Problem 9.1

(a) Notice how consecutive 90° rotations around the \( x_i \) axis bring us from (1a) to (1b) to (1d) to (1c).

With Rubik's cube hindsight we figure that we can get \( \pm \frac{\pi}{2}, \pi \) rotations around \( x_2 \) from combinations of \( \pm \frac{\pi}{2} \), \( \pi \) rotations around \( x_1 \) and \( x_3 \) axes.

The Euler angles \( \psi, \Theta \) and \( \gamma \) need to be around \( x_3, x'_1 \) and \( x''_3 \), so we need to keep track of where the axes go as well.
3.1 cont'd...

We can get to (b) by choosing \((\psi, \theta, \psi) = (\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2})\)

or \((\psi, \theta, \psi) = (-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})\)

We can get to (c) with \((\psi, \theta, \psi) = (\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2})\) or \((-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})\)

We can get to (d) with \((\psi, \theta, \psi) = (\pi, \pi, 0)\) for instance.

b) \((b) \rightarrow (c)\) is \(180^\circ\) around \(x_2\) like \((a) \rightarrow (d)\), so \((\psi, \theta, \psi) = (\pi, \pi, 0)\).

c) \((c) \rightarrow (d)\) is \(-90^\circ\) around \(x_2\) like \((a) \rightarrow (c)\), so \((\psi, \theta, \psi) = (\frac{\pi}{2}, -\frac{\pi}{2}, -\frac{\pi}{2})\)

would do the trick.
Extra: let's look at the resulting rotation matrix for 
\[ (q, \theta, y) = \left( \frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2} \right) \]

\[ \hat{R}(q, \theta, y) = R_y R_\theta R_\gamma \]

\[ = \begin{pmatrix} \cos y & -\sin y & 0 \\ -\sin y & \cos y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos y & \sin y & 0 \\ -\sin y & \cos y & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \hat{R}\left( \frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2} \right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix} \]

which is a rotation around the z-axis with rotation angle 
\[ \alpha = \frac{\pi}{2} \]
Problem 9.2

General idea in this problem is to conserve (linear) momentum and angular momentum before and after the collision.

This naturally splits into a center-of-mass and relative-to-center-of-mass parts.

In this problem we will ignore the mass \( m \) compared with \( M \), but we will not ignore the (angular) momentum that the small particle carries.

The center-of-mass will thus for all practical purposes be at the center of the ellipsoid with magnitude \( M \).

Before the collision total momentum is \(-m\mathbf{v}\mathbf{\hat{q}}\). So after the collision the center-of-mass has linear momentum \(-m\mathbf{v}\mathbf{\hat{q}}\) and velocity \(-\frac{m\mathbf{v}}{M}\).

After the collision there is zero linear momentum within the C-M-S frame, with \( m \) sticking at \( M \).

In the C-M-S frame, before the collision there is angular momentum (with respect to the center-of-mass):

\[
\mathbf{M} = \mathbf{r}_m \times \mathbf{p}_m \quad \text{with} \quad \mathbf{r}_m = (x, y, z) \quad \mathbf{p}_m = (0, -m\mathbf{v}, 0)
\]

\[
\mathbf{M} = (0, m\mathbf{v}, 0 - 0, -m\mathbf{v}) = \begin{pmatrix} 0 \\ m\mathbf{v} \end{pmatrix}
\]

After the collision:

\[
M_1 = g_2 \mathbf{v} = I_1 \omega_1
\]

with \( I_1 = I_2 = I_{xx} = I_{yy} = \frac{1}{5} M (a^2 + b^2) \)

\[
M_2 = 0 = I_2 \omega_2
\]

and \( I_3 = I_{zz} = \frac{1}{5} M (a^2 + a^2) \)

\[
M_3 = -g_1 \mathbf{v} = I_3 \omega_3
\]

ignoring the contribution of mass \( m \) to inertia tensor \( I_{ij} \).
9.2 cont'd

The Euler equations will tell us how \( \mathbf{\omega}(t) \) will evolve with time in the principal-axes coordinate system with origin at the center of mass.

\[
\begin{align*}
I_1 \mathbf{\dot{w}}_1 &= (I_2 - I_3) w_2 w_3 \\
I_2 \mathbf{\dot{w}}_2 &= (I_3 - I_1) w_1 w_3 \\
I_3 \mathbf{\dot{w}}_3 &= (I_1 - I_2) w_1 w_2
\end{align*}
\]

since \( I_1 = I_2, \ w_3 = 0, \ w_2 = \text{constant in time} \)

\[
I_3 \mathbf{\dot{w}}_3 = -\frac{\tau_3}{I_3} m \mathbf{v}
\]

\[
\omega_3 = -\frac{\tau_3}{I_3} \frac{V}{a^2} \left( \frac{m}{M} \right)
\]

\[
I_2 - I_3 = \frac{1}{2} M \left( b^2 - a^2 \right) = -(I_3 - I_1)
\]

\[
\begin{align*}
\mathbf{\dot{w}}_1 &= \left( \frac{I_2 - I_3}{I_1} \right) w_2 w_3 \\
\mathbf{\dot{w}}_2 &= \left( \frac{I_3 - I_1}{I_2} \right) w_3 w_1 \\
\mathbf{\dot{w}}_3 &= \left( \frac{I_1 - I_2}{I_3} \right) w_1 w_2
\end{align*}
\]

\[
\dot{w}_1 = \frac{3}{2} M \left( b^2 - a^2 \right) \left( \frac{\tau_3}{I_3} \frac{V}{a^2} \frac{m}{M} \right) w_2 \equiv \alpha w_2
\]

\[
\dot{w}_3 = \frac{3}{2} \frac{m}{M} \frac{\tau_3}{a^2} \left( \frac{V}{a^2} \right) \frac{1}{a^2 + b^2}
\]

Let the collision happen at \( t = 0 \), then, with \( \omega_1(0) = \frac{\tau_2}{I_1} m \mathbf{v} \) and \( \omega_2(0) = 0 \)

the solution is

\[
\omega_1(t) = 5 \frac{\tau_2}{I_1} \frac{V}{a^2 + b^2} \frac{m}{M} \cos(\alpha t)
\]

\[
\omega_2(t) = 5 \frac{\tau_2}{I_3} \frac{V}{a^2 + b^2} \frac{m}{M} \sin(\alpha t)
\]

\( \frac{m}{M} = "\text{small}" \)

So, an observer in an inertial system will see the ellipsoid move away with small velocity, rotate with small angular velocity, and rotation axis changes with small frequency.
Problem 9.3

(a) \[ l_i = \sqrt{a^2 + \left(q_{i+1} - q_i\right)^2} \]

Increase in length from \( q_{i+1} = q_i \) is \( \Delta l_i \):

\[ \Delta l_i = l_i - a = \sqrt{a^2 + \left(q_{i+1} - q_i\right)^2} - a \approx a \left(1 + \frac{1}{2} \left(\frac{q_{i+1} - q_i}{a}\right)^2\right) - a \]

when \( q_{i+1} - q_i << a \)

\[ = a \left(\frac{1}{2} \left(\frac{q_{i+1} - q_i}{a}\right)^2\right) \]

(b) Kinetic energy is \( \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \)

Potential energy for i\textsuperscript{th} mass is \( T \Delta l_i \); if we define our potential energy to be zero when \( l_i = a \), \( U = \sum_{i=1}^{N} T \Delta l_i = \sum_{i=1}^{N} a \frac{1}{2} \left(\frac{q_{i+1} - q_i}{a}\right)^2 \)

(Note that tension is a force; corresponding potential energy is integral over distance \( U = \int dx \ T \); e.g. for a spring with \( F = kx \), \( U = \int dx \ F = \frac{1}{2} kx^2 \), not worrying about the sign.)

The boundary conditions determine what happens to the potential energy for \( q_1 \) and \( q_N \). Here the boundary conditions are not specified, so we leave our result as a sum over \( i \), where the exact range for \( i \) is left undetermined (e.g. \( 1 \leq i \leq N \) or \( 1 \leq i \leq N-1 \) etc.)

\[ L = \sum_{i=1}^{N} \left[ \frac{1}{2} m_i v_i^2 - a \frac{1}{2} T \left(\frac{q_{i+1} - q_i}{a}\right)^2 \right] \]
3.3 contd.

c) \( N \to \infty \) but \( \xi \left\{ \sum_{i} \right\} \) constant finite requires \( a \to 0 \), \( a \propto \frac{1}{N} \).

Note that we cannot leave the mass \( m \) as is, as that would make the string infinitely heavy, and infinite-mass strings do not move. So replace \( m = \lambda a \) where \( \lambda \) is a mass density.

\[
L = \sum_{i} \left[ a \frac{1}{2} \lambda q_i^2 - a \frac{1}{2} T (\frac{q_i - q_{i-1}}{a})^2 \right]
\]

Now replace \( q_i \to \dot{q}(x,t) \), \( (\frac{q_i - q_{i-1}}{a}) \to q'(x,t) \), \( \xi a \to \int dx \)

to find in limit \( a \to 0 \):

\[
L = \int dx \left[ \frac{1}{2} \lambda q'^2 - \frac{1}{2} T (q'^2) \right] = \int dx \ L
\]

\[ L (\dot{q}(x,t), q'(x,t) ; x,t) = \frac{1}{2} \lambda \dot{q}^2 - \frac{1}{2} T q'^2 \]

\[
\frac{\partial L}{\partial \dot{q}} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial q'} \right) = 0
\]

\[
0 - \lambda \frac{\partial \dot{q}}{\partial t} + T \frac{\partial^2 q'}{\partial x^2} = 0
\]

\[
\lambda \ddot{q} = T q'' \rightarrow \text{wave equation with } V = \sqrt{\frac{T}{\lambda}}.
\]
Substitute $\tilde{\eta} = A \sin (\vec{k} \cdot \vec{x} + wt + \phi)$ into the wave equation

$$\frac{\partial^2 \eta_i}{\partial t^2} - \frac{\lambda}{c} \nabla^2 \eta_i = 0$$

$$\frac{\partial^2}{\partial t^2} \left( A_i \sin \left( k_i x_i + wt + \phi \right) \right) - \frac{\lambda}{\lambda} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} \left( A_i \sin \left( k_i x_i + wt + \phi \right) \right)$$

$$= -w^2 A_i \sin \left( k_i x_i + wt + \phi \right) - \frac{\lambda}{\lambda} - k_i k_i \int A_i \sin \left( k_i x_i + wt + \phi \right)$$

$$= -\left( w^2 - \frac{\lambda}{\lambda} \right) A_i \sin \left( k_i x_i + wt + \phi \right)$$

$$= 0 \quad \forall t, \vec{x}, i$$

so $w^2 - \frac{\lambda}{\lambda} \vec{k}^2 = 0$ (or $A_i = 0$, but that is a trivial solution)

$$w = \frac{\lambda |\vec{k}|}{\lambda} \quad \text{(we choose } w \text{ to be positive)}$$