**Instructions**

1. The exam is **closed book. Calculators are not permitted.**

2. Answer all questions on this exam book in the spaces that are marked. Nothing will be graded outside of those spaces.

3. If you want to use scrap paper, please use the backside of papers on the exam.

4. Justifications are only needed for those problems for which we ask it. For all other problems, justifications need not be given, nor will they result in partial credit. (Please check your arithmetic carefully to avoid careless mistakes. A careless mistake may result in no credit.)

5. Budget your time. If a problem (or a part of a problem) is taking too long, you may want to go on to the next one.

6. If you think that there is an ambiguity on a question, please state the assumption you are making inside the space provided for the solution. Try to make the assumption reasonable.

7. The exam sums to 100 points.

Name: ____________________________________________
<table>
<thead>
<tr>
<th>Problem</th>
<th>Max</th>
<th>Subtracted</th>
<th>Assigned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td>12</td>
<td></td>
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<td>3</td>
<td>14</td>
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<td>4</td>
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<td>5</td>
<td>6</td>
<td></td>
<td></td>
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<td>6</td>
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<td>12</td>
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<tr>
<td>Midterm</td>
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</tr>
</tbody>
</table>
Problem 1: (12 points)

Look at the figure that follows and answer the set of questions based on the figure. The feasible region is “striped”. The feasible region is infinite. The constraints are as follows:

\[
\begin{align*}
  x - 2y & \leq 4 \\
  x - y & \geq -2 \\
  x & \geq 0; \quad y \geq 0
\end{align*}
\]

a) (4 points) Please list all corner points, if there are none, please write “none”

Corner points (or “none”):

b) (4 points) Please give an example of an objective function such that both (2,0) and (0,0) are optimal, but (1,1) is not optimal (assume you are maximizing).

An Objective Function:

An Objective Function:

c) (4 points) Please give an example of an objective function such that the optimal solution value is unbounded above (assume you are maximizing).

An Objective Function:
Problem 2 (12 points)

Ulaanbaatar City National Bank is open Monday-Friday from 10 a.m. to 4 p.m. From past experience the bank knows that it needs the number of tellers shown in the following table.

<table>
<thead>
<tr>
<th>Time Period</th>
<th>Tellers Required</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-11</td>
<td>5</td>
</tr>
<tr>
<td>11-12</td>
<td>4</td>
</tr>
<tr>
<td>12-1</td>
<td>6</td>
</tr>
<tr>
<td>1-2</td>
<td>5</td>
</tr>
<tr>
<td>2-3</td>
<td>4</td>
</tr>
<tr>
<td>3-4</td>
<td>8</td>
</tr>
</tbody>
</table>

The bank hires two types of tellers, full-time and part-time.

Full-time tellers work 10-4, except for 1 hour off for lunch. The bank determines when a full-time employee takes lunch hour, but each teller must go between noon and 1 p.m. or between 1 p.m. and 2 p.m. Full-time employees are paid $8/hour (they do get paid for lunch hour).

The bank may also hire part-time tellers. Each part-time teller must work exactly 3 consecutive hours each day. A part-time teller is paid $7/hour.

You may ignore any integrality constraints in this problem.

a) (5 points) Formulate an LP to meet the teller requirements at minimum cost.
The bank feels that the part-time tellers are badly trained and their services are just not as good as those of full-time tellers. To maintain adequate quality of service, the bank has decided that at most 25% of workers may be part-time.

b) (3 points) Formulate this additional constraint.

Now assume that it is legitimate to schedule fewer tellers than the specified requirements; however, the bank assesses a penalty of $10 for every teller short of meeting the requirements in an hour.

c) (4 points) With this new information, formulate an LP that minimizes the cost of workers and penalties. (You may ignore the additional constraint from b).
Problem 3: (14 points)

Consider the following tableau obtained while solving a linear maximization problem with the simplex algorithm.

<table>
<thead>
<tr>
<th></th>
<th>z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>x₆</th>
<th>x₇</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>w₄</td>
<td>w₅</td>
<td>w₆</td>
<td>0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-3</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

a) (2 points) What is the basic feasible solution associated with this tableau? Give the values of all of the decision variables as well as the objective value.

The basic feasible solution is:

b) (3 points) Now assume that the tableau is optimal. Which of the following statements could be true:
   i. there exists a unique optimal solution
   ii. there are multiple optimal solutions
   iii. \( w₄ + w₅ + w₆ < 0 \).

List all statements that could be true, or write “none” if none of the statements could be true:

c) (3 points) Assume that the current solution is non-optimal and that \( x₆ \) is the variable to be pivoted into the basis. What is the variable that is pivoted out of the basis?

The variable is:
d) (3 points) What is the column for the RHS in the tableau following the pivot of variable $x_6$ into the basis?

The RHS column:

<table>
<thead>
<tr>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>


e) (3 points) Is it possible that the feasible region for the LP is unbounded? Please provide a short explanation for your answer. (HINT: look at the constraints one at a time.)

Is it possible that the feasible region for the LP is unbounded? (please circle one)

YES

NO

A short explanation:
Problem 4: (24 points)

a) (2 points) If the feasible region is two dimensional and 3 corner points are optimal then:
   i. These are the only optimal points
   ii. Only these points and points along the line segments that join the corners are optimal.
   iii. All points in the feasible region are optimal
   iv. None of the above.

Please circle your answer:
   i  ii  iii  iv

b) (2 points) If one of the tableaus for the simplex method is degenerate, then every tableau for the simplex method is degenerate.

Please circle your answer:
   True  False

c) (4 points) Consider the following optimization problem:

\[
\begin{align*}
\text{Min } & \ (\text{Max} \{ 2x + 3, |x-ay| , 3x+3y\}) \\
\text{subject to } & \ x + y = 16 \\
& \ x \geq 0, y \geq 0.
\end{align*}
\]

Assume that \(a\) is some positive constant. Show how to convert the above optimization problem to a linear program. Do not solve it.
d) (2 points) Every feasible linear program in standard form has at least one corner point.

Please circle your answer:

| True | False |

- True
- False

e) (2 points) The second best basic feasible solution to an LP with a single optimal solution must always be one pivot away from the optimal solution. (Assume the feasible region is bounded and no basis is degenerate.)

Please circle your answer:

| True | False |

- True
- False

f) (2 points) If there are no degenerate bases, then the simplex algorithm for a maximization problem is finite regardless of how one chooses the entering variable among those with negative z-row coefficient.

Please circle your answer:

| True | False |

- True
- False

g) (2 points) It is possible that a basic feasible solution is not a corner point.

Please circle your answer:

| True | False |

- True
- False

h) (2 points) If there is a feasible solution to the original problem, then the optimal solution value to the phase 1 problem must be 0.

Please circle your answer:

| True | False |

- True
- False

i) (2 points) Assume that there are no degenerate bases. If there is a non basic variable whose reduced cost is zero at an optimal point, then multiple optimal solutions exist.

Please circle your answer:

| True | False |

- True
- False
j) (4 points) The following tableau was obtained during the simplex algorithm.

<table>
<thead>
<tr>
<th></th>
<th>z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Is the tableau optimal?

Please circle your answer:

Yes  No

If your answer was yes, please explain why (1-2 sentences). If your answer was no, please use the simplex algorithm to find the optimal tableau.

An explanation or tableau(s) obtained from the simplex algorithm:
Problem 5: (6 points)

JB was performing the simplex method; however, at exactly one intermediary step, he made the following error: instead of pivoting in a variable with a negative objective row coefficient, he instead pivoted in a variable with a positive objective row coefficient. He chooses the exiting variable using the usual minimum ratio test. Assume after this initial error JB continues to solve the problem using the simplex method and does not make any additional errors. Also assume that no basis is degenerate.

HINT: Some of these are easier to solve by thinking algebraically about what is happening, others are easier to solve by thinking geometrically about what is happening. Thus we recommend if you can’t figure out a question using one approach try the other.

a) (2 points) It is possible that the pivot that JB took will result in a basic solution that is not feasible.

Please circle your answer:

| True | False |

b) (2 points) It is possible that the pivot JB took will result in an optimal basis.

Please circle your answer:

| True | False |

c) (2 points) It is possible that there are fewer simplex pivots with the error than if the error had not been made.

Please circle your answer:

| True | False |
Problem 6: (20 points)

The NCAA is making plans for distributing tickets to the upcoming regional basketball championships. Up to 10,000 available seats will be divided between the media, the competing universities, and the general public, subject to the following conditions:

- Media people are admitted free, and at least 1000 tickets must be reserved for the media.
- The NCAA receives $45 per ticket from competing universities and $100 per ticket from the general public.
- At least half as many tickets should go to competing universities as the general public.

With these restrictions, the NCAA proposes the following LP to maximize their profit:

\[
\begin{align*}
\text{max} & \quad 45x_2 + 100x_3 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \leq 10 \\
& \quad x_2 - 0.5x_3 \geq 0 \\
& \quad x_1 \geq 1 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Here, \(x_1\) is the number of media tickets (in thousands), \(x_2\) is the number of university tickets (in thousands), and \(x_3\) is the number of public tickets (in thousands). The profit is in $1000s.

On solving this problem, the NCAA obtains the following sensitivity report:

<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
<th>Reduced</th>
<th>Objective</th>
<th>Allowable Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Media Tickets</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>81.667</td>
<td>1E+30</td>
<td></td>
</tr>
<tr>
<td>University Tickets</td>
<td>3</td>
<td>0</td>
<td>45</td>
<td>55</td>
<td>245</td>
<td></td>
</tr>
<tr>
<td>Public Tickets</td>
<td>6</td>
<td>0</td>
<td>100</td>
<td>1E+30</td>
<td>55</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
<th>Shadow</th>
<th>Constraint</th>
<th>Allowable R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacity Constraint</td>
<td>10</td>
<td>81.667</td>
<td>10</td>
<td>1E+30</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Univ./Public Constraint</td>
<td>0</td>
<td>-36.667</td>
<td>0</td>
<td>9</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>Media Constraint</td>
<td>1</td>
<td>-81.667</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

a) (2 points) What is the optimal value of the decision variables, and what is the optimal profit?

\[
x_1 = \quad x_2 = \quad x_3 = \quad \text{profit (in $1000s)} =
\]
b) (3 points) Suppose there is an alternate arrangement of the stadium that can provide 20,000 seats in total, and thus 20,000 tickets may be sold. How much additional revenue would be gained from the expanded seating?

Additional revenue (in $1000s) =

---

c) (3 points) In order to make the games more affordable for fans, another proposal would reduce the price of public tickets to $50. (Ignore part b here.) Will the same solution be optimal if we implement this change? (By ‘solution’, we mean the values of the decision variables.) If the same solution is used, how much revenue would we lose?

Is the solution still optimal? (please circle one)

Yes
No

Lost revenue (in $1000s) =

---

d) (4 points) To accommodate high demand from student supporters of participating universities, the NCAA is considering marketing a new “scrunch seat” that costs $15 and consumes only 50% of a regular seat, but counts fully against the “university ≥ half public” rule. (So there can be two tickets sold for each available seat.)

Let $x_4$ indicate the number of seats of this type (in thousands). Formulate the revised LP.
e) (4 points) What is the reduced cost of $x_4$, and would selling such seats be profitable? (Use the information in part d and the sensitivity report.)

Reduced cost of $x_4$ (you do not need to simplify):

Would selling such seats be profitable? (please circle one)

Yes  No

f) (4 points) Suppose that the number of seats is changed to 13,000. What would be the optimal solution? (Hint: $x_1$ will still be equal to 1.) What is the optimal profit? Verify that the change in the optimal solution value is what would have been predicted from the Excel Sensitivity Report.

Optimal solution:

$x_1 =$  $x_2 =$  $x_3 =$

Optimal profit (in $1000s) =

Change in optimal profit (in $1000s) =

Change predicted by sensitivity report =
Problem 7: (12 points)

Consider the following linear program:

\[
\begin{align*}
\text{max } & \quad 3x_1 + x_2 \\
\text{s.t. } & \quad -2x_1 + x_2 + x_3 = 2 \\
& \quad x_1 + x_2 + x_4 = 6 \\
& \quad x_1 + x_5 = 4 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*}
\]

The initial simplex tableau is:

<table>
<thead>
<tr>
<th></th>
<th>z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Solving this problem, we obtain the following optimal tableau:

<table>
<thead>
<tr>
<th></th>
<th>z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

a) (4 points) What is the shadow price of the second constraint, \(x_1 + x_2 + x_4 = 6\)? And what is the range of the change \(\Delta\) to the RHS for which this shadow price is valid?

Shadow Price =

Range of \(\Delta\):

b) (4 points) Suppose the cost coefficient of variable \(x_5\) changes to \(\Delta\). What is the range of \(\Delta\) such that the current basis remains optimal?

Range of \(\Delta\):

c) (4 points) Suppose the cost coefficient of variable \(x_2\) changes to \(1 + \Delta\). What is the range on \(\Delta\) such that the current basis remains optimal?
Range of $\Delta$: 