Massachusetts Institute of Technology
Department of Physics

Course: 8.09 Classical Mechanics and Special Relativity
Term: Fall 2000

Final Examination
December 13, 2002 – Trial

Instructions

• Do not start until you are told to do so.
• Solve all problems.
• Put your name and recitation section number on the covers of all notebooks you are using.
• Show all work neatly in the blue book, label the problem you are working on.
• Mark the final answers.
• Books and notes are not to be used. Calculators are unnecessary.
Useful Formulae

Newton and Basic Kinematics:

\[ \vec{F} = \vec{v} = m \vec{a} \quad \text{for} \quad v \ll c \]

\[ \vec{v} = \vec{v}_0 + \int_{t=0}^{t'=t} dt' \vec{a} \]

\[ \vec{r} = \vec{r}_0 + \vec{v}_0 t + \int_{t=0}^{t'=t} dt' \int_{t''=0}^{t'} dt'' \vec{F}(t'')/m \]

Gravitational Law:

\[ \vec{F} = \frac{-G m_1 m_2}{r_{12}^2} \vec{r}_{12} \]

Lagrangian and Hamiltonian:

\[ L(q, \dot{q}) = T - U; \quad H(p, q) = T + U = p \frac{\partial q}{\partial t} - L \]

Hamilton Equation of Motion:

\[ \frac{\partial H}{\partial q} = -\dot{p}; \quad \frac{\partial H}{\partial p} = \dot{q} \]

Euler-Lagrange (without and with constraints):

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0; \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial q}{\partial x} = 0 \]

Orbit Equation:

\[ u'' + u = -\frac{\mu}{\ell^2 u^2} F(u) \quad \text{with} \quad u = \frac{1}{r} \]

Effective Potential:

\[ V(r) = U(r) + \frac{\ell^2}{2 \mu r^2} \]

Keplerian Orbits:

\[ U(r) = -\frac{k}{r}; \quad \frac{\alpha}{r} = \varepsilon \cos \theta + 1 \]

\[ \alpha = \frac{\ell^2}{\mu k}; \quad \varepsilon = \sqrt{1 + \frac{2 \ell^2}{\mu k^2}}, \quad \tau^2 = \frac{4 \pi^2 \mu}{k} \]

\[ r_{\text{min}} = a(1 - \varepsilon) = \frac{\alpha}{1 + \varepsilon}; \quad r_{\text{max}} = a(1 + \varepsilon) = \frac{\alpha}{1 - \varepsilon} \]

Polar Coordinates:

\[ x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta \]

Scattering:

\[ \sigma(\theta) = \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\phi} \right| \]

\[ \tan \psi = \frac{\sin \theta}{\cos \theta + (m_1/m_2)}; \quad \zeta = \frac{\pi}{2} - \frac{\theta}{2} \]
Problem 1: Hamiltonian (15 points)

A particle of mass \(m\) moves in a 1-dimensional potential of the form which is displayed in Figure 1:

\[
U = \begin{cases} 
k(x - 2na) & \text{for } 2na < x < (2n + 1)a 
k(2(n + 1)a - x) & \text{for } (2n + 1)a < x < 2na \end{cases}
\]

where \(a\) is a positive constant and \(n\) is an integer.

![Potential Diagram](Image)

Figure 1: Potential.

a) Find a Hamiltonian for the system in term of \(x\), \(p\), \(m\) and \(a\).

b) Give Hamilton equation of motion.

c) Find the condition of the particle to remain confined to a finite interval of space.

d) Sketch typical trajectories in phase space for confined and and unconfined particle. Also plot the limiting trajectory, i.e. the trajectory where the particle becomes just unconfined.
Problem 2: Lagrangian (15 points)

A pendulum which is exposed to gravity consists of a mass $m$ which is suspended by a massless ideal elastic string\(^1\) of unextended length $l$ and a string constant $k$. The string is constrained by a frictionless fixture which still allows the string to extend, as shown in the figure.

\[ \phi \]

\[ \theta \]

\[ r \]

\[ l \]

\[ g \]

a) Give the kinetic and potential energies of the mass $m$, using the coordinates $r$, $\theta$ and $\phi$ as indicated in the picture.

b) Find the Lagrangian for this system.

c) Find the equations of motion and quote them in the form:

\[
\begin{align*}
m \ddot{r} & = \ldots \\
m \ddot{\theta} & = \ldots \\
m \ddot{\phi} & = \ldots 
\end{align*}
\]

Please write them in exactly this form and mark your final answers with a box.

d) Which momentum or momenta are conserved and which coordinate or coordinates are related to each?

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\(^1\) The ideal elastic string follows Hooke's Law like an ideal spring.
Problem 3: Tides (15 points)

We consider the observation of tides on Earth and the significance of Newton's explanation of their twelve hour period. Answer the following questions with one sentence of less than 25 words (i.e. we will only count the first 25 words of the answer!).

a) Why do tides occur every twelve hours instead of every twenty four hours?

b) Why was the ability of Newton’s theory to describe the twelve hour period of the tides particularly significant?
Problem 4: Flywheel (15 points)

A flywheel of length $l$ and radius $R$ has a density of $\rho$ and mass $M$. The flywheel is used to store energy in the following way: every $\tau_0$ seconds, the wheel, rotating at $\omega_0$ is clutched into a DC generator for 2 seconds, during which time electricity is generated at a constant rate such that over the next $\tau_1$ seconds a fraction $f$ of the energy of the flywheel is used. for the next $\tau_2$ seconds, and AC motor clutches into the flywheel and restores the energy extracted. The cycle repeats, $\tau_0 = \tau_1 + \tau_2$.

a) Find the flywheels momentum of inertia around the rotation axis.

b) For the energy stored in the flywheel when it is rotating at and angular velocity $\omega_0$.

c) Find the power output to the generator (i.e. $dT_{\text{flywheel}}/dt$) during extraction.

d) Find the torque necessary to bring the flywheel back to $\omega_0$ after extraction.
Problem 5: Lagrangian (15 points)

The Lagrangian for a free particle of mass $m$ in the fixed frame is

$$L = \frac{1}{2}mv^2$$

a) Use the relation for $v$ to give $L$ in the rotating frame, $L(\bar{v}, \bar{V}, \bar{r})$.

b) Take $\bar{\omega} = 0$. Find the three equations of motion from the resulting Lagrangian in terms of $x, \dot{x}, \ddot{x}, \ldots$ etc.

c) Let $\bar{V} = gt\bar{z}$ and $\bar{\omega} = 0$. Find the resulting equations of motion and give the general solution.

d) Comment in one sentence on the result in part c) and its relation to motion in the gravitational field near the Earth.