 + m_3) + m_2 m_3
\]

\[
H = [(m_1 + m_2 + m_3) x + (m_3 - m_2) y] \dot{x} + [(m_3 - m_2) x + (m_2 + m_3) y] \dot{y} = T + U
\]

\[
= \frac{1}{2} \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} m_2 + m_3 & m_2 - m_3 \\ m_3 - m_2 & m_2 + m_3 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + q [(m_2 + m_3 - m_3)x + (m_3 - m_2) y]
\]

\[
= \frac{1}{\text{det} M} \frac{1}{2} \begin{pmatrix} p_x & p_y \end{pmatrix} \begin{pmatrix} m_2 + m_3 & m_2 - m_3 \\ m_3 - m_2 & m_2 + m_3 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + q [(m_2 + m_3 - m_3)x + (m_3 - m_2) y]
\]

...and we leave the Hamiltonian in this matrix-style notation.

Hamilton equations of motion:

\[
p_x = -\frac{\partial H}{\partial x} = q (m_1 - (m_2 + m_3)) = q (m_1 - (m_2 + m_3)) t
\]

where we used that we started from rest

\[
p_y = -\frac{\partial H}{\partial y} = q (m_2 - m_3) = q (m_2 - m_3) t
\]

at \(t = 0\).

\[
\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{\text{det} M} \left[ (m_2 + m_3) p_x + (m_2 - m_3) p_y \right]
\]

\[
\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{\text{det} M} \left[ (m_2 - m_3) p_x + (m_1 + m_2 + m_3) p_y \right]
\]
4.1. cont'd

In matrix-style notation:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \frac{1}{\det M} \begin{pmatrix}
w_2 + w_3 & w_2 - w_3 \\
w_2 - w_3 & w_1 + w_2 + w_3
\end{pmatrix} \begin{pmatrix}
m_1 - (w_2 + w_3) \\
w_2 - w_3
\end{pmatrix} \frac{1}{2} \dot{q}^2
\]

which we can easily integrate.

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \frac{1}{\det M} \begin{pmatrix}
w_2 + w_3 & w_2 - w_3 \\
w_2 - w_3 & w_1 + w_2 + w_3
\end{pmatrix} \begin{pmatrix}
m_1 - (w_2 + w_3) \\
w_2 - w_3
\end{pmatrix} \frac{1}{2} \dot{q}^2
\]

\[
x = \frac{1}{\det M} \left[ (w_2 + w_3)(m_1 - (w_2 + w_3)) + (w_2 - w_3)^2 \right] \frac{1}{2} \dot{q}^2
\]

\[
y = \frac{1}{\det M} \left[ (w_2 - w_3)(m_1 - (w_2 + w_3)) + (m_1 + w_2 + w_3)(w_2 - w_3) \right] \frac{1}{2} \dot{q}^2
\]

\[
x = \frac{1}{2} \frac{m_1 (w_2 + w_3) - 4 m_2 m_3}{m_1 (w_2 + w_3) + 4 m_2 m_3} \dot{q}^2 = \frac{1}{2} \frac{m_1 - 4 m_2 m_3}{m_1 + 4 m_2 m_3} \dot{q}^2
\]

\[
y = \frac{1}{2} \frac{2 m_1 (w_2 - w_3)}{m_1 (w_2 + w_3) + 4 m_2 m_3} \dot{q}^2 = \frac{1}{2} \frac{2 m_1}{m_1 + 4 m_2 m_3} \left( \frac{w_2 - w_3}{w_2 + w_3} \right) \dot{q}^2
\]

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \frac{1}{2} \left( \frac{2 m_1}{m_1 + 4 m_2 m_3} \right) \dot{q} \left( \frac{w_2 - w_3}{w_2 + w_3} \right) \dot{q}^2
\]

\[
\frac{q'}{q'} = \frac{2 m_1}{m_1 + 4 m_2 m_3} = q + \frac{m_1 - 4 m_2 m_3}{m_1 + 4 m_2 m_3} q = q \left( 1 + \frac{m_1 - 4 m_2 m_3}{m_1 + 4 m_2 m_3} \right)
\]

Comparing with the single pulley system, we see that for

\[
m_1, x : \text{ behaves as if } m_A = m_1, \quad m_B = 4 m_2 m_3, \quad q' = q
\]

\[
m_2, m_3, 3 : \text{ behaves as if } m_A = m_2, \quad m_B = m_3 \quad \text{but with acceleration } q' = q + \frac{m_1 - 4 m_2 m_3}{m_1 + 4 m_2 m_3} q
\]
Problem 4.2:

We are given \( H(p,q) = H(p_x, p_y, x, y) = \frac{p_x^2 + p_y^2}{2m} + \frac{mw^2}{2}(x^2 + y^2) \)

and for this Hamiltonian we have the equations of motion

\[
\begin{align*}
\dot{p}_x &= -\frac{\partial H}{\partial x} \quad \dot{x} = \frac{\partial H}{\partial p_x} \\
\dot{p}_y &= -\frac{\partial H}{\partial y} \quad \dot{y} = \frac{\partial H}{\partial p_y}
\end{align*}
\]

Now we need to show that for the new variables \( P, Q \) these new equations of motion with respect to the new Hamiltonian \( H'(P, Q) \) give us the same physics.

So we need to determine \( H'(P, Q) = H'(P_x, P_y, X, Y) \)

and show that

\[
\begin{align*}
\dot{P}_x &= -\frac{\partial H'}{\partial X} \quad \dot{X} = \frac{\partial H'}{\partial P_x} \quad \text{are equivalent with the original equations of motion.} \\
\dot{P}_y &= -\frac{\partial H'}{\partial Y} \quad \dot{Y} = \frac{\partial H'}{\partial P_y}
\end{align*}
\]

First, write \( P_x, P_y, X \) and \( Y \) in terms of \( p_x, p_y, x \) and \( y \) by elimination/substitution to find:

\[
\begin{align*}
P_x &= \omega \lambda p_x + mw \sin \lambda y \\
P_y &= \omega \lambda p_y + mw \sin \lambda x \\
X &= \omega \lambda x - \frac{\sin \lambda}{mw} P_y \\
Y &= \omega \lambda y - \frac{\sin \lambda}{mw} p_x
\end{align*}
\]
Next, we calculate \( H'(P_x, P_y, X, Y) \) from \( H(p_x, p_y, x, y) \):

\[
H'(P_x, P_y, X, Y) = \frac{(-mw Y \sin \lambda + P_x \cos \lambda)^2 + (-mw X \sin \lambda + P_y \cos \lambda)^2}{2m} + \frac{mw^2}{2} \left( (X \omega \lambda + \frac{P_y}{mw} \sin \lambda)^2 + (Y \omega \lambda + \frac{P_x}{mw} \sin \lambda)^2 \right)
\]

= after a lot of cancellations

\[
= \frac{P_x^2 + P_y^2}{2} + \frac{mw^2}{2} \left[ X^2 + Y^2 \right]
\]

Note that the new Hamiltonian \( H' \) has the exact same form as the old \( H \).

The old equations of motion were:

\[
\begin{align*}
\dot{p}_x &= -\frac{\partial H}{\partial x} = -mw^2x \quad & \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -mw^2y \quad & \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m}
\end{align*}
\]

The new equations of motion are

\[
\begin{align*}
\dot{P}_x &= -\frac{\partial H'}{\partial X} = -mw^2X \quad & \dot{X} &= \frac{\partial H'}{\partial P_x} = \frac{P_x}{m} \\
\dot{P}_y &= -\frac{\partial H'}{\partial Y} = -mw^2Y \quad & \dot{Y} &= \frac{\partial H'}{\partial P_y} = \frac{P_y}{m}
\end{align*}
\]

The new equations certainly look the same as the old ones, but in order to see if they correspond to the same physics we need to write the new equations in terms of the old variables.
4.2. continued

\[ \dot{P}_x = -m w^2 X \text{ in old variables:} \]

L.H.S.: \[\dot{P}_x = \cos \lambda \dot{p}_x + m w \sin \lambda \dot{q} \]

R.H.S.: \[-m w^2 X = -m w^2 \cos \lambda x + m w \sin \lambda p_y \]

If we now substitute the old equations of motion then the new equations describe the same physics as the old equations if the left hand side is equal to the right hand side.

(i.e., the question we are answering here is: Are the new equations true given that the old ones are true).

\[ \dot{P}_x = \cos \lambda p_x + m w \sin \lambda \dot{q} \Rightarrow -m w^2 X = -m w^2 \cos \lambda x + m w \sin \lambda p_y \]

\[ = \cos \lambda (-m w^2 x) + m w \sin \lambda \left( \frac{P_y}{m} \right) \]

\[ \Rightarrow \]

\[ \dot{P}_y = \cos \lambda \dot{p}_y + m w \sin \lambda x \Rightarrow -m w^2 Y = -m w^2 \cos \lambda y + m w \sin \lambda p_x \]

\[ = \cos \lambda (-m w^2 y) + m w \sin \lambda \left( \frac{P_x}{m} \right) \]

\[ \Rightarrow \]

\[ X = \cos \lambda x - \frac{\sin \lambda}{m w} \dot{P}_y \Rightarrow \frac{\dot{P}_x}{m} = \frac{\cos \lambda p_x + m w \sin \lambda q}{m} \]

\[ = \cos \lambda \frac{P_x}{m} - \frac{\sin \lambda}{m w} (-m w^2 q) \]

\[ \Rightarrow \]

\[ Y = \cos \lambda \dot{q} - \frac{\sin \lambda}{m w} \dot{p}_x \Rightarrow \frac{\dot{P}_y}{m} = \frac{\cos \lambda p_y + m w \sin \lambda x}{m} \]

\[ = \cos \lambda \frac{P_y}{m} - \frac{\sin \lambda}{m w} (-m w^2 x) \]

So, yes, all new equations of motion describe the same physics as the old ones.
4.2. cont'd

The shortcut to this problem comes through Poisson brackets.

We know that \([x, p_x] = 1 = [y, p_y]\) and all other brackets are zero.

The transformation to \(x, p_x, y, p_y\) is a canonical transformation if we have \(\{q_i, p_j\} = \delta_{ij}\) and all other Poisson brackets are zero.

(Read Goldstein Chapter 9 for the proof and more background information on the so-called symplectic structure of phase-space.)

For this problem it means showing that \([x, p_x] = 1 = [y, p_y]\)

and \([x, y] = [x, p_y] = [y, p_x] = [p_y, p_x] = 0\).

\[
[x, p_x] = \cos^2 \lambda \left[ x, p_x \right] - \sin^2 \lambda \left[ p_y, y \right] + \cos \lambda \sin \lambda \left( mw \left[ x, y \right] - \frac{1}{mw} \left[ p_y, p_x \right] \right) \\
= \cos^2 \lambda (1) - \sin^2 \lambda (-1) + \cos \lambda \sin \lambda \left( mw (0) - \frac{1}{mw} (0) \right) \\
= 1 \checkmark
\]

\[
[y, p_y] = \cos^2 \lambda \left[ y, p_y \right] - \sin^2 \lambda \left[ p_x, x \right] + 0 = 1 \checkmark
\]

\[
x, y = \cos^2 \lambda \left[ x, y \right] + \sin^2 \lambda \left[ p_y, p_x \right] + \cos \lambda \sin \lambda \left( \frac{[x, p_x]}{mw} + \frac{[p_y, y]}{mw} \right) \\
= 0 + 0 + \cos \lambda \sin \lambda \left( -1 + 1 \right) = 0 \checkmark
\]

\[
[p_y, p_x] = (\ldots) [x, x] + (\ldots) [x, p_y] + (\ldots) [p_y, p_y] + (\ldots) [p_y, x] = 0 + 0 + 0 = 0 \checkmark
\]

\[
[y, p_x] = (\ldots) [y, y] + (\ldots) [y, p_x] + (\ldots) [p_x, p_x] + (\ldots) [p_x, y] = 0 + 0 + 0 = 0 \checkmark
\]

\[
[p_y, p_x] = (\ldots) [p_y, p_x] + (\ldots) [x, y] + mw \cos \lambda \sin \lambda \left( [x, p_x] + [p_y, y] \right) = 0 + 0 + (\ldots) (1-1) = 0 \checkmark
\]
Problem 4.3

(a) \( E_{ijk} \) is the fully antisymmetric tensor with \( E_{123} = +1 \).

Any permutation of its indices yields a minus sign (i.e., if you exchange two indices you get minus what you had before).

\[
E_{ijk} = -E_{ikj} = E_{kij} = -E_{jki} = -E_{kji}
\]

Note that a so-called cyclic permutation yields a plus sign (or no minus sign in other words).

So \( E_{123} = E_{312} = E_{231} = +1 \), \( E_{132} = E_{321} = E_{213} = -E_{123} = -1 \).

All other of the \( 3! = 6 \) elements of \( E_{ijk} \) are zero, because they can be shown to be minus itself, e.g., \( E_{123} = E_{ijk} \) with \( i = 1 \), \( j = 2 \), \( k = 3 \) \( \implies -E_{123} \).

\[
E_{ijk} \, P_j \, P_k = -E_{ikj} \, P_j \, P_k = -E_{ijk} \, P_j \, P_k = -E_{ijk} \, P_j \, P_k = -E_{ijk} \, P_j \, P_k
\]

Since \( E_{ijk} \, P_j \, P_k = -E_{ijk} \, P_j \, P_k \)
we have to conclude that it is equal to zero.

Note that \( E_{ijk} \, P_j \, P_k = (\vec{p} \times \vec{p})_i \) is the \( i \)th component of the cross product \( \vec{p} \times \vec{p} \).
4.3 Cont'd

And \( \hat{p} \times \hat{p} = 0 \) for any vector \( \hat{p} \).

Last remark, \( \varepsilon_{ijk} p_j p_k = \sum_{j,k} (\varepsilon_{ijk})(P_j P_k) \)

you are contracting the indices of an object, \( \varepsilon_{ijk} \), that is antisymmetric in those contracted indices (\( j \) and \( k \)), with another object, \( p_j p_k \), that is symmetric with respect to those indices \( (P_j P_k = P_k P_j) \), and this will always give zero.

Let \( A_{ij} = \frac{1}{2} A_{ji} \) and \( S_{ij} = S_{ji} \)

then \( A_{ij} S_{ij} = 0 \)

(but if \( B_{ij} = -B_{ji} \) then \( A_{ij} B_{ij} \neq 0 \) in general

and if \( T_{ij} = T_{ji} \) then \( S_{ij} T_{ij} \neq 0 \) in general)

\[ b) \quad r = \sqrt{r_i^2 + r_j^2 + r_s^2} = \sqrt{r_i^2 r_j^2} \quad \text{note that we cannot, strictly, write this as } \sqrt{r_i^2}, \text{ since the summation is only implied when the same index appears twice.} \]

using \( \frac{\partial}{\partial r_i} r = \delta_{k,i} \), with \( \delta \) the Kronecker-delta.

Also we have \( \frac{\partial}{\partial r_i} \frac{1}{r} = \frac{\partial}{\partial r_i} \frac{1}{\sqrt{r_j r_s}} = -\frac{\partial}{\partial r_i} \left( \frac{1}{(r_j r_s)^{3/2}} \right) \frac{(r_j r_s)}{r_j} \)

yikes! the same index \( i \) now appears 4 times, which is not allowed!
4.3. Cont'd

We relabel one of the \( r_j \) to \( r_k \)

\[
-\frac{1}{2 \, (r_j r_j)^{1/2}} \frac{\partial}{\partial r_i} (r_j r_k) = - \frac{r_k}{(r_j r_j)^{1/2}} \frac{\partial}{\partial r_i} \frac{r_k}{(r_j r_j)^{3/2}} = - \frac{r_k}{(r_j r_j)^{3/2}} \delta_{k,i} = - \frac{r_k}{r^3} = - \frac{r_i}{r^3}
\]

(c) Poisson bracket \([f, g] = \frac{\partial f}{\partial r_i} \frac{\partial g}{\partial \varrho_i} - \frac{\partial f}{\partial \varrho_i} \frac{\partial g}{\partial r_i}\)

(we again have the same index repeated four times, however in each term in the sum the index appears only twice, and the "no more than twice repeated index" rule applies to single terms only).

\[
\frac{\partial r_i}{\partial r_j} = \delta_{k,j} \quad \frac{\partial r_i}{\partial \varrho_j} = 0 \quad \frac{\partial \varrho_i}{\partial r_j} = 0 \quad \frac{\partial \varrho_i}{\partial \varrho_j} = \delta_{k,j}
\]

\[
\left[ r_i, r_j \right] = \left( \frac{\partial}{\partial r_i} \right)^{(r)} \frac{\partial r_j}{\partial \varrho_i} - \left( \frac{\partial}{\partial \varrho_i} \right)^{(r)} \frac{\partial r_j}{\partial r_i} = - \delta_{i,j} \cdot \frac{1}{\varrho_i} \frac{1}{r^3} = 0
\]

\[
\left[ r_i, \varrho_j \right] = \left( \frac{\partial}{\partial r_i} \right)^{(r)} \frac{\partial \varrho_j}{\partial \varrho_i} - \left( \frac{\partial}{\partial \varrho_i} \right)^{(r)} \frac{\partial \varrho_j}{\partial r_i} = \delta_{i,j} \left( \frac{1}{\varrho_i} \right) = \delta_{i,j} \cdot - \frac{\varrho_i}{r^3} = - \frac{r_i}{r^3} \quad \text{using result from } \textbf{b)}
\]

(d) \( \vec{M} = \vec{r} \times \vec{p} \), so \( (\vec{M})_i = (\vec{r} \times \vec{p})_i = E_{ijk} r_j \, p_k \)
\( (\vec{M})_c = \vec{M}_i = \varepsilon_{ijk} r_j P_k \)

\[ H = \frac{P_r^2}{2m} + U(r) = \frac{P_r P_k}{2m} + U(\sqrt{r^2 + (r e)^2}) \]

\[ [\vec{M}_i, \vec{H}] = \left[ \varepsilon_{ijk} r_j P_k, \frac{P_m P_n}{2m} + U(\sqrt{r^2 + (r e)^2}) \right] \]

\[ = \varepsilon_{ijk} \left\{ \frac{\partial}{\partial r_s} \left( r_j P_k \right) \right\} \left[ \frac{\partial}{\partial P_s} \left( \frac{P_m P_n}{2m} + U(\sqrt{r^2 + (r e)^2}) \right) \right] - \left( \frac{\partial}{\partial P_s} \left( r_j P_k \right) \right) \left[ \frac{\partial}{\partial r_s} \left( \frac{P_m P_n}{2m} + U(\sqrt{r^2 + (r e)^2}) \right) \right] \]

\[ = \varepsilon_{ijk} \left\{ \left( P_k \delta_{j,s} \right) \left( \frac{P_m}{m} \delta_{s,m} \right) - \left( r_j \delta_{k,s} \right) \left( \frac{\partial U(r)}{\partial r} \frac{dr}{dr_s} \right) \right\} \]

\[ = \varepsilon_{ijk} \left\{ \frac{P_k \delta_{j,s} P_s}{m} - r_j \delta_{k,s} \left( \frac{\partial U(r)}{\partial r} \right) \frac{r_s}{r} \right\} \text{ using } \frac{dr}{dr_s} = \frac{\sqrt{r^2 + (r e)^2}}{r} = \frac{r_s}{r} \]

\[ = \varepsilon_{ijk} \frac{P_k P_j}{m} - \varepsilon_{ijk} r_j r_k \frac{1}{r} \left( \frac{\partial U(r)}{\partial r} \right) \]

\[ = 0 - 0 \]

Since \( \varepsilon_{ijk} P_k P_j = 0 \) as in part (d).

And \( \varepsilon_{ijk} r_j r_k = 0 \)
Problem 4.4:

(a) A free, non-interacting, particle has Hamiltonian $H = H(p, q) = \frac{p^2}{2m}$.

\[
\frac{\partial H}{\partial q} = 0 \Rightarrow p = \text{const} = p_0 \quad \frac{\partial H}{\partial p} = \frac{E}{m} \Rightarrow q = \frac{p_0 t + q_0}{m}.
\]

\[
\begin{array}{c}
\text{PP} \\
\frac{q_0}{p_0} \quad \overset{q=x}{\rightarrow} \quad \frac{p_0}{q_0} \\
\end{array}
\]

: is the path in phase space of a free particle starting at $(p, q) = (p_0, q_0)$ at $t = 0$.

\[
\begin{array}{c}
\text{PP} \\
\frac{-p_0}{q_0} \quad q \rightarrow \quad \overset{q=x}{\rightarrow} \\
\end{array}
\]

: is the path when starting at $t = 0$ at $(p, q) = (-p_0, q_0)$ (with $p_0 > 0$).

Now let $t_0 = 0$, $t_1$ same time $t_1 > t_0$, $t_2$ same time $t_2 > t_1$.

\[
\begin{array}{c}
\frac{t_0}{p_0} \quad t_1 \quad \ldots \quad t_2 \\
\frac{q_0}{q_0} \\
\end{array}
\]

Now look at a particle, indicated by $\square$, that started with the same momentum $p_0$, but a slightly different $q_0$.

\[
\begin{array}{c}
\frac{t_0}{p_0} \quad t_1 \quad t_2 \\
\frac{q_0}{q_1} \quad q_0 \\
\end{array}
\]
Next, look at a particle, \( \Delta \), that started at \( t = t_0 \) with \( q = q_0 \) but \( p = p_0' \) with \( p_0' \) slightly larger than \( p_0 \). Since this particle has a larger momentum, it will go faster than the first two.

Include * with initial values \((p, q) = (p_0', q_0')\), and connect the dots to figure out what happens to a group of particles.

Note that the volume (area) stays the same. \( \Delta x \Delta p = \text{constant with time} \)

(We will not bother to calculate a precise form of \( \Delta x(t) \) and \( \Delta p(t) \) since the picture shows \( \Delta x \Delta p = \text{const best.} \))

If we started at \( p = -p_0 \) it would look like:

If we started at \( p = -p_0 \), it would look like:
4.4. contd.

b) Now we add walls at \( q = x = 0 \) and \( q = x = L \).

A single particle will at some time \( t \) hit the wall and its momentum will instantly change sign.

For a group of particles, it is the same as in part a) except that when the group hits the wall the area will be "eaten" by the wall to reappear at negative momentum, with total area being conserved at all times.

Of course, the same process happens when the next wall is met, and you can imagine what the phase-space diagram looks like at the specific time where one of the fastest particles laps/catches up with the slowest ones.

c) Harmonic oscillator \( \rightarrow \) ellipses in phase-space

\[
\begin{align*}
\text{Remember: } & \quad H = \frac{p^2}{2m} + \frac{1}{2}mω^2q^2 \\
& \quad \dot{p} = \frac{\partial H}{\partial q} = -mω^2q \\
& \quad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \\
& \quad \ddot{q} = -ω^2q \\
& \quad q = A \cos(ωt + q₀) \\
& \quad p = mAω \sin(ωt + q₀)
\end{align*}
\]
A single harmonic oscillator will perform a clockwise (why?) ellipse.

Next, look at two particles, \( \circ \) and \( \square \), that oscillate in phase, but with a slightly different amplitude.

Finally, make the sketch for a whole group of particles.

Although the sketch already suggests that the area might be conserved, this is not immediately obvious, and thus we have to show it.

With \( q = q_0 \cos(\omega t + \varphi) \)
\[
p = -m \omega q_0 \sin(\omega t + \varphi)
\]

Area = \( \int \int dp dq \)

Switch from \( dp dq \) to \( dq \cdot dq \) (for fixed time \( t \))

\[
\left| \begin{array}{cc}
p & dq \\frac{\partial}{\partial q} & \end{array} \right| \int dq \cdot dq = \left| m \omega q_0 \right| dq \cdot dq \quad \text{which is independent of } t
\]

and thus the area is conserved.

(Of course using polar coordinates, \( r = \sqrt{p^2 + q^2} \) and \( \theta = \arctan \frac{p}{q} \), as intermediate step gives the same answer.)