LECTURE 16
OTHER TOPICS IN RESOURCE ALLOCATION

Integer Programming
Analyzing Queues in Project Management

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Today’s conversation ...

Integer Programming as a Tool for Resource Allocation in Project Management
- Activities only
- Activities and Resources

Queuing in Project Management
- Examples of Queuing Systems
- Components of a Queuing System
- The Capacity of a Queue
- Queuing System Performance
- Attributes of a Queuing System
PART 1

Integer Programming for Resource Allocation in Project Management
Consider the Following Simple Project Activity Diagram (Activities Only)

Types of relationships

1. Complementary: $X_i$ should be done with (before or after $X_j$ )
2. Substitutionary: Either $X_i$ must be done or $X_j$ must be done.
3. Precedence: $X_i$ must be done before $X_j$
   \[ X_i \text{ must be done after } X_j \]
4. Mandatory: $X_i$ must be done “no matter what”
Benefit $= b_{X_2}$
Cost $= c_{X_2}$

Benefit $= b_{X_3}$
Cost $= c_{X_3}$

Benefit $= b_{X_1}$
Cost $= c_{X_1}$

Benefit $= b_{X_4}$
Cost $= c_{X_4}$

Benefit $= b_{X_5}$
Cost $= c_{X_5}$

Benefit $= b_{X_6}$
Cost $= c_{X_6}$

Diagram showing relationships between $X_1$, $X_2$, $X_3$, $X_4$, $X_5$, and $X_6$. The diagram illustrates the flow of benefits and costs between these variables. 
Examples of Benefits: Equipment Utilization (%), Labor Utilization, etc.

Benefit = $b_{X_1}$
Cost = $c_{X_1}$

Benefit = $b_{X_2}$
Cost = $c_{X_2}$

Benefit = $b_{X_3}$
Cost = $c_{X_3}$

Benefit = $b_{X_4}$
Cost = $c_{X_4}$

Benefit = $b_{X_5}$
Cost = $c_{X_5}$

Benefit = $b_{X_6}$
Cost = $c_{X_6}$
Examples of Benefits: Equipment Utilization (%), Labor Utilization, etc.
Examples of Costs: Monetary ($), Duration (days), etc.

Benefit = $b_{X2}$
Cost = $c_{X2}$

Benefit = $b_{X3}$
Cost = $c_{X3}$

Benefit = $b_{X1}$
Cost = $c_{X1}$

Benefit = $b_{X4}$
Cost = $c_{X4}$

Benefit = $b_{X5}$
Cost = $c_{X5}$

Benefit = $b_{X6}$
Cost = $c_{X6}$
Examples of Benefits: Equipment Utilization (%), Labor Utilization, etc.
Examples of Costs: Monetary ($), Duration (days), etc.
Useful to convert all benefits and cost into a single unit
Examples of Benefits: Equipment Utilization (%), Labor Utilization, etc.
Examples of Costs: Monetary ($), Duration (days), etc.
Useful to convert all benefits and cost into a single unit

Benefit $= b_{X2}$
Cost $= c_{X2}$

Benefit $= b_{X3}$
Cost $= c_{X3}$

Benefit $= b_{X4}$
Cost $= c_{X4}$

Benefit $= b_{X5}$
Cost $= c_{X5}$

Benefit $= b_{X6}$
Cost $= c_{X6}$

Scaling,
Metricization
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<tr>
<th>Activity</th>
<th>Benefit (Utility)</th>
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Some Mathematical Notations and Formulations

(a) Constraints
(b) Objective function
(a) Constraints
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$X_i = 1, 0 \quad i = 1, 2, \ldots, 6$
Activity Benefit

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$X_i = 1,0 \quad i = 1,2,\ldots,6$  
Integer programming formulation. Carry out Activity $i$ or do not.
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$X_1 = 1$  
Activity 1 is mandatory
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$X_6 = 1$  

Activity 6 is mandatory
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<td>TOTAL</td>
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Carry out Activity 2 or 4 but not both.
Activity | Benefit (Utility) | Cost (Dis-utility)  
---|---|---  
$X_1$ | $b_{x1}$ | $c_{x1}$  
$X_2$ | $b_{x2}$ | $c_{x2}$  
$X_3$ | $b_{x3}$ | $c_{x3}$  
$X_4$ | $b_{x4}$ | $c_{x4}$  
$X_5$ | $b_{x5}$ | $c_{x5}$  
$X_6$ | $b_{x6}$ | $c_{x6}$  
TOTAL | $B$ | $C$  

$x_2 + x_4 = 1$ 

Carry out Activity 2 or 4 but not both.
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\[
\frac{1}{4} \sum_{i=1}^{4} b_{X_i} \geq b^* 
\]
The average benefit of all selected activities should be at least $b^*$

$$\frac{1}{4} \sum_{i=1}^{4} b_{X_i} \geq b^*$$
The average cost of all selected activities should not exceed $c^*$.

\[
\frac{1}{4} \sum_{i=1}^{4} c_{X_i} \leq c^*
\]
\[ \min(\tilde{b}_{X_i}) = b^* \]

The least benefit of any selected activity should be \( b^* \)

OR

No activity selected should have a benefit that is less than \( b^* \)
\[
\max(\tilde{c}_{X_i}) = c^*
\]

The highest cost of any selected activity should be \( c^* \)

OR

No activity selected should have a cost that is more than \( c^* \)
(b) Objective Function
Objective Function

*What is our goal?*

*What are we seeking to maximize/minimize?*

- Maximize the sum of all benefits
  
  *E.g. The set of activities that involve the lowest total duration*

- Minimize the sum of all costs
  
  *E.g. The set of activities that involve the least resources*

- Maximize all benefits and minimize all costs
Objective Function

*What is our goal?*

*What are we seeking to maximize/minimize?*

- Maximize the sum of all benefits
  
  *E.g. The set of activities that involve the lowest total duration*

- Minimize the sum of all costs
  
  *E.g. The set of activities that involve the least resources*

- Maximize all benefits and minimize all costs

  \[
  OBJ = B + (-C)
  \]

  *Linear additive*
Objective Function

What is our goal?

What are we seeking to maximize/minimize?

- Maximize the sum of all benefits
  
  \[ OBJ = B + (-C) \]

  \[ OBJ = 1B - 1C \]

  Linear additive

  Linear additive, equal weight of 1

- Minimize the sum of all costs

  \[ OBJ = B + (–C) \]

  \[ OBJ = 1B - 1C \]

  Linear additive

- Maximize all benefits \textbf{and} minimize all costs

  \[ OBJ = B + (–C) \]

  \[ OBJ = 1B - 1C \]
Objective Function

What is our goal?

What are we seeking to maximize/minimize?

- Maximize the sum of all benefits
  
  \[ E.g. \text{The set of activities that involve the lowest total duration} \]

- Minimize the sum of all costs
  
  \[ E.g. \text{The set of activities that involve the least resources} \]

- Maximize all benefits \textbf{and} minimize all costs

\[
OBJ = B + (-C) \quad \text{Linear additive}
\]
\[
OBJ = 1B - 1C \quad \text{Linear additive, equal weight of 1}
\]
\[
OBJ = w_B B - w_C C \quad \text{Linear additive, non-equal weights}
\]
Objective Function

What is our goal?

What are we seeking to maximize/minimize?

- Maximize the sum of all benefits
  
  E.g. The set of activities that involve the lowest total duration

- Minimize the sum of all costs
  
  E.g. The set of activities that involve the least resources

- Maximize all benefits and minimize all costs

  \[ OBJ = B + (-C) \]  \hspace{1cm} \text{Linear additive}

  \[ OBJ = 1B - 1C \]  \hspace{1cm} \text{Linear additive, equal weight of 1}

  \[ OBJ = w_B B - w_C C \]  \hspace{1cm} \text{Linear additive, non-equal weights}

  \[ OBJ = e^{w_B B} e^{w_C C} \]  \hspace{1cm} \text{Linear multiplicative}
EXTENSION OF THE PROBLEM TO RESOURCE ALLOCATION
EXTENSION OF THE PROBLEM TO RESOURCE ALLOCATION

Previous formulations:

$X$ represents an activity
EXTENSION OF THE PROBLEM TO RESOURCE ALLOCATION

Previous formulations:

$X$ represents an activity

New formulations (Resource Allocation formulations):

$X$ represents an activity+resource bundle
EXTENSION OF THE PROBLEM TO RESOURCE ALLOCATION

\( X_{ij} \) is a Resource-activity pair
Resource \( j \) is allocated to Activity \( i \)

Activities: \( i = 1, 2, \ldots, I \)
Resources: \( j = 1, 2, \ldots, J \)

\( X_{23} \) means
Resource 3 is allocated to Activity 2

\( X_{55} \) means
Resource 5 is allocated to Activity 5

New formulations (Resource Allocation formulations):
\( X \) represents an activity+resource bundle
What if more than one resource is allocated to an activity?

Example, Resources 3 and 5 are allocated to Activity 5

Means that $X_{55}$ and $X_{53}$ should exist in the mathematical formulation.
Again, let’s see some Mathematical Notations and Formulations

(a) Constraints
(b) Objective function
(a) Constraints
<table>
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<tr>
<th>Activity-Resource Pair</th>
<th>Benefit (Utility)</th>
<th>Cost (Dis-utility)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{15}$</td>
<td>$b_{X15}$</td>
<td>$c_{X15}$</td>
</tr>
<tr>
<td>$X_{23}$</td>
<td>$b_{X23}$</td>
<td>$c_{X23}$</td>
</tr>
<tr>
<td>$X_{31}$</td>
<td>$b_{X31}$</td>
<td>$c_{X31}$</td>
</tr>
<tr>
<td>$X_{44}$</td>
<td>$b_{X44}$</td>
<td>$c_{X44}$</td>
</tr>
<tr>
<td>$X_{55}$</td>
<td>$b_{X55}$</td>
<td>$c_{X55}$</td>
</tr>
<tr>
<td>$X_{64}$</td>
<td>$b_{X64}$</td>
<td>$c_{X64}$</td>
</tr>
<tr>
<td>TOTAL</td>
<td>$B$</td>
<td>$C$</td>
</tr>
</tbody>
</table>
\( X_{ij} = 1,0 \)  
\[ i = 1,2,\ldots, I \]  
\[ j = 1,2,\ldots, J \]
\(X_{15} = 1\)  
Resource-Activity Pair 1-5 definitely needs to be carried out
\[ X_{23} + X_{44} = 1 \] Carry out Resource-Activity Pairs 2-3 or 4-4 but not both.

<table>
<thead>
<tr>
<th>Activity-Resource Pair</th>
<th>Benefit (Utility)</th>
<th>Cost (Dis-utility)</th>
</tr>
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<td>( X_{15} )</td>
<td>( b_{X15} )</td>
<td>( c_{X15} )</td>
</tr>
<tr>
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<td>( b_{X23} )</td>
<td>( c_{X23} )</td>
</tr>
<tr>
<td>( X_{31} )</td>
<td>( b_{X31} )</td>
<td>( c_{X31} )</td>
</tr>
<tr>
<td>( X_{44} )</td>
<td>( b_{X44} )</td>
<td>( c_{X44} )</td>
</tr>
<tr>
<td>( X_{55} )</td>
<td>( b_{X55} )</td>
<td>( c_{X55} )</td>
</tr>
<tr>
<td>( X_{64} )</td>
<td>( b_{X64} )</td>
<td>( c_{X64} )</td>
</tr>
<tr>
<td>TOTAL</td>
<td>( B )</td>
<td>( C )</td>
</tr>
</tbody>
</table>
The average benefit of all selected Resource-activity pair should be at least $b^*$.
The average cost of all selected Resource-activity pairs should not exceed $c^*$.
The least benefit of any selected Resource-activity pair should be $b^*$ OR
No Resource-activity pair selected should have a benefit that is less than $b^*$
The highest cost of any selected Resource-activity pair should be $c^*$

OR

No Resource-activity pair selected should have a cost that is more than $c^*$

\[
\max(\tilde{c}_{X_{ij}}) = c^*
\]
(b) Objective Function
Objective Function

*What is our goal?*

*What are we seeking to maximize/minimize?*

- Maximize the sum of all benefits
  
  *E.g. The set of resource-activity pairs that involve the lowest total duration*

- Minimize the sum of all costs

  *E.g. The set of resource-activity pairs that involve the least resources*

- Maximize all benefits and minimize all costs
Objective Function

What is our goal?

What are we seeking to maximize/minimize?

- Maximize the sum of all benefits
  
  *E.g. The set of activities that involve the lowest total duration*

- Minimize the sum of all costs
  
  *E.g. The set of activities that involve the least resources*

- Maximize all benefits and minimize all costs

  \[ OBJ = B + (-C) \quad \text{Linear additive} \]

  \[ OBJ = 1B - 1C \quad \text{Linear additive, equal weight of 1} \]

  \[ OBJ = w_BB - w_CC \quad \text{Linear additive, non-equal weights} \]

  \[ OBJ = e^{w_BB} e^{w_CC} \quad \text{Linear multiplicative} \]
EXAMPLE: A Project Manager seeks to carry out a certain task that involves a series of activities. There is more than one way of carrying out this task as some resources and activities can be substituted by others.

The table below shows the set of feasible activities and their associated costs and benefits.

What is the optimal set (not series) of activities if the PM seeks to maximize the benefits and minimize the costs in a linear additive fashion.

<table>
<thead>
<tr>
<th>RESOURCE-ACTIVITY PAIRS</th>
<th>BENEFITS</th>
<th>COSTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Resource Utilization</td>
<td>$ Equivalent (in 1000’s)</td>
</tr>
<tr>
<td>$X_{15}$</td>
<td>65%</td>
<td>$17.1$</td>
</tr>
<tr>
<td>$X_{23}$</td>
<td>85%</td>
<td>$19.4$</td>
</tr>
<tr>
<td>$X_{31}$</td>
<td>45%</td>
<td>$14.0$</td>
</tr>
<tr>
<td>$X_{44}$</td>
<td>50%</td>
<td>$15.0$</td>
</tr>
<tr>
<td>$X_{55}$</td>
<td>62%</td>
<td>$16.2$</td>
</tr>
<tr>
<td>$X_{64}$</td>
<td>76%</td>
<td>$18.0$</td>
</tr>
</tbody>
</table>
Constraints:

Minimum average resource utilization = 70%

Maximum total duration = 28 days

\[ X_{23} + X_{44} = 1 \]

\[ X_{31} + X_{55} = 1 \]

\[ X_{31} + X_{55} = 1 \]

\[ X_{15} = 1 \]

\[ X_{64} = 1 \]

Assume no precedence conditions
Objective Function:

(a) Maximize benefits (resource utilizations)

\[ b_{x15} + b_{x23} + b_{x31} + b_{x44} + b_{x55} + b_{x64} \]
Objective Function:

(a) Maximize benefits (resource utilizations)

\[ b_{X15} + b_{X23} + b_{X31} + b_{X44} + b_{X55} + b_{X64} \]
Objective Function:

(a) Maximize benefits (resource utilizations)

\[ b_{X15} + b_{X23} + b_{X31} + b_{X44} + b_{X55} + b_{X64} \]
Objective Function:

(a) Maximize benefits (resource utilizations)

\[ b_{X_{15}} + b_{X_{23}} + b_{X_{31}} + b_{X_{44}} + b_{X_{55}} + b_{X_{64}} \]

Note that for any path, some resource-activity pairs are inconsequential!

Correct benefit maximization function is:

\[ b_{X_{15}X_{15}} + b_{X_{23}X_{23}} + b_{X_{31}X_{31}} + b_{X_{44}X_{44}} + b_{X_{55}X_{55}} + b_{X_{64}X_{64}} \]
Correct benefit maximization function is:

\[ b_{X_{15}}X_{15} + b_{X_{23}}X_{23} + b_{X_{31}}X_{31} + b_{X_{44}}X_{44} + b_{X_{55}}X_{55} + b_{X_{64}}X_{64} \]

Example: For the path \( X_{15} - X_{23} - X_{31} - X_{64} \)

The objective function is:

\[ b_{X_{15}}X_{15} + b_{X_{23}}X_{23} + b_{X_{31}}X_{31} + b_{X_{44}}X_{44} + b_{X_{55}}X_{55} + b_{X_{64}}X_{64} \]

\[ \begin{align*}
&1 &1 &1 &0 &0 &1 \\
\end{align*} \]

The objective function becomes:

\[ b_{X_{15}}X_{15} + b_{X_{23}}X_{23} + b_{X_{31}}X_{31} + b_{X_{64}}X_{64} \]

Therefore the Resource-activity pairs \( X_{44} \) and \( X_{55} \) and their benefits become inconsequential!
Benefit maximization function is:

\[ b_{x15}X_{15} + b_{x23}X_{23} + b_{x31}X_{31} + b_{x44}X_{44} + b_{x55}X_{55} + b_{x64}X_{64} \]

This is the benefit expressed in terms of % resource utilization
Objective Function:

(b) Minimize the total costs (cost of durations)

In a similar reasoning as done for benefits, the cost minimization function is:

\[ c_{X_{15}}X_{15} + c_{X_{23}}X_{23} + c_{X_{31}}X_{31} + c_{X_{44}}X_{44} + c_{X_{55}}X_{55} + c_{X_{64}}X_{64} \]

This is the cost in terms of duration (days)
Combined Objective Function:

(a) Maximize total benefits in %RU

(b) Minimize total costs (cost of durations)

If functional form is assumed to be linearly additive, then:

\[
OBJ = B + C \\
= B + \left( -C \right) \\
= b_{x15}X_{15} + b_{x23}X_{23} + b_{x31}X_{31} + b_{x44}X_{44} + b_{x55}X_{55} + b_{x64}X_{64} \\
+ c_{x15}X_{15} + c_{x23}X_{23} + c_{x31}X_{31} + c_{x44}X_{44} + c_{x55}X_{55} + c_{x64}X_{64}
\]
But there is a problem!

These are in different units (b’s are in %RU, c’s are in days.
How do we add or subtract?

Convert them into a single metric of utility, … in this case, dollars!
Recall data given in the problem

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Benefits:
\[ S = 8415.4 \ln(RU) - 18451 \]
\[ R^2 = 0.9722 \]

Costs:
\[ S = 3259.1 \times 0.2591(DAYS) \]
\[ R^2 = 0.9863 \]
Combined Objective Function:

\[ \text{OBJ} = b_{X_{15}}X_{15} + b_{X_{23}}X_{23} + b_{X_{31}}X_{31} + b_{X_{44}}X_{44} + b_{X_{55}}X_{55} + b_{X_{64}}X_{64} \]
\[ + c_{X_{15}}X_{15} + c_{X_{23}}X_{23} + c_{X_{31}}X_{31} + c_{X_{44}}X_{44} + c_{X_{55}}X_{55} + c_{X_{64}}X_{64} \]

Replace all \( b \)'s by the equivalent monetary function:
Replace all \( c \)'s by the equivalent monetary function

Let’s try using MS Excel Solver to solve this problem.
PART 2
Resource Allocation for Queues Encountered in Project Management
What is a Queue?

Examples.
What is a Queue?

Simply: A line waiting to be served

Figure by MIT OCW.
Examples of Queuing Systems in Everyday Life

- Vehicles waiting to be served at a drive-through pharmacy, fast-food restaurant, bank, toll-booth

- Individuals (in person) waiting to be served at various service counters and stations, such as check-out lanes for groceries, supermarkets, etc.

- Patients scheduled for use of hospital operation theater
Queuing Systems in Everyday Life (cont’d)

- Football fans waiting to get into stadium
- Football fans waiting to get out of stadium
- Operations of a vending machine
- Candy dispenser
- Human urinary system
Examples of Queuing Systems in Project Management

Vendors supply vehicles waiting to unload
Project vehicles waiting to unload finished materials on site
Construction trucks waiting to be loaded with raw materials

*Queuing in Project Management ... is it a big deal?*

*Yes and No*
The flow entity (queuing unit) in a queuing system is typically a discrete element, and is represented by a discrete random variable (trucks, cars, people, etc.).

But ...
Queuing Units

.... queuing systems may also involve continuous flow entities, (and hence, continuous random variables, such as:

- Water (in gallons, say) in a large reservoir “waiting” to be served daily to a project site

- Aggregates, cement, etc. in storage bins, “waiting” to be shipped to site or to processing plants
Components of a Queuing System
Components of a Queuing System

- Flow Entity
- Arrival Pattern (of the queuing units)
- Queue Multiplicity (Nr. of Queues)
- Queue Discipline
- Number of Servers
- Service Arrangement
- Service Pattern
Components of a Queuing System

- Flow Entity (vehicles, people, materials, etc.)
- Arrival Pattern (of the queuing units)
- Queue Multiplicity (Nr. of Queues)
- Queue Discipline
- Number of Servers
- Service Arrangement
- Service Pattern
Components of a Queuing System

- Queue Discipline
- Arrival Pattern
- Service Facility
  - Equipment for loading or unloading
    - Number of servers
    - Service arrangement
    - Service pattern
- Queue dissipation (vehicles leaving the queuing system after being served)

Figure by MIT OCW.
A: Arrival Pattern

- Describes the way (usually a rate) in which the arrivals enter the queuing system

- May be Frequency-based or Interval-based. That is, arrivals can be described on the basis of:
  --- the number of arrivals that arrive in a given time interval
  --- the average interval of time that passes between successive arrivals.

- Conversion between (a) and (b) is possible.
Arrival Pattern (cont’d)

- Maybe deterministic or probabilistic:

(a) **Deterministic**: fixed number of arrivals per unit time or fixed length of time interval between arrivals

(b) **Probabilistic**: Stochastic number of arrivals per unit time or stochastic length of time interval between arrivals (e.g., Negative exponential)
Arrival Pattern (cont’d)

- Probabilistic frequency of arrivals may be described by Poisson distribution or other appropriate *discrete* probability distribution.

- Probabilistic interval of time between arrivals may be described by the negative exponential distribution or other appropriate *continuous* probability distribution.
B: Service Facility Characteristics

(i) Number of servers
   (1 or more?)

(ii) Arrangement of servers
     (parallel or series or combo?)

(iii) Service pattern
     What distribution? How fast (average), etc.)
B: Service Facility Characteristics (cont’d)

(i) **Number of Servers:**

- **Single Server:**
  - Examples: Only one counter open at bank
  - Candy dispenser,
  - Coke vending machine
  - Single truck loader at project site

- **Multi-server:**
  - Examples: Several counters open at bank
  - Multi-lane freeway toll booth
  - Multiple truck loader at project site
(ii) Arrangement of servers

Parallel arrangement of servers
- e.g., Bank counters

Serial arrangement of servers
- e.g., Some McDonald drive-thrus
  - $S_1$: PLACE ORDER
  - $S_2$: PAY MONEY
  - $S_3$: COLLECT FOOD
(ii) Arrangement of servers (cont’d)

Combination of Parallel and Serial Arrangements

S3 → S2

S3 → S4
(ii) Arrangement of servers (cont’d)

Combination of Parallel and Serial Arrangements

2 channels and 2 phases

Arrivals

S3 → S2 → Departures

Channel 1

S3 → S4 → Departures

Channel 2
(ii) Arrangement of servers (cont’d)

Combination of Parallel and Serial Arrangements

2 channels and 2 phases

Arrivals

\[ S_3 \rightarrow S_2 \rightarrow \text{Departures} \]

\[ S_3 \rightarrow S_4 \rightarrow \text{Departures} \]

Phase 1 Phase 2
(ii) Arrangement of servers (cont’d)

Combination of Parallel and Serial Arrangements

2 channels and 2 phases

Arrivals

Channel 1

S₃ → S₂ → Departures

Phase 1

Channel 2

S₃ → S₄ → Departures

Phase 2
Arrivals \[\xrightarrow{1}\] Serve \[\xrightarrow{1}\] Departure

*Single Phase, Single Channel*
Queuing Systems

Single Phase, Single Channel

Multiple Phase, Single Channel
Queuing Systems

Single Phase, Single Channel

Single Phase, Multiple Channel

Multiple Phase, Single Channel
Queuing Systems

Single Phase, Single Channel

Multiple Phase, Single Channel

Single Phase, Multiple Channel

Multiple Phase, Multiple Channel
Service Pattern

Describes the way (usually a rate) by which arrivals are processed

- May be frequency-based or interval-based. That is, service can be described based on:
  - the number of arrivals that are served in a given time interval
  - The average interval of time that is used to serve the arrivals

- Conversion between (a) and (b) is possible
(iii) Service pattern (cont’d)

- Maybe deterministic or probabilistic:

(a) Deterministic: fixed number of served arrivals per unit time or fixed length of time interval between services
(b) Probabilistic: Stochastic number of servings per unit time or stochastic length of time interval between servings
C: Queue Multiplicity

Refers to the number of queues being served simultaneously

- Single queue (Examples: most drive thrus, banks, narrow toll bridges, traffic green lights serving only one lane)
- Multiple queue (assuming no preference between each queue)
  Examples: Most dining counters, toll booths traffic green light serving two or more lanes)
  Typically number of queues ≤ number of servers, but when number of queues > number of servers, then some extra rules for queue discipline are needed
Some Queuing Configurations

1 queue, 1 server

$S_1$
Some Queuing Configurations

1 queue, 2 servers
Some Queuing Configurations

2 queues, 2 servers
Some Queuing Configurations

2 queue, 5 servers
Some Queuing Configurations

4 queues, 1 server
D: Queue Discipline

This refers to the rules by which the queue is served

Relates serving priority to:

(i) Order of arrival times, or  
(ii) Order of arrival urgencies  
(iii) Order of expected length of service time  
(iv) Order of “desirability” of arrival of specific flow entities
D: Queue Discipline (cont’d)

(i) Serving priority by order of arrival times

FIFO (First in, first out)
  First come, first served
  Last in, last out
  e.g., Truck in front is always served first.

FIFO is a non-discriminatory queue discipline, very fair

LIFO (Last in, first out)
  e.g., Truck at tail end of queue always served first
  e.g., Candy dispenser
  e.g., Often crowded elevator mostly serving 2 floors
D: Queue Discipline (cont’d)

(ii) Serving priority by order of arrival urgencies

- Trucks needing attention most urgently is served first, regardless of when they arrived
- Examples in everyday life:
  - scheduling patients for surgery in order of sickness severity
  - Giving way to fire trucks at intersections
D: Queue Discipline (cont’d)

(iii) Serving priority by order of expected service period

Trucks whose service will take shorter times are served first, regardless of the time they joined the queue

Examples:
- Express lanes at supermarkets (shoppers with less than 5 items)
- Trucks taking away items that take a very short time to load
- Trucks delivering items that take very short time to unload
Performance of Queuing Systems
Performance of Queuing Systems

Class Question:
- How would you assess the performance of a queuing system?
- That is, what criteria would you use?
MOE

Performance criteria for queuing systems:
- Average queue length
- Maximum queue length
- Average waiting period per truck
- Maximum waiting period per truck
- % of time each server is idle
- Physical and operating cost of the queuing systems
- Number of customers served per unit time

\[ \text{Minimize this, or truck time is wasted} \]

\[ \text{Minimize this, or project resources are wasted} \]

\[ \text{Maximize this, or both truck time and project resources are wasted} \]
Attributes of a Queuing System
QUEUING SYSTEM ATTRIBUTES

These describe the way the system is structured, its operating procedure, and how well it performs.
Consists of system components, and other attributes

- **Physical System Components**
  - Flow Entities (trucks), Queues, Servers (serving facilities/equipment)

- **Operational System Components**
  - Pattern of Arrivals, Number of Queues, Pattern of Service, Number of Servers, Queue Discipline, Queue Capacity

- **System Performance**
  - Queue length, waiting time, server idle time, etc.