We encourage you to work together, as long as you write your own solutions.

1 Intertemporal Labor Supply

Consider the following problem. The consumer problem is:

$$\max_{(C_t),(N_t)} E_0 \left( \sum_{t=0}^{T} \beta^t \left\{ \chi_1 C_t^{\psi_1} + \chi_2 N_t^{\psi_2} \right\} \right)$$

s.t.

$$A_{t+1} = R(A_t + N_tW_t - C_t)$$

Where $C$ is consumption, $N$ is labor supply, $A_0$ is initial wealth, $R = 1 + r$, and the greek letters are parameters. Only $W$ is stochastic

1). Derive and interpret the first-order conditions for $C_t$ and $N_t$.

What are the intertemporal and intratemporal optimal conditions

There are at least two (equivalent ways of setting up and solving this problem). I will implement both of them

1. Lagrangian method I:

In this case the Lagrangian is:

$$\max_{\{C_t\},\{N_t\},\{A_{t+1}\}} \mathcal{L} = E_0 \left( \sum_{t=0}^{T} \beta^t \left\{ \chi_1 C_t^{\psi_1} + \chi_2 N_t^{\psi_2} - \lambda_t (A_{t+1} - R (A_t + N_t W_t - C_t)) \right\} \right)$$

s.t. $A_0$

F.O.C.

$$\{C_t\} : \frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \chi_1 \psi_1 C_t^{\psi_1 - 1} = \lambda_t R \quad (1)$$
The interpretation is more or less natural. The first condition tells you that the marginal utility of $C_t$ has to be equal to the marginal utility of wealth (do not worry too much about $R$—it is an implication of the fact that $C$ in this case is realized at the beginning of the period $t$—so the value of saving one unit of consumptions is $1$ times $R$). The second condition tells you that marginal disutility of work has to be equal to the marginal utility of wealth times how much consumption you can get from one our of work (the wage rate). Why no expectation? This is important, at time $t$, the only source of uncertainty is realized ($W_t$) and, roughly speaking, $\lambda_t$ is a function of expected future wages (see below).

Combining both equations we get the intratemporal condition:

$$-\chi_2 N_t^{\varpi - 1} = W_t \chi_1 C_t^{\varpi - 1}$$

To get the intertemporal condition, we need another FOC:

$$\frac{\partial L}{\partial A_{t+1}} = 0 \iff \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} R$$

Using the FOC for consumption, we get our intertemporal condition:

$$\beta^t \chi_1 C_t^{\varpi - 1} = \beta^{t+1} RE(\chi_1 C_{t+1}^{\varpi - 1})$$

$$1 = \beta RE\left[ \left( \frac{C_{t+1}}{C_t} \right)^{\varpi - 1} \right]$$

2. Lagrangian method II:

Notice that you can iterate the constraint and write the intertemporal budget constraint.

$$\sum_{t=0}^{T} R^{-t} C_t = A_0 + \sum_{t=0}^{T} R^{-t} N_t W_t$$

Two comments: (1) Why no expectation? This has to hold exactly. (2) what about $A_{T+1}$?

$$\max_{\{C_t, N_t\}} L = E_0 \left( \sum_{t=0}^{T} \beta^t \{ \chi_1 C_t^{\varpi_1} + \chi_2 N_t^{\varpi_2} \} - \lambda \left( \sum_{t=0}^{T} R^{-t} C_t - A_0 - \sum_{t=0}^{T} R^{-t} N_t W_t \right) \right)$$

F.O.C.
If you combine both equations, you get the same intratemporal condition as before.

To get the intertemporal condition write the FOC for \( t + 1 \) taking expectations at \( t \).

Using the FOC for \( C_t \):

\[
E_t \left[ \beta^{t+1} \chi_1 \omega_1 C_{t+1}^{\omega_1 - 1} - \chi_1 \beta^t \omega_1 C_t^{\omega_1 - 1} R^{-t-1} \right] = 0
\]

and you get the same as before:

\[
1 = \beta RE \left[ \left( \frac{C_{t+1}}{C_t} \right)^{\omega_1 - 1} \right]
\]

What is more interesting about this method is that \( \lambda \) can be interpreted as the sufficient statistic you need to solve the problem.

Notice that (1) and (2) imply that:

\[
\lambda_t R = \lambda (\beta R)^{-t}
\]

How can you interpret this condition?

We can plug (1) and (3) into the budget constraint and we will get a decreasing implicit function of \( \lambda \) as a function of \( \{W_i\} \) with \( i \geq t \) and, of course, some constant terms (among them \( R \)). Thus, the only thing you need to compute \( \lambda \) is to know something about \( \{W_i\} \). Therefore, you will keep your \( \lambda \) constant in the future if the realizations of \( \{W_i\} \) were more or less what you expected in the past.

2). What is the link between \( C_t \) and \( N_t \) in this model? What assumption(s) is (are) producing this result?

The only link between both variables is \( \lambda \) (or \( \lambda_t \)). This implies the intratemporal condition and the negative correlation between both variables if \( W \) is kept constant. And, the lack of correlation if \( W \) increases without affecting \( \lambda \).

Assumption: (1) both terms are additively separable in the utility function and (2) perfect capital markets (what may happen if we introduce frictions as \( A_t > 0 \))
3). How can you analyze changes to wages that do not affect the expected wealth of the consumers? (Hint: what is $\lambda$—the Lagrange multiplier—in this model?). Let’s define the wealth-constant elasticity of labor supply as $\eta = \frac{\partial \ln N_t}{\partial \ln W_t}$ when wealth is constant. What is $\eta$ in this model? Is it positive or negative? Why?

Given our previous discussion, these are changes that do not affect $\lambda$, therefore we can take logs in (3) and get:

$$\log\left(\frac{-\chi_2 \bar{w}_2}{\lambda (\beta R)}\right) + (\bar{w}_2 - 1) \log(N_t) = \log W_t$$

Then

$$\eta = \frac{\log(N_t)}{\log W_t} = \frac{1}{(\bar{w}_2 - 1)}$$

Notice that $\eta > 0$. There are several ways of understanding this. Here is one:

First, notice that obviously $\chi_1 > 0, \chi_2 < 0, \bar{w}_1 < 1$. Why? People like $C$, dislike $N$, and marginal utility of consumption is decreasing in general. Therefore if the instantaneous utility function is to be concave (and, therefore, the solution we just proposed has sense), it has to be true that:

$$\frac{\partial^2 U}{\partial N^2} = \chi_2 \bar{w}_2 (\bar{w}_2 - 1) N_t \bar{w}_2^{-2} < 0 \Rightarrow (\bar{w}_2 - 1) > 0$$

4). Take the FOCs and discuss the effect of the following situations on $\{C_t\}$ and $\{N_t\}$.

1. **Cross-sectional differences associated with permanent differences in human capital (assume $W$ varies in the cross-section)**

Higher human capital will probably imply higher $W$. Therefore, $\lambda \downarrow$, $C \uparrow$.

What about $N$? Assume all other parameters do not vary in the cross-section and take two individuals $i$ and $j$

$$\log\left(\frac{N_j}{N_i}\right) = \frac{1}{(\bar{w}_2 - 1)} \left(\log \frac{W_j}{W_i} + \log \frac{\lambda_j}{\lambda_i}\right)$$

So effect is ambiguous. One of them is richer, but at the same time the alternative cost of leisure is higher.

**Life-cycle changes in wages (i.e. if Mincerian equations are correct, wages follow an inverted-U behavior over the life-cycle)**

Notice that these evolutionary changes were predictable at the beginning of the life-cycle and, therefore, as $W$ changes, $\lambda$ is constant. Thus $C$ is also constant—see (2). But given that $\eta > 0$ and $\lambda$ is constant, $N$ moves together with $W$. 

4
• **Unexpected temporary shocks to wages**
  
  If change is temporary, it shouldn’t affect λ by much, then C is constant, and N moves together with W.

• **Unexpected permanent shocks to wages**
  
  In this case λ should be affected. Therefore, C moves in the same direction as W and the effect on N is unclear.

2 Log-Linear RBC

Consider the following problem. Consumers maximize

$$\max E_t \left[ \sum_{j=0}^{\infty} b^j U(\bar{C}_{t+j}, N_{t+j}) \right]$$

Subject to

$$\bar{K}_{t+1} = R_t \bar{K}_t + \bar{W}_t N_t - \bar{C}_t$$

$$U(\bar{C}_t, N_t) = \log(\bar{C}_t) + \phi \log(1 - N_t)$$

while firms solve

$$\left\{ N_t, \bar{K}_t \right\} = \arg \max_{N, \bar{K}} Z_t F(A_t N, \bar{K}) - (R_t + \delta - 1) \bar{K} - \bar{W}_t N$$

$$F(A_t N, \bar{K}) = Y = Z \bar{K}^{1-\alpha} (A_t N)^\alpha$$

and we assume

$$A_{t+1} = \gamma A_t.$$ 

This is the standard RBC model. Trending variables are those with a hat.

1) **De-trend the model**

Define $V = \frac{\bar{V}}{\bar{A}}$.

1. Utility function:

$$U(C_t, N_t) = \log(C_t A_t) + \phi \log(1 - N_t) = \underbrace{\log(A_t)}_{\text{Does not matter}} + \log(C_t) + \phi \log(1 - N_t)$$

2. Consumers B.C. $K_{t+1} A_{t+1} = R_t K_t A_t + W_t A_t N_t - C_tA_t$

$$K_{t+1} A_{t+1} = R_t K_t A_t + W_t A_t N_t - C_tA_t$$

$$\Leftrightarrow \quad K_{t+1} \frac{A_{t+1}}{A_t} = R_t K_t + W_t N_t - C_t$$

$$\Leftrightarrow \quad \gamma K_{t+1} = R_t K_t + W_t N_t - C_t$$
3. Profits

\[ \{ N_t, K_t \} = \arg \max_{N,K} Z_t (K_t A_t)^{1-\alpha} (A_t N)^{\alpha} - (R_t + \delta - 1) A_t K_t - A_t W_t N_t \]

\[ \Leftrightarrow \{ N_t, K_t \} = \arg \max_{N,K} Z_t (K_t)^{1-\alpha} (N)^{\alpha} - (R_t + \delta - 1) K_t - W_t N_t \]

2) Solve for the F.O.C. of the consumer problem, either using a Lagrangian or a Bellman Equation. You should find two final equations, the Euler Equation and the labor supply.

1. Bellman equation

\[ V(K) = \max_{C,N} \{ \log(C) + \phi \log(1-N) + bE[V(K')] \} \]

\[ s.t. \quad \gamma K_t = RK + WN - C \]

FOC:

\[ \frac{\partial V}{\partial C} = \frac{1}{C} + bE \left[ \frac{\partial [V(K')] \partial K'}{\partial C} \right] = \frac{1}{C} - \frac{b}{\gamma} E \left[ \frac{\partial [V(K')] \partial K'}{\partial C} \right] = 0 \quad (4) \]

\[ \frac{\partial V}{\partial N} = -\frac{\phi}{1-N} + bE \left[ \frac{\partial [V(K')] \partial K'}{\partial N} \right] = -\frac{\phi}{1-N} + \frac{Wb}{\gamma} E \left[ \frac{\partial [V(K')] \partial K'}{\partial N} \right] = 0 \quad (5) \]

Envelope theorem:

\[ \frac{\partial V}{\partial K} = bE \left[ \frac{\partial [V(K')] \partial K'}{\partial C} \right] = \frac{Rb}{\gamma} E \left[ \frac{\partial [V(K')] \partial K'}{\partial C} \right] \quad (6) \]

(a) Euler Equation:

Using (15) and (4), we get:

\[ \frac{\partial V}{\partial K} = \frac{R}{C} \]

Then, we get:

\[ \frac{1}{C} = \frac{b}{\gamma} E \left[ \frac{R'}{C'} \right] = 0 \]

(a) Labor supply

(5) and (4) imply:

\[ 1 - N = \frac{C \phi}{W} \quad (7) \]
2. Lagrangean:

$$\max_{\{C_t\}, \{N_t\}} \mathcal{L} = E \left( \sum_{t=0}^{t=T} b^t \{ \log(C_t) + \phi \log(1 - N_t) - \lambda_t (\gamma K_{t+1} - (R_t K_t + W_t N_t - C_t)) \} \right)$$

FOC

$$\{C_t\} : \frac{1}{C_t} = \lambda_t$$ \quad (8)

$$\{C_t\} : -\frac{\phi}{1 - N_t} = \lambda_t W_t$$ \quad (9)

From (5) and (4) we get:

$$1 - N_t = \frac{C_t \phi}{W_t}$$ \quad (10)

Take additional FOC:

$$\{K_{t+1}\} : \beta^t \gamma \lambda_t = E \left[ \beta^{t+1} \lambda_{t+1} R_{t+1} \right]$$ \quad (11)

Then, combining (16) and (11), we get our Euler equation:

$$\beta^t \gamma \frac{1}{C_t} = E \left[ \beta^{t+1} \frac{1}{C_{t+1}} R_{t+1} \right] \iff 1 = \beta^t \gamma \frac{1}{C_t} = E \left[ \frac{\beta}{\gamma} \frac{C_t}{C_{t+1}} R_{t+1} \right]$$

3) Solve the firm problem to find the labor and capital demands.

Give the law of motion for capital (re-write $K_{t+1} = R_t K_t + \tilde{W}_t N_t - \tilde{C}_t$ to get the usual equation)

This should be easy.

$$\Leftrightarrow \{N_t, K_t\} = \arg \max_{N,K} \Pi = \arg \max_{N,K} Z_t \left( K_t \right)^{1-\alpha} (N)^{\alpha} - (R_t + \delta - 1) K_t - W_t N_t$$

FOC:

$$\{N_t\} : \frac{\partial \Pi_t}{\partial N_t} = 0 \iff \alpha Z_t \left( \frac{K_t}{N_t} \right)^{1-\alpha} = W_t$$

$$\{K_t\} : \frac{\partial \Pi_t}{\partial K_t} = 0 \iff (1 - \alpha) Z_t \left( \frac{N_t}{K_t} \right)^{\alpha} = (R_t + \delta - 1)$$

4) We have a total of 5 equations. Discuss how would you proceed to log-linearize our system (you don’t need to do all the math, just apply the method to one or two equations).

There is no unique way of log-linearize things. I will use here my way (which I think is simpler and does not use any math trick you didn’t know). Let go one by one:
1. Consumers BC:

\[ \gamma K_{t+1} = R_t K_t + W_t N_t - C_t \]

\[ \iff \gamma K_{t+1} = (1 - \delta) K_t + Y_t - C_t \]

It is also true that (where $\overline{X}$ denotes the steady state of $X$):

\[ \gamma \overline{X} = (1 - \delta) \overline{X} + Y - \overline{C} \]

Then:

\[ \gamma \left( K_{t+1} - \overline{X} \right) = (1 - \delta) \left( K_t - \overline{X} \right) + \left( Y_t - \overline{Y} \right) - \left( C_t - \overline{C} \right) \]

Define \( x_t = \log (X_t) - \log (\overline{X}) \approx \left( \frac{x_t - \overline{x}}{\overline{X}} \right) \)

\[ \iff \gamma k_{t+1} = (1 - \delta) \frac{Y}{K} k_t + y_t - \frac{C}{K} c_t \quad (12) \]

But, take the production function (I'll skip some steps):

\[ \iff y_t = z_t + \alpha n_t + (1 - \alpha) k_t \]

Then (12) becomes:

\[ -\gamma k_{t+1} + \left( 1 - \delta + (1 - \alpha) \frac{Y}{K} \right) k_t - \frac{C}{K} c_t + \alpha \frac{Y}{K} n_t + \frac{Y}{K} z_t = 0 \]

2. The wage equation:

\[ \alpha Z_t \left( \frac{K_t}{N_t} \right)^{1-\alpha} = W_t \]

It is also true that

\[ \alpha Z \left( \frac{\overline{K}}{\overline{N}} \right)^{1-\alpha} = \overline{W} \]

Then

\[ Z_t \left( \frac{K_t}{N_t} \right)^{1-\alpha} = W_t \overline{W} \]

Take logs and use small letters to denote log deviations from the steady state and you get:

\[ z_t + (1 - \alpha)(k_t - n_t) = w_t \]
3. Labor supply

\[ 1 - N_t = \frac{C_t}{W_t} \]

\[ 1 - N = \frac{C\phi}{W} \]

Then,

\[ \frac{1 - N}{1 - N_t} = \frac{C_t W_t}{C_W} \]

Take logs:

\[ \log \left( \frac{1 - N}{1 - N_t} \right) = w_t - c_t \]  \hspace{1cm} (13)

Let's define \( L = 1 - N \) and \( l = \log(L) - \log(L) \).

Then, the LHS can be written as:

\[ \log \left( \frac{1 - N}{1 - N_t} \right) = \frac{1}{l} \approx \frac{1 - N}{N - N} \approx \frac{1 - N}{-nN} = \frac{n(1 - N)}{N} \]

Then, (13) becomes:

\[ \frac{n(1 - N)}{N} = w_t - c_t \]

So there is a mistake in the log-linearization that appears in the program.

4. Price of capital

As with the wage equation:

\[ (1 - \alpha) Z_t \left( \frac{N_t}{K_t} \right)^\alpha = (R_t + \delta - 1) \]

Then (after some steps)

\[ Z_t \frac{N_t}{Z} \left( \frac{N}{N_t} \frac{K}{K_t} \right)^\alpha = \frac{(R_t + \delta - 1)}{(R + \delta - 1)} \]

Take logs:

\[ z_t + \alpha n_t - \alpha k_t \approx \frac{(R_t + \delta - 1) - (R + \delta - 1)}{(R + \delta - 1)} \]

\[ = \frac{R_t - R}{(R + \delta - 1)} \approx \frac{R}{(R + \delta - 1)} r_t \]
5. Euler equation

\[ 1 = E \left[ \frac{b}{C_{t+1}} R_{t+1} \right] \]

Then,

\[ 1 = \frac{b}{C_{t+1}} R_{t+1} \]

5) Our log-linearized system is the following (small variables denote logs):

The law of motion and the market clearing conditions:

\[-\gamma k_{t+1} + \left( 1 - \delta + (1 - \alpha) \frac{Y}{K} \right) k_t - \frac{C_t}{K} c_t + \alpha \frac{Y}{K} n_t + \frac{Y}{K} z_t = 0\]
\[ \alpha \left( \frac{1 - \alpha}{\alpha} (k_t - n_t) - w_t + z_t \right) = 0 \]
\[ \frac{1}{N} \phi c_t + n_t - \frac{1}{N} w_t = 0 \]
\[ \alpha k_t + \frac{R}{R + \delta - 1} r_t - \alpha n_t - z_t = 0 \]

The Euler Equation (or any other asset pricing equation):

\[ E_t [r_{t+1} - c_{t+1}] + c_t = 0. \]

We assume

\[ z_t = \rho z_{t-1} + \epsilon_t. \]

We want to write the system of six equations in order to solve it numerically using the process in Uhlig(1997). Let \( s_t \) be the vector of endogenous state variables (capital stock), \( x_t = [c_t; r_t; n_t; w_t]' \) the endogenous control variables and \( z_t \) the stochastic processes. To avoid confusion, matrices are named using two capitalized letters. There are two conceptually different sets of equations. The first set describes the law of motion of the economy, together with the market equilibrium conditions: I use \( M_\_ \) for these matrices (Motion and Market). In the second set, the equations are forward looking and they involve the expectations of the agents: I use \( F_\_ \) for these equations (Forward). Finally, we have the equation for the stochastic process \( z_t \). Write the system in the following way:

\[ MS_1 * s_{t+1} + MS * s_t + MX * x_t + MZ * z_t = 0 \]
\[ E_t [FS_2 s_{t+2} + FS_1 s_{t+1} + FS s_t + FX x_{t+1} + FX x_t + FZ z_{t+1} + FZ z_t] = 0 \]
\[ z_{t+1} - ZZ * z_t - \epsilon_{t+1} = 0. \]
In other words, what does each matrix contain? 

Just by inspecting the log-linearize system we get:

\[
\begin{align*}
MS_1 &= \begin{bmatrix} -\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
MS &= \begin{bmatrix} (1-\delta + (1-\alpha) \frac{Y}{K}) & 0 \\ (1-\alpha) & 0 \\ 0 & \alpha \end{bmatrix}, \\
MX &= \begin{bmatrix} -\frac{C}{R} & 0 & 0 \\ 0 & 0 & -1 \frac{C}{R} \\ 0 & \frac{1-N}{R} & -\alpha \end{bmatrix}, \\
MZ &= \begin{bmatrix} \frac{Y}{K} \\ 1 \\ 0 \\ -1 \end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
FS_2 &= 0, FS_1 = 0, FS = 0, FX &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
FZ_1 &= 0, FZ = 0' \\
\end{align*}
\]

6) We want to find numerically a solution of the type 
\[
\begin{align*}
s_{t+1} &= \hat{S} \ast s_t + \hat{SZ} \ast z_t \\
x_t &= \hat{X} \ast s_t + \hat{XZ} \ast z_t, \\
\end{align*}
\]

because we are looking at small deviations from steady state. You have to solve that using matlab. I give you a program, linear, containing the function linear which does the solution of the system, so basically what you have to do is create a program containing the following:

\begin{itemize}
\item[a)] Values of the parameters of the model. We will assume
\[
\begin{align*}
\delta &= 0.025, \ \gamma = 1.004, \ \alpha = \frac{2}{3}, \ \phi = 1, \ \rho = .979, \\
R &= 1.0163, \ \frac{Y}{K} = 1/2/4, \ K = 1, \ C = Y \ast 68/100, \\
N &= .2, \ \sigma_c = .0072 \\
\end{align*}
\]

Before solving the problem, notice that we need a few clarifications here:

There are some conditions that the steady state values have to satisfy:

\[
(1-\alpha) \frac{Z}{K} = (\frac{Y}{K}) = (R + \delta - 1) \Rightarrow (\frac{Y}{K}) = \frac{(R + \delta - 1)}{(1-\alpha)} = 3 (0.0163 + 0.025) \approx \frac{1}{8.0701}
\]

So \( \left( \frac{\kappa}{\pi} \right) \) is not a parameter that you can change.

\[
\gamma \kappa = (1 - \delta) \kappa + \gamma - \bar{C} \iff \frac{\bar{C}}{\kappa} = (1 - \delta - \gamma) + \frac{\bar{Y}}{\kappa}
\]

\[
\Rightarrow \frac{\bar{C}}{\bar{Y}} = (1 - \delta - \gamma) \frac{\kappa}{\bar{Y}} + 1 \approx 8.0701(0.025 - 0.004) + 1 \approx 0.7629
\]

So the value that is in the PS is inconsistent with the budget constraint.

Sorry about that.

Moreover,

\[
1 - \bar{N} = \frac{\bar{C} \phi}{\bar{W}} = \frac{\bar{C} \phi}{\alpha \left( \frac{\bar{Y}}{\pi} \right)} = \left( \frac{\bar{C}}{\bar{Y}} \right) \phi \bar{N} \iff \bar{N} = \frac{1}{1 + \left( \frac{\bar{C}}{\bar{Y}} \right) \phi} \approx .3180
\]

So, if we want to have \( \bar{N} = 0.2 \), we need \( \phi \neq 1 \)! Fortunately, this does not affect the simulations Olivier showed in class because it is the sum of two mistakes that cancel out each other. In any case, keep in mind that if you change \( \bar{N} \) you have to change \( \phi \). This is completely intuitive because \( \phi \) is a measure of how much you value leisure. Notice that if \( \phi = 0 \), then \( \bar{N} = 1 \).

b) The matrices derived in 5).

c) Then you have to call the function linear.

7) Now we want to do impulse-response functions. For that, attach to your program (copy and paste after part c)) the program linear 2. This program calls two other functions, impulse and impulse-plot. The first finds the reduced form model and the second hits it with a shock. That should be the end of your program. Explain the results. How do they depend on the values assumed in a)? (run the program for different values of at least two parameters and compare results). Interpret.

Several exercises can be implemented:

- Base case (Figure 1)–Minor differences with the figure Olivier showed in class, same intuition.
Baseline case

- Change $\rho$ to 0.999 and 0.300 (Figures 2 and 3)
\( \rho = 0.9 \)

- Change \( N = 0.8 \) (Figure 4).

\( N = 0.8 \)

- \( R = 1.1 \) (Figure 5)
\( R = 1.1 \)

- \( \gamma = 1.01 \) (Figure 6)

3 (Optional) Discrete Time Dynamic Program-
Use the discrete time setup of Philippon and Segura-Cayuela (2003) (up to section 5) and the MATLAB programs contained in the folder DynamicProgramming. Read tutorial.doc and the comments in DP.m.

1) Derive the Euler equation in discrete time in the model with exogenous labor, first using dynamic programming and then using the Lagrange multiplier method. Compare it to the continuous time Euler equation.

The problem is:

\[
\begin{align*}
\max L_{(C_t)} &= \sum_{t=0}^{\infty} b^t \left( \frac{C_t}{\sigma - 1} \right)^{\frac{\sigma - 1}{\sigma}} \\
\text{s.t.} & \quad \tilde{K}_{t+1} = R_t \tilde{K}_t + \tilde{W}_t - \tilde{C}_t \\
A_{t+1} &= \gamma A_t
\end{align*}
\]

Divide all the trending variables by \( A_t \). Define \( V = \frac{\tilde{V}}{A} \) and notice that \( A_t = A_0 \gamma^t \).

Utility function:

\[
\begin{align*}
\max V &= \max \sum_{t=0}^{\infty} b^t \left( C_t A_0 \gamma^t \right)^{\frac{\sigma - 1}{\sigma}} \\
&= \max A_0^{\frac{\sigma - 1}{\sigma}} \sum_{t=0}^{\infty} \left( b \gamma^t \right)^t \left( C_t \right)^{\frac{\sigma - 1}{\sigma}} \\
&= \max \sum_{t=0}^{T} \left( \beta^t \right) \left( C_t \right)^{\frac{\sigma - 1}{\sigma}}
\end{align*}
\]

The budget constraint (same as before):

\[
\gamma K_{t+1} = R_t K_t + W_t - C_t
\]

Thus, the problem becomes \( \max \sum_{t=0}^{T} \left( \beta^t \right) \left( C_t \right)^{\frac{\sigma - 1}{\sigma}} \) s.t. \( \gamma K_{t+1} = R_t K_t + W_t - C_t \)

- Bellman equation

\(^1\)This exercise is optional in the sense that you could skip it if you feel comfortable with stochastic DP, and do it if you want to get more training.
\[ V(K) = \max_C \left\{ t \frac{\sigma}{\sigma - 1} (C^{\frac{\sigma}{\sigma - 1}}) + \beta \mathbb{E}[V(K')] \right\} \]

s.t.

\[ \gamma K' = RK + WN - C \]

FOC:

\[ \frac{\partial V}{\partial C} = C^{-\frac{1}{\sigma}} + \beta \mathbb{E} \left[ \frac{\partial [V(K')] \partial K'}{\partial C} \right] = C^{-\frac{1}{\sigma}} - \frac{\beta}{\gamma} \mathbb{E} \left[ \frac{\partial [V(K')]}{\partial K'} \right] = 0 \quad (14) \]

Envelopment theorem:

\[ \frac{\partial V}{\partial C} = \beta \mathbb{E} \left[ \frac{\partial [V(K')] \partial K'}{\partial C} \right] = \frac{R \beta}{\gamma} \mathbb{E} \left[ \frac{\partial [V(K')]}{\partial K'} \right] \quad (15) \]

Using (15) and (4), we get:

\[ \frac{\partial V}{\partial K} = RC^{-\frac{1}{\sigma}} \]

Then, we get:

\[ 1 = \frac{\beta}{\gamma} \mathbb{E} \left[ R' \left( \frac{C'}{C} \right)^{-\frac{1}{\sigma}} \right] \]

But notice that:

\[ \frac{\beta}{\gamma} = b \gamma^{-\frac{1}{\sigma}} \]

Then

\[ 1 = b \mathbb{E} \left[ R' \left( \frac{\gamma C'}{C} \right)^{-\frac{1}{\sigma}} \right] \]

• Lagrangean:

\[ \max_{\{C_t\}} \mathbb{E} \sum_{t=0}^{t=\infty} \beta^t \left\{ \frac{\sigma}{\sigma - 1} (C_t^{\frac{\sigma}{\sigma - 1}}) - \lambda_t (\gamma K_{t+1} - (R_t K_t + W_t N_t - C_t)) \right\} \]

FOC

\[ \{C_t\} : C_t^{\frac{1}{\sigma}} = \lambda_t \quad \text{(16)} \]

\[ \{K_{t+1}\} : \beta^t \gamma \lambda_t = \mathbb{E} [\beta^{t+1} \lambda_{t+1} R_{t+1}] \quad \text{(17)} \]

Then, combining (16) and (11), we get our Euler equation:

\[ \beta^t \gamma C_t^{\frac{1}{\sigma}} = \mathbb{E} [\beta^{t+1} C_{t+1}^{\frac{1}{\sigma}} R_{t+1}] \Leftrightarrow 1 = \mathbb{E} \left[ \frac{\beta}{\gamma} \left( \frac{C_t}{C_{t+1}} \right)^{\frac{1}{\sigma}} R_{t+1} \right] \]
Continuous time

\[
\max_{\{C_t\}} \int_0^\infty \frac{\sigma}{\sigma - 1} \left(\tilde{C}_t\right)^{\sigma-1} \exp(-\theta t) \, dt
\]

s.t.

\[
\tilde{K}_t = (R_t - 1)\tilde{K}_t + \tilde{W}_t - \tilde{C}_t
\]

\[
A_{t+1} = \gamma A_t
\]

Detrending:

\[
\max \int_0^\infty \frac{\sigma}{\sigma - 1} \left(\tilde{C}_t\right)^{\sigma-1} \exp(-\theta t) \, dt = \max \int_0^\infty \frac{\sigma}{\sigma - 1} \left(C_tA_t\right)^{\sigma-1} \exp(-\theta t) \, dt
\]

\[
= \max A_0^{\sigma-1} \int_0^\infty \frac{\sigma}{\sigma - 1} \left(\gamma t\right)^{\sigma-1} \exp(-\theta t) \, dt
\]

Notice that \((\gamma)^{\sigma-1} = \exp \left(\log (\gamma^{\sigma-1})\right) = \exp \left(\left(\frac{\sigma-1}{\sigma}\right) \log (\gamma)\right) \approx \exp \left(\left(\frac{\sigma-1}{\sigma}\right) \mu\right),
\]

Then

\[
\max A_0^{\sigma-1} \int_0^\infty \frac{\sigma}{\sigma - 1} \left(C_t\right)^{\sigma-1} \left(\gamma t\right)^{\sigma-1} \exp(-\theta t) \, dt
\]

\[
= \max \int_0^\infty \frac{\sigma}{\sigma - 1} \left(C_t\right)^{\sigma-1} \exp \left(- \left(\theta - \left(\frac{\sigma-1}{\sigma}\right) \mu\right) t\right) \, dt
\]

\[
= \max \int_0^\infty \frac{\sigma}{\sigma - 1} \left(C_t\right)^{\sigma-1} \exp(-\rho t) \, dt
\]

with \((\theta - \left(\frac{\sigma-1}{\sigma}\right) \mu) \equiv \rho
\]

The budget constraint becomes

\[
K_t = (R_t - \mu - 1)K_t + W_t - C_t
\]

Then the Euler equation will be:

\[
\frac{\tilde{C}_t}{C_t} = \sigma(R_t - \mu - \rho - 1) = \sigma(R_t - \mu - \left(\theta - \left(\frac{\sigma-1}{\sigma}\right) \mu\right) - 1)
\]

\[
= \sigma(R_t - \frac{\mu}{\sigma} - \theta - 1)
\]

18
2) Numerical analysis using $DP.m$. Describe the dynamics of the economy when it starts far away from its steady state (say 30% below). How does the recovery depend on the evolution of productivity? What does it tell you about the rapid growth in Europe after the second world war? (Hint: for the question on the evolution of productivity you may want to replicate the model, say 100 times, and see what happens with the recovery path)

See figures

- Evolution of output after the recovery (Figure 7):

- Mean and median and max and min in 100 simulations (Figures 8 and 9)
3) $DP.m$ produces 5 plots. Describe briefly each of them (what they represent and how they are computed. I am not asking for the details of the computations.
Simply show me that you understand the architecture of the program).

- Figure 1: 3D graph showing optimal consumption as a function of realizations of Z and initial capital stock. By-product of the search for the policy rule.
- Figure 2: Policy rule. $K_{t+1}$ as a function of $K_t$. Capital in the steady state goes from 4.6 to around 5.4.
- Figure 3: Evolution of consumption in a stochastic simulation after starting from some initial capital that is below the steady state value.
- Figure 4: Evolution of consumption in the same simulation.
- Figure 5: Stationary distribution of capital in the steady state.

5) Numerical analysis using $DP.m$. We have seen in the continuous time setup that the reaction of consumption to a positive technology shock depends on the elasticity $\sigma$. Use the policy function estimated by $DP.m$ to illustrate this finding. (note: in the program, they define $\gamma$, the CRRA instead of $\sigma$. How are they related? Why?) Hint: $DP.m$ produces 5 plots. The first one is called consumption, and it is 3 dimensional. Consumption is measured on the vertical axis. The two horizontal axis are for $K$ and $Z$. If you do not see well, you can rotate the picture with the mouse.

Notice that $CRRA = \sigma^{-1}$

- Different $\sigma = \{0.1, 1, 10\}$ Figures 10 to 12.
$\sigma = 0.1$

$\sigma = 1$

CONSUMPTION, CRRA=0.1
6) Show numerically how the behavior of consumption is affected by the persistence of the shocks. Interpret.

- Different $\rho = \{0.05, 0.99\}$
\rho = 0.99