Answer keys for problem set 3

Exercise 7.2
Start by drawing the indifference curves to check what’s going on with this utility.

\[
\max_{x_1, x_2} \max \{x_1, x_2\} \\
p_1 x_1 + p_2 x_2 \leq m
\]

- if \( p_2 \geq p_1 \) then \( x_1 = \frac{m}{p_1}, x_2 = 0 \)
- if \( p_2 \leq p_1 \) then \( x_1 = 0, x_2 = \frac{m}{p_2} \)

The indirect utility function is

\[
v(p, m) = \frac{m}{\min(p_1, p_2)}
\]

Use the fact that \( v(p, e(p, u)) = u \) to invert it and find the expenditure function

\[
e(p, u) = \min(p_1, p_2) u
\]

Exercise 7.4
By Roy’s equality

\[
x_i(p, m) = -\frac{\partial u}{\partial p_i} = \frac{m}{p_1 + p_2}
\]

Invert the indirect utility function to find the expenditure function \( (v(p, e(p, u)) = u) \)

\[
e(p, u) = (p_1 + p_2) u
\]

This expenditure function shows that to get \( u \) utils it requires to buy \( u \) units of good 1 and \( u \) units of good 2. Hence the direct utility function is (up to an increasing transformation)

\[
u(x_1, x_2) = \min(x_1, x_2)
\]
Exercise 7.6
This consumer has homothetic preferences. Indeed his demand functions are
\[ x_i = -\frac{\partial \ln A(p)}{\partial p_i} m \]
showing that his indifference curves are parallel (along a ray through the origin the indifference curves have the same slope, increase the income by a factor \( t \) and his optimal consumption of any good is increased by the same factor \( t \)).

Invert the indirect utility function to find the expenditure function \( (v(p, e(p, u)) = u) \)
\[ e(p, u) = \frac{u}{A(p)} \]

The indirect money metric utility is the income necessary to achieve the same level of utility when facing the prices \( p \) as when facing prices \( q \) and having an income equal to \( m \).
\[ \mu(p; q, m) = e(p, v(q, m)) = \frac{A(q)}{A(p)} m \]

If \( v(p, m) = A(p) m^b \) then
\[ e(p, u) = \left( \frac{u}{A(p)} \right)^{\frac{1}{b}} \]
and the indirect money metric becomes
\[ \mu(p; q, m) = e(p, v(q, m)) = \left( \frac{A(q)}{A(p)} \right)^{\frac{1}{b}} m \]

Exercise 8.2
The demand functions for a Cobb Douglas utility are:
\[ x_1 = \frac{am}{p_1} \]
\[ x_2 = \frac{(1-a) m}{p_2} \]
The substitution matrix is:
\[ \begin{pmatrix}
\frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial m} x_1 & \frac{\partial x_2}{\partial p_1} + \frac{\partial x_2}{\partial m} x_1 \\
\frac{\partial x_1}{\partial p_2} + \frac{\partial x_1}{\partial m} p_2 & \frac{\partial x_2}{\partial p_2} + \frac{\partial x_2}{\partial m} p_2 \\
\end{pmatrix} = \begin{pmatrix}
-a(1-a) \frac{m}{p_1} & a(1-a) \frac{m}{p_1 p_2} \\
as(1-a) \frac{m}{p_1} & -(1-a) \frac{m}{p_2} \\
\end{pmatrix} \]
This matrix is indeed symmetric and negative semidefinite as the diagonal elements are negative and the determinant is equal to 0.
Exercise 8.4
From the property of a "cost function", we know that
\[
\frac{\partial e(p,u)}{\partial p} = h(p,u)
\]
and we also have
\[
x(p, e(p,u)) = h(p,u)
\]
which implies
\[
\frac{\partial e(p,u)}{\partial p} = x(p, e(p,u))
\]
Let’s pick a point \(x^0 = x(p^0, m^0) = e^{ap^0 + bm^0 + c}\) and set the utility to \(u^0\) at this point. The expenditure function solves
\[
\frac{\partial e(p,u^0)}{\partial p} = e^{ap^0 + bm^0 + c}
\]
\[
e(p^0, u^0) = p^0 x^0 = m^0
\]
The solution is
\[
e(p, u^0) = m^0 - \frac{1}{b} \ln \left[ 1 - \frac{b}{a} x^0 (e^{a(p^0 - p^0)} - 1) \right]
\]

Exercise 8.6
After the increasing transformation \(g(u) = u^a\), the utility is a Cobb Douglas utility with \(a = \frac{a}{b}\). Hence the demand functions are
\[
x_1(p, m) = \frac{3}{5} \frac{m}{p_1}
\]
\[
x_2(p, m) = \frac{2}{5} \frac{m}{p_2}
\]
Plug those demand functions back in the utility to get the indirect utility function
\[
v(p, m) = \left( \frac{m}{5} \right)^{\frac{a}{b}} \left( \frac{3}{5} \frac{m}{p_1} \right)^{\frac{1}{b}} \left( \frac{2}{5} \frac{m}{p_2} \right)^{\frac{1}{b}}
\]
Invert to get the expenditure function
\[
e(p, u) = 5u^{\frac{a}{b}} \left( \frac{p_1}{3} \right)^{\frac{1}{b}} \left( \frac{p_2}{2} \right)^{\frac{1}{b}}
\]
Differentiate wrt \(p\) to get the Hicksian demand functions
\[
h_1(p, u) = u^a \left( \frac{3 p_2}{2 p_1} \right)^{\frac{1}{b}}
\]
\[
h_2(p, u) = u^a \left( \frac{2 p_1}{3 p_2} \right)^{\frac{1}{b}}
\]
Exercise

Let’s add some assumptions to solve this exercise. Each generation lives for one period during which they have offspring. A female can only have $N$ (choose $N = 2$ if you want) babies, otherwise the optimal solution would be to have an infinite number of babies. It is also implicit that the mother actually picks the gender of her babies.

When a female decides the gender of her baby, she knows that her offspring will maximize the number of their descendants. Hence she only needs to focus on how easy it will be for her offspring to become parent. That is to say how easy will it be for a male to find a female and vice versa. To translate that in utility, a mother maximizes the number of expected grandchildren.

A female also thinks that her own choices will not change the total proportion of male/female in the society. Call $x$ the proportion of male in the next generation.

- $x > 1 - x$, in the next generation there will be more males than females. The probability for a male to find a female is $\frac{1}{1-x}$, but the probability for a female to find a male is 1. Therefore having one son corresponds to $\frac{1}{1-x}N$ grandchildren in expectation and having one daughter to $N$. The mother is maximizing the number of her grandchildren, she will pick to have only daughters.

  Every mother chooses to have daughters, and the proportion of male in the next generation will be $x = 0$, which contradicts the assumption that $x > 1 - x$.

- $x < 1 - x$, in the next generation there will be fewer males than females.

  Let’s imagine that in this society you cannot have a shortage of males. Therefore having one son corresponds to having more than $N$ grandchildren in expectation and having one daughter to $N$. The mother is maximizing the number of her grandchildren, she will pick to have only sons.

  Each mothers pick to have sons, the proportion of male in the next generation will be $x = 1$, which contradicts the assumption that $x < 1 - x$.

- $x = 1 - x$, i.e. $x = 0.5$, males and females are in equal proportion. In that case a mother is strictly indifferent between having a son or a daughter, which leads to $x = 0.5$. 
