Problem 5.1

a) \[ L = T - U \]
\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \]
\[ = \frac{1}{2} m (\dot{r}^2 + \dot{r}^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \]
\[ U = U(r) \]
\[ L (r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}) = \frac{1}{2} m (\dot{r}^2 + \dot{r}^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - U(r) \]

b) \[ H = \sum_i \frac{p_i^2}{2} - L = \dot{p}_r \dot{r} + \dot{p}_\theta \dot{\theta} + \dot{p}_\phi \dot{\phi} - L = H (p_r, p_\theta, p_\phi, r, \theta, \phi) \]
\[ \dot{r} = \frac{\partial L}{\partial \dot{r}} = \frac{p_r}{m} \]
\[ \dot{\theta} = \frac{\partial L}{\partial \dot{\theta}} = \frac{m r^2 \dot{\theta}}{m} \]
\[ \dot{\phi} = \frac{\partial L}{\partial \dot{\phi}} = \frac{m r^2 \sin^2 \theta \dot{\phi}}{m r^2 \sin^2 \theta} \]
\[ p_r = \frac{\partial L}{\partial \dot{r}} = \frac{p_r}{m} \quad \dot{r} = \frac{p_r}{m} \]
\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{m r^2 \dot{\theta}}{m} \quad \dot{\theta} = \frac{p_\theta}{m r^2} \]
\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{m r^2 \sin^2 \theta \dot{\phi}}{m r^2 \sin^2 \theta} \quad \dot{\phi} = \frac{p_\phi}{m r^2 \sin^2 \theta} \]
\[ H = T + U \]
\[ = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + U(r) \]

c) Since \( \dot{\phi} \) does not appear in Lagrangian and Hamiltonian, we can immediately claim that \( \frac{\partial L}{\partial \phi} \) should be conserved in Lagrangian formalism (\( p_\phi \) conserved in Hamiltonian formalism). However, we still have the freedom to orient our spherical coordinate system.
In the case of a single particle we can choose it such that \( \theta = 0 \) as an initial condition, which leaves \( \phi = \text{constant} \), but in fact we have the freedom to choose \( \phi = 0 \) as initial condition as well.

Continuing in Hamiltonian style:

\[
\begin{align*}
\dot{P}_r &= -\frac{\partial H}{\partial r} = -\frac{\partial U(r)}{\partial r} + \frac{2}{r^3} \left( \frac{P_r^2}{2m} + \frac{P_\theta^2}{2m \sin^2 \theta} \right) \\
\dot{P}_\theta &= \frac{\partial H}{\partial \theta} = \frac{P_r \sin \theta}{m r^2} \\
\dot{P}_\phi &= -\frac{\partial H}{\partial \phi} = 0 \\
\dot{r} &= \frac{\partial H}{\partial P_r} = \frac{P_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{m r^2} \\
\dot{\phi} &= \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{m r^2 \sin^2 \theta}
\end{align*}
\]

With our freedom to set \( \phi = 0, \dot{\phi} = 0 \) at initial time \( t = 0 \)

we have that \( P_\phi = 0 \) and therefore \( \dot{P}_\theta = 0 \Rightarrow P_\theta \) is conserved.

Angular momentum of course is the constant of motion, recall \([M_i, H] = 0\) on problem 4.3 with \( i = i(r) \) and \( \vec{M} = \vec{r} \times \vec{p} \).

With our choice of orientation we have set two components of \( \vec{M} \) equal to zero and the third one is \( P_\theta \). Energy is also conserved of course.
Problem 5.2

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

\[ \frac{x}{r} = \varepsilon \cos \theta + 1 \]
\[ \frac{x}{r} = \varepsilon \frac{x}{r} + 1 \]
\[ \frac{x - \varepsilon x}{r} = 1 \]
\[ x - \varepsilon x = r \]
\[ (\alpha - \varepsilon x)^2 = r^2 \]
\[ \alpha^2 - 2\varepsilon \alpha x + \varepsilon^2 x^2 = x^2 + y^2 \]
\[ x^2(1 - \varepsilon^2) + 2\varepsilon \alpha x + y^2 = \alpha^2 \]

complete the squares

\[ x^2(1 - \varepsilon^2) + 2\varepsilon \alpha x = (1 - \varepsilon^2)(x - x_0)^2 - (1 - \varepsilon^2)x_0^2 \]

with \[ z \varepsilon \alpha x = (1 - \varepsilon^2)(-z\varepsilon x_0) \]

\[ X_0 = \frac{-\varepsilon \alpha}{1 - \varepsilon^2} \]
(1 - \varepsilon^2)x_0 = \frac{\varepsilon^2 \alpha^2}{1 - \varepsilon^2}

\[ y^2 = (y - y_0)^2 \]
with \[ y_0 = 0 \]

(1 - \varepsilon^2)(x - x_0)^2 + (y - y_0)^2 = \alpha^2 + \frac{x^2}{1 - \varepsilon^2} = \frac{\alpha^2}{1 - \varepsilon^2}

\[ \frac{(x - x_0)^2}{\alpha^2} + \frac{(y - y_0)^2}{1 - \varepsilon^2} = 1 \]

\[ a = \frac{\alpha}{1 - \varepsilon^2} \]
[ \[ b = \frac{\alpha}{\sqrt{1 - \varepsilon^2}} \]

\[ \varepsilon = 0: \ a = b = \alpha \]
\[ \varepsilon \to 1: \ a \text{ and } b \text{ and } x_0 \text{ diverge} \]
The velocity is found from the kinetic energy:
\[ T = E - U(r) = \begin{cases} E & \text{outside} \\ E + V & \text{inside} \end{cases} \]

\[ V = \sqrt{\frac{2}{m} T} = \begin{cases} \sqrt{\frac{2}{m} E} & \text{outside well} \\ \sqrt{\frac{2}{m} (E + V)} & \text{inside well} \end{cases} \]

b) Inside the well we have \( M = m \lim_{\min} V \)

\[ \lim_{\min} = \frac{M}{mV} = \frac{M}{\sqrt{2m(E + V)}} \]

since inside and outside, of the well the particle travels in a straight line.

Alternatively, look at the effective potential \( V_{\text{eff}} = \frac{M^2}{2mr^2} + U(r) \)

at \( r = r_{\min} \) we have \( V_{\text{eff}} = E \)

\[ \frac{M}{r_{\min}} = \sqrt{2m(E + V)} \]

once more.

c) \( E < V_{\text{eff}}(r = R) \)

\[ E < \frac{M^2}{2mR^2} \]
d) \( E < \frac{M^2}{2mR^2} \): either the particle is trapped inside the well and never gets out (see part e) for trajectory) or the particle is outside the well and never enters the well:

Since outside it is a free particle moving along a line, the trajectory is simply a straight line which passes the well.

Now the particle will enter and exit the well. Both inside and outside it will follow a straight line, but at the boundary the direction will change: refraction.

The remaining question then is: which way does it diffract, closer to the origin or away from it?

You could consider the following situation, with a smooth potential:

Then you can imagine lines of force in the region where the potential drops from 0 to \(-V\).
Force $F = -\frac{\partial U}{\partial r}$ draws you into the well, and direct you closer to the origin than your original path would take you.

The explanation for this refraction is of course conservation of angular momentum.

Outside $M = m_b \cdot V_{out}$

Inside $M = m r_{min} V_{in}$

Since $V_{in} > V_{out}$
we need $r_{min} < b$
to have $M$ constant.

Trapped inside, the particle will bounce off the walls with $r_{min} \leq r \leq R$. 
Problem 5.4

\[ F(r) = -\frac{a}{r^4} - \frac{b}{r^4} \quad F(r) = -\frac{d}{dr} U(r) = 0 \quad U(r) = -\frac{a}{r} - \frac{b}{3r^3} \]

\[ V_{\text{eff}}(r) = U(r) + \frac{\ell^2}{2mr^2} = -\frac{b}{3r^3} + \frac{\ell^2}{2mr^2} - \frac{a}{r} \]

A stable circular orbit corresponds with a local minimum of the effective potential.

Mathematically, a local minimum \( r = R \) requires

\[ \left. \frac{dV_{\text{eff}}(r)}{dr} \right|_{r=R} = 0 \quad (\star) \]

and

\[ \left. \frac{d^2V_{\text{eff}}}{dr^2} \right|_{r=R} > 0 \quad (\star\star) \]

\[ \frac{dV_{\text{eff}}}{dr} = -\frac{b}{r^4} - \frac{\ell^2}{mr^3} + \frac{a}{r^2} \]

\[ \frac{d^2V_{\text{eff}}}{dr^2} = -\frac{4b}{r^5} - \frac{3\ell}{mr^4} - \frac{2a}{r^3} \]

Furthermore, we require \( R \) to be real and positive.

Although it is tempting to solve (\star) and (\star\star) all at once for arbitrary real-valued \( a \) and \( b \) and \( \ell \), this can become very complicated because different signs of \( a \) and \( b \) can give very different outcomes. It is therefore wise to systematically work through the different cases \( a = 0, a < 0, a > 0, b = 0, b < 0, b > 0 \).
5.4. cont'd

For the angular momentum term, \( l^2 \), we know that this has to be positive, \( l^2 > 0 \), since when \( l^2 = 0 \) there can be no circular, or even orbital, motion, since \( \theta \) would be constant.

It is also very instructive to make a sketch of the effective potential; the behavior of \( V_{\text{eff}}(r) \) as \( r \to 0 \) and \( r \to \infty \) plus the number of extreme points, plus the fact that \( V_{\text{eff}}(r) \) is continuous will tell you if there can be a local minimum.

Let \( a = 0, b > 0 \)

\[
V_{\text{eff}} = -\frac{16l}{3r^3} \, \frac{r^2}{2m^2}
\]

The equation \( \frac{dV_{\text{eff}}}{dr} = 0 = \frac{b}{r^n} - \frac{e^2}{2mr^2} + \frac{a}{r^2} \)

has at most two solutions for real \( R > 0 \), (i.e. 0, 1 or 2 solutions)

The only way to connect \( -\frac{l}{r^3} \) 0 at \( r \to 0 \)

to \( +\frac{l}{r^2} \) at \( r \to \infty \) with at most two extreme points is to have one (local) maximum.

This \( V_{\text{eff}} \) has no local minima and thus no stable circular orbits.

\( V_{\text{eff}} \) 0 stable circular orbits
\( a = 0 \ b < 0 \) \( V_{\text{eff}} = \frac{|b|}{3r^3} + \frac{b^2}{2mr^2} \)

If there would be two extreme points at positive \( r \), this would yield a local minimum.

Extreme points \( \frac{b}{r^4} - \frac{b^2}{mr^3} = 0 \quad b - \frac{b^2 r}{m} = 0 \)

Zero extreme points and \underline{NO} stable circular orbits.

\( a < 0 \ b > 0 \) \( V_{\text{eff}} = -\frac{|b|}{3r^3} + \frac{b^2}{2mr^2} + \frac{1|a|}{r} \)

Can only connect with one maximum.

\underline{NO} stable circular orbits.

\( a < 0 \ b = 0 \) \( V_{\text{eff}} = \frac{b^2}{2mr^2} + \frac{|a|}{r} \)

If two \underline{positive} extreme points with \( r > 0 \) then there is a local minimum.

\( -\frac{b^2}{mr^3} + \frac{a}{r^2} = 0 \)

\( aR = \frac{b^2}{m} \)

\( R = \frac{b^2}{ma} < 0 \) since \( a < 0 \)

\underline{No} stable circular orbits.
5.4. Cont'd

\( a < 0 \) \( b < 0 \) \( \text{V}_{\text{eff}} = \frac{161}{3r^3} + \frac{r^2}{2mr^2} + \frac{1a1}{r} \)

**Extreme Points:**
\[
\frac{b}{r^4} - \frac{r^2}{mr^3} + \frac{a}{r^2} = 0
\]
\[
\frac{b}{m} - \frac{r^2}{r} + a \frac{r^2}{2} = 0
\]

\[ R_+ = \frac{r^2}{2ma} \pm \sqrt{\left(\frac{r^2}{2ma}\right)^2 - \frac{1a1}{(1a)}} \]

No solutions, \( R_+ > 0 \)
and No stable circular orbit possible

\( a > 0 \) \( b < 0 \) \( \text{V}_{\text{eff}} = \frac{161}{3r^3} + \frac{r^2}{2mr^2} - \frac{1a1}{r} \)

Only way to connect is to have one local minimum

\[ R_+ = \frac{r^2}{2ma} \pm \sqrt{\left(\frac{r^2}{2ma}\right)^2 + \frac{161}{(1a)}} \]

\( R = R_+ > 0 \) is the local minimum

\[ \boxed{\text{Yes}} \], stable circular orbit is possible!

\( a > 0 \) \( b = 0 \) \( \text{V}_{\text{eff}} = \frac{r^2}{2mr^2} - \frac{1a1}{r} \)

\[ \boxed{\text{Yes}} \], stable circular orbit

with \( -\frac{r^2}{mr^3} + \frac{a}{r^2} = 0 \)
\( aR = \frac{r^2}{m} \)
\( R = \frac{r}{ma} \)
A local minimum is possible when there are two extreme points

\[ R_\pm = \frac{\ell^2}{2mla} \pm \sqrt{\left(\frac{\ell^2}{2mla}\right)^2 - \frac{16l}{la}} \]

has \( R_\pm > 0 \) if \( \left(\frac{\ell^2}{2mla}\right)^2 - \frac{16l}{la} > 0 \)

with \( R_+ \) being the radius of the stable circular orbit.

Recapping, we have a stable circular orbit when:

- \( a > 0 \) \( b > 0 \) at \( R = \frac{\ell^2}{2mla} + \sqrt{\left(\frac{\ell^2}{2mla}\right)^2 - \frac{16l}{la}} \)

- \( a > 0 \) \( b = 0 \) at \( R = \frac{\ell^2}{ma} \)

- \( a > 0 \) \( b > 0 \) at \( R = \frac{\ell^2}{2mla} + \sqrt{\left(\frac{\ell^2}{2mla}\right)^2 - \frac{16l}{la}} \) if \( \left(\frac{\ell^2}{2mla}\right)^2 - \frac{16l}{la} > 0 \)

which can be condensed as:

- \( a > 0 \)
- \[ \frac{b}{a} < \left(\frac{\ell^2}{2ma}\right)^2 \Rightarrow b < \frac{\ell^4}{4m^2a} \] at \( R = \frac{\ell^2}{2ma} + \sqrt{\left(\frac{\ell^2}{2ma}\right)^2 - \frac{b}{a}} \).