Probabilistic Planning Part I

1.040/1.401 - Project Management

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Recall …

Project Management Phases

- FEASIBILITY
  - Project Evaluation
  - Project Finance

- DESIGN PLANNING
  - Project Organization
  - Project Cost Estimation
  - Project Planning and Scheduling

- DEVELOPMENT

- CLOSEOUT

- OPERATIONS
Outline for this Lecture

- Deterministic systems decision-making – the general picture
- Probabilistic systems decision-making – the general picture
  – the special case of project planning
- Why planning is never deterministic
- Simplified examples of deterministic, probabilistic planning
  - everyday life
  - project planning
- Illustration of probabilistic project planning
- PERT – Basics, terminology, advantages, disadvantages, example
Deterministic Systems Planning and Decision-making

Influence of deterministic inputs on the outputs of engineering systems -- the general picture

Input variables and their FIXED VALUES

- $X^*$
- $Y^*$
- $Z^*$

Examples: costs, time duration, quality, interest rates, etc.

Deterministic Analysis of Engineering System

FIXED VALUES of the Outputs (system performance criteria, etc.)

- $O_1^*$
- $O_2^*$
- $O_m^*$
- $O^*_{COMBO}$

Examples: queuing systems, network systems, etc.

Final evaluation result and decision

- Alt. 1
- Alt. $k$...
- Alt. $n$

VALUE OF combined output (index, utility or value) representing multiple performance measures of the system

A SINGLE evaluation outcome
But engineering systems are never deterministic!

Why?

For sample, for Project Planning Systems …

- Variations in planning input parameters
- Beyond control of project manager
- Categories of the variation factors
  - Natural (weather -- good and bad, natural disasters, etc.)
  - Man-made (equipment breakdowns, strikes, new technology, worker morale, poor design, site problems, interest rates, etc.)
- Combined effect of input factor variation is a variation in the outputs (costs, time, quality)
Why planning is never deterministic -- II

Influence of stochastic inputs on the outputs of engineering systems -- the general picture

Input variables and their probability distributions

- $f_x$
- $f_y$
- $f_z$

Examples: costs, time durations, quality, interest rates, etc.

Outputs (system performance criteria, etc.) and their probability distributions

- $f(O_1)$
- $f(O_2)$
- ...
- $f(O_n)$

Probability distribution for combined output (index, utility or value) representing multiple performance measures

- $f(O_{COMBO})$

Discrete probability distribution for evaluation outcome

$P_{\text{Alt. } i}$ is the probability that an alternative turns out to be most desirable or most critical

Variability of final evaluation result and decision

Alt. 1  Alt. 2  ***  Alt. n
Why planning is never deterministic -- III

Probability distribution?

What exactly do you mean by that?

See survey for dinner durations (times) of class members
Probability Distribution for your Dinner Times:

![Graph showing the probability distribution for dinner times. The x-axis represents the time spent for dinner in minutes, and the y-axis represents the probability. The graph peaks around 17.5 minutes.]
Probability that a randomly selected student spends less than $T^*$ minutes for dinner is:

\[ P(T < T^*) = P\left( \frac{T - \mu}{\sigma} < \frac{T^* - \mu}{\sigma} \right) = P(Z < Z^*) \]
Probability that a randomly selected student spends less than $T^*$ minutes for dinner is:

\[ P(T < T^*) = P(T - \mu < \frac{T^* - \mu}{\sigma}) = P(Z < Z^*) \]
Probability that a randomly selected student spends …

… less than $T_1 = P(T < T^*) = P\left(\frac{T - \mu}{\sigma} < \frac{T^* - \mu}{\sigma}\right) = P(Z < Z^*)$

… more than $T_1 = P(T > T^*) = P\left(\frac{T - \mu}{\sigma} > \frac{T^* - \mu}{\sigma}\right) = P(Z > Z^*)$
Probability that a randomly selected student spends …

… less than \( T^* = P(T < T^*) = P\left(\frac{T - \mu}{\sigma} < \frac{T^* - \mu}{\sigma}\right) = P(Z < Z^*) \)

… more than \( T^* = P(T > T^*) = P\left(\frac{T - \mu}{\sigma} > \frac{T^* - \mu}{\sigma}\right) = P(Z > Z^*) \)

… between \( T_{1}^* \) and \( T_{2}^* = P(T_{1}^* < T < T_{2}^*) = P\left(\frac{T_{1}^* - \mu}{\sigma} < \frac{T - \mu}{\sigma} < \frac{T_{2}^* - \mu}{\sigma}\right) = P(Z_{1}^* < Z < Z_{2}^*) \)
Probability is the area under the probability distribution/density curves)

Probability can be found using any one of three ways:

- coordinate geometry
- calculus
- statistical tables
Like-wise, we can build probability distributions for project planning parameters by ...

- using historical data from past projects, OR
- computer simulation

And thus we can find the probability that project durations falls within a certain specified range
Probabilistic planning of project management systems can involve uncertainties in:

- Need for an Activity (need vs. no need)
- Durations
  - Activity durations
  - Activity start-times and end-times
- Cost of activities
- Quality of Workmanship and materials
- Etc.
Probabilistic planning of project management systems can involve uncertainties in:

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- Durations
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- Quality of Workmanship and materials
- Etc.
Influence of **stochastic inputs** -- the specific picture of **project planning**

- **Input variable and its probability distribution**
- **Project activity durations**
- **Probabilistic Analysis of Project Planning**
- **Outputs (system performance criteria, etc.) and their probability distributions**
- **Probability distributions for the output**
- **Variability of final evaluation result and decision**
- **Frequency distribution or probability distribution for evaluation outcome**
- **$P_{Alt,i}$** is the probability that a given path turns out to be the critical path

In this case, ... "Output" is the identification of the critical path for the project.
SAM Waking up and meditating
START = 7AM
Duration 1hr
FINISH = 8AM

SAM Bathing, Breakfast, Reading,
START = 8AM
Duration 5hrs
FINISH = 1 PM

US Meeting in Class
For this Lecture
START = 1 PM
Duration 1.5hr
FINISH = 2:30

YOU Preparing for Classes, etc.
START = 7 AM
Duration 1 hr
FINISH = 8 AM

YOU showing up at Other Classes
START = 8 AM
Duration 5 hr
FINISH = 1 PM

YOU missing the lecture
START = 1 PM
Duration 1.5hr
FINISH = 2:30

Perfectly deterministic
Probabilistic planning: An example in everyday life

KEY

<table>
<thead>
<tr>
<th>Activity</th>
<th>Start Time</th>
<th>Activity Duration</th>
<th>Finish Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>SAM Waking up and meditating</td>
<td>7AM</td>
<td>1hr</td>
<td>8AM</td>
</tr>
<tr>
<td>SAM Bathing, Breakfast, Reading</td>
<td>8AM</td>
<td>5hrs</td>
<td>1 PM</td>
</tr>
<tr>
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<td>1.5hr</td>
<td>2:30</td>
</tr>
<tr>
<td>YOU Preparing for Classes, etc.</td>
<td>7 AM</td>
<td>1 hr</td>
<td>8 AM</td>
</tr>
<tr>
<td>YOU showing up at Other Classes</td>
<td>8 AM</td>
<td>5 hr</td>
<td>1 PM</td>
</tr>
<tr>
<td>YOU missing the lecture</td>
<td>1 PM</td>
<td>1.5hr</td>
<td>2:30</td>
</tr>
</tbody>
</table>
Probabilistic planning: *An example in everyday life*

Partly deterministic, Partly probabilistic

**SAM Waking up and meditating**

- **ES = 7AM**
- **LS = 7AM**
- **Duration = 1 hour**
- **EF = 8AM**
- **LF = 8AM**

**SAM Bathing, Breakfast, Reading,**

- **ES = 7AM**
- **LS = 8AM**
- **Duration = 5 hours**
- **EF = 1PM**
- **LF = 1PM**

**US Meeting in Class**

For this Lecture

- **ES = 12:55**
- **LS = 1PM**
- **Duration = 1.5 hours**
- **EF = 2:25**
- **LF = 2:30**

**YOU Preparing for Classes, etc.**

- **ES = 7AM**
- **LS = 7AM**
- **Duration = 1 hour**
- **EF = 8AM**
- **LF = 8AM**

**YOU showing up at Other Classes**

- **ES = 7:45**
- **LS = 8AM**
- **Duration = 5 hours**
- **EF = 1PM**
- **LF = 1PM**

**YOU missing the**

- **ES = 6AM**
- **LS = 7AM**
- **Duration = 1 hour**
- **EF = 8AM**
- **LF = 8AM**

**KEY**

- **Activity**
  - Earliest Start
  - Duration of Activity
  - Earliest Finish
  - Latest Start
  - Latest Finish
SAM Waking up and meditating

ES = 7AM
LS = 6AM

Duration

μ = 1 hr
σ = 0.25 hr

EF = 8AM
LF = 8AM

SAM Bathing, Breakfast, Reading,

ES = 8AM
LS = 7AM

Duration

μ = 5 hr
σ = 0.6 hr

EF = NOON
LF = 1 PM

US Meeting in Class For this Lecture

ES = 12:55
LS = 1 PM

Duration

μ = 1.5 hr
σ = 0.31 hr

EF = 2:25
LF = 2:30

YOU Preparing for Classes, etc.

ES = 7:45
LS = 7AM

Duration

μ = 1 hr
σ = 0.15 hr

EF = 8AM
LF = 8AM

YOU showing up at Other Classes

ES = 7:45
LS = 7AM

Duration

μ = 5 hr
σ = 0.35 hr

EF = 12:45
LF = 1 PM

YOU missing the

ES = 6AM
LS = 7AM

Duration

μ = 1 hr
σ = 0.2 hr

EF = 8AM
LF = 8AM

Probabilistic planning: An example in everyday life
Fully probabilistic

Activity

Earliest Start | Duration of Activity | Earliest Finish
Latest Start | Latest Finish

KEY
Probabilistic planning: An example in everyday life

Fully probabilistic

<table>
<thead>
<tr>
<th>Activity</th>
<th>Earliest Start</th>
<th>Duration of Activity</th>
<th>Latest Finish</th>
</tr>
</thead>
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<td></td>
</tr>
<tr>
<td>YOU missing the</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

KEY

Fully probabilistic

\[ \mu = 1 \text{ hr} \]
\[ \sigma = 0.25 \text{ hr} \]

\[ \mu = 5 \text{ hr} \]
\[ \sigma = 0.6 \text{ hr} \]

\[ \mu = 1.5 \text{ hr} \]
\[ \sigma = 0.31 \text{ hr} \]

\[ \mu = 1 \text{ hr} \]
\[ \sigma = 0.15 \text{ hr} \]

\[ \mu = 5 \text{ hr} \]
\[ \sigma = 0.35 \text{ hr} \]

\[ \mu = 1 \text{ hr} \]
\[ \sigma = 0.2 \text{ hr} \]
Probabilistic planning: An example in Project Management

### Activity Name

<table>
<thead>
<tr>
<th>Activity</th>
<th>Early Start</th>
<th>Late Start</th>
<th>Duration</th>
<th>Early Finish</th>
<th>Late Finish</th>
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<tbody>
<tr>
<td>Activity R</td>
<td>Month 0</td>
<td>Month 0</td>
<td>4 months</td>
<td>Month 4</td>
<td>Month 4</td>
</tr>
<tr>
<td>Activity G</td>
<td>Month 4</td>
<td>Month 7</td>
<td>3 months</td>
<td>Month 10</td>
<td>Month 10</td>
</tr>
<tr>
<td>Activity J</td>
<td>Month 10</td>
<td>Month 15</td>
<td>5 months</td>
<td>Month 15</td>
<td>Month 15</td>
</tr>
<tr>
<td>Activity W</td>
<td>Month 15</td>
<td>Month 15</td>
<td>2 months</td>
<td>Month 15</td>
<td>Month 17</td>
</tr>
<tr>
<td>Activity C</td>
<td>Month 4</td>
<td>Month 4</td>
<td>6 months</td>
<td>Month 10</td>
<td>Month 10</td>
</tr>
<tr>
<td>Activity M</td>
<td>Month 10</td>
<td>Month 12</td>
<td>3 months</td>
<td>Month 13</td>
<td>Month 15</td>
</tr>
</tbody>
</table>
Probabilistic planning: *An example in Project Management*

<table>
<thead>
<tr>
<th>Activity Name</th>
<th>Early Start</th>
<th>Late Start</th>
<th>Duration (O-M-P)</th>
<th>Early Finish</th>
<th>Late Finish</th>
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</thead>
<tbody>
<tr>
<td>Activity R</td>
<td>Month 0</td>
<td>Month 11</td>
<td>4 Months (3-4-8)</td>
<td>Month 4</td>
<td>Month 8</td>
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<tr>
<td>Activity G</td>
<td>Month 4</td>
<td>Month 7</td>
<td>3 Months (2-3-5)</td>
<td>Month 10</td>
<td>Month 15</td>
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<td>Month 10</td>
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<td>5 Months (2-5-6)</td>
<td>Month 15</td>
<td>Month 12</td>
</tr>
<tr>
<td>Activity C</td>
<td>Month 4</td>
<td>Month 10</td>
<td>6 Months (3-6-7)</td>
<td>Month 15</td>
<td>Month 18</td>
</tr>
<tr>
<td>Activity M</td>
<td>Month 10</td>
<td>Month 12</td>
<td>3 Months (1-3-5)</td>
<td>Month 15</td>
<td>Month 15</td>
</tr>
<tr>
<td>Activity W</td>
<td>Month 15</td>
<td>Month 17</td>
<td>2 Months (1-2-4)</td>
<td>Month 17</td>
<td>Month 17</td>
</tr>
</tbody>
</table>

O: Optimistic (earliest time)
M: Most probable time
P: Pessimistic (latest time)
Probabilistic planning involving activity durations

An Illustration

<table>
<thead>
<tr>
<th>Activity</th>
<th>Optimistic</th>
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<th>Pessimistic</th>
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<tbody>
<tr>
<td>R</td>
<td>3</td>
<td>4</td>
<td>8</td>
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<tr>
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<td>2</td>
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<tr>
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<td>3</td>
<td>6</td>
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</tr>
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<td>M</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>W</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
Probabilistic planning involving activity durations

An Illustration

Let’s say we have lots of data on the durations of each activity. Such data is typically from:

- Historical records (previous projects)
- Computer simulation (Monte Carlo)
Probabilistic planning involving activity durations

An Illustration

<table>
<thead>
<tr>
<th>Activity</th>
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<th>Pessimistic</th>
<th>Mean</th>
<th>Standard Dev.</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>3</td>
<td>4</td>
<td>8</td>
<td>4.50</td>
<td>0.833</td>
<td>0.69444</td>
</tr>
<tr>
<td>A</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>4.00</td>
<td>0.333</td>
<td>0.11111</td>
</tr>
<tr>
<td>G</td>
<td>2</td>
<td>3</td>
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<td>3.17</td>
<td>0.500</td>
<td>0.25000</td>
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<tr>
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Calculate the Expected Duration of each path, and Identify the Critical Path on the basis of the mean only:

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</table>

Path: Deterministic Duration: Probabilistic Duration:

- R-A-W: 10: 10.67
- R-G-J-W: 14: 14.50
- R-C-J-W: 17: 17.00
- R-C-M-W: 15: 15.33

Critical Path: R-C-M-W
Calculate the Expected Duration of each path, and Identify the Critical Path on the basis of both the mean and the std dev:

<table>
<thead>
<tr>
<th>Activity</th>
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<td>2.17</td>
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</tr>
</tbody>
</table>

![Network Diagram]
Critical Path

Path R-A-W

Path R-G-J-W

Path R-C-J-W

Path R-C-M-W

Mean Duration

0 0.5 1 1.5 2 2.5 3 3.5 4 4.5

0 5 10 15 20 25
Critical path here can be considered as that with:
- Longest duration (mean)
- Greatest variation (stdev)
Consider the following hypothetical project paths:

<table>
<thead>
<tr>
<th>Path</th>
<th>Mean Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-Q-R</td>
<td>20</td>
</tr>
<tr>
<td>P-F-W-R</td>
<td>25</td>
</tr>
<tr>
<td>P-H-W-R</td>
<td>30</td>
</tr>
<tr>
<td>P-H-M-R</td>
<td>35</td>
</tr>
</tbody>
</table>

On the basis of mean duration only, Path P-H-W-R is the critical path.

On the basis of the variance of durations only, Path P-F-W-R is the critical path.

How would you decide the critical path on the basis of both mean duration and variance of durations?
Better way to identify critical path is using the amount of slack in each path (see later slides)
Another Example of Probabilistic Project Scheduling

- Monte Carlo Simulation
- Similar activity structure as before, but Start and End activities are dummies (zero durations).

See Excel Sheet Attached
Benefits of Probabilistic Project Planning

- Helps identify likely critical paths in situations where there is great uncertainty.

- Helps ascertain the likelihood (probability) that overall project duration will fall within a given range.

- Helps establish a scale of “criticality” among the project activities.

Discussed in previous slides.
Benefits of Probabilistic Project Planning

- Helps identify likely critical paths in situations where there is great uncertainty.
- Helps ascertain the likelihood (probability) that overall project duration will fall within a given range.
- Helps establish a scale of “criticality” among the project activities.

*Discussed in subsequent slides*
Probabilistic planning …

Is it ever used in real-life project management?
A Tool for Stochastic Planning: PERT

- Program Evaluation and Review Technique (PERT)
  - Need for PERT arose during the Space Race, in the late fifties
  - Developed by Booz-Allen Hamilton for US Navy, and Lockheed Corporation
    - Polaris Missile/Submarine Project
    - R&D Projects
    - Time Oriented
    - Probabilistic Times
    - Assumes that activity durations are Beta distributed
PERT Parameters

- Optimistic duration \( a \)
- Most Likely duration \( m \)
- Pessimistic duration \( b \)
- Expected duration 
  \[ \bar{d} = \frac{1}{3} \left[ 2m + \frac{1}{2} (a + b) \right] = \frac{a + 4m + b}{6} \]
- Standard deviation 
  \[ s = \frac{b - a}{6} \]
- Variance 
  \[ \nu = s^2 \]
Steps in PERT Analysis

- Obtain $a, m$ and $b$ for each activity
- Compute Expected Activity Duration $d = t_e$
- Compute Variance $v = s^2$
- Compute Expected Project Duration $D = T_e$
- Compute Project Variance $V = S^2$ as Sum of Critical Path Activity Variance
- In Case of Multiple Critical Path Use the One with the Largest Variance
- Calculate Probability of Completing the Project
### PERT Example

<table>
<thead>
<tr>
<th>Activity</th>
<th>Predecessor</th>
<th>a</th>
<th>m</th>
<th>b</th>
<th>d</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>1</td>
<td>2</td>
<td>4</td>
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<td>0.25</td>
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<tr>
<td>B</td>
<td>-</td>
<td>5</td>
<td>6</td>
<td>7</td>
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<td>0.11</td>
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<tr>
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<td>-</td>
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<td>4</td>
<td>5</td>
<td>3.83</td>
<td>0.25</td>
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<td>A</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>2.83</td>
<td>0.25</td>
</tr>
<tr>
<td>E</td>
<td>C</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>5.17</td>
<td>0.25</td>
</tr>
<tr>
<td>F</td>
<td>A</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>4.00</td>
<td>0.11</td>
</tr>
<tr>
<td>G</td>
<td>B,D,E</td>
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<td>2</td>
<td>3</td>
<td>2.00</td>
<td>0.11</td>
</tr>
</tbody>
</table>
PERT Example

Finding the Standard Deviation of the duration of a given path comprising Activities $C$, $E$, and $G$.

\[
T_e = 11 \\
= 0.25 + 0.25 + 0.1111 \\
= 0.6111 \\
S = \sqrt{0.6111} \\
= 0.7817
\]
PERT Analysis

Finding probability that project duration is less than some value
Example: probability that project ends before 10 months

\[ P(T \leq T_d) = P(T \leq 10) \]
\[ = P\left( z \leq \frac{10 - T_e}{S} \right) \]
\[ = P\left( z \leq \frac{10 - 11}{0.7817} \right) \]
\[ = P(z \leq -1.2793) \]
\[ = 1 - P(z \leq 1.2793) \]
\[ = 1 - 0.8997 \]
\[ = 0.1003 \]
\[ = 10\% \]
Probability that the project will end before 13 months

\[
P(T \leq 13) = P\left(z \leq \frac{13 - 11}{0.7817}\right)
\]

\[
= P(z \leq 2.5585)
\]

\[
= 0.9948
\]
Probability that the project will have a duration between 9 and 11.5 months

\[ P(T_L \leq T \leq T_U) = P(9 \leq T \leq 15) \]
\[ = P(T \leq 11.5) - P(T \leq 9) \]
\[ = P\left(z \leq \frac{11.5 - 11}{0.7817}\right) - P\left(z \leq \frac{9 - 11}{0.7817}\right) \]
\[ = P(z \leq 0.6396) - P(z \leq -2.5585) \]
\[ = P(z \leq 0.6396) - [1 - P(z \leq 2.5585)] \]
\[ = 0.7389 - [1 - 0.9948] \]
\[ = 0.7389 - 0.0052 \]
\[ = 0.7337 \]
PERT Advantages

- Includes Variance

- Assessment of Probability of Achieving a Goal
PERT Disadvantages

- Data intensive - Very Time Consuming
- Validity of Beta Distribution for Activity Durations
- Only one Critical Path considered
- Assumes independence between activity durations
PERT Monte Carlo Simulation

- Determine the Criticality Index of an Activity
- Used 10,000 Simulations, Now from 1000 to 400 Have Been Reported as Giving Good Results
PERT Monte Carlo Simulation Process

- Set the Duration Distribution for Each Activity
- Generate Random Duration from Distribution
- Determine Critical Path and Duration Using CPM
- Record Results
# Monte Carlo Simulation Example

## Statistics for Example Activities

<table>
<thead>
<tr>
<th>Activity</th>
<th>Optimistic Time, a</th>
<th>Most Likely Time, m</th>
<th>Pessimistic Time, b</th>
<th>Expected Value, d</th>
<th>Standard Deviation, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>0.66</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>8</td>
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<tr>
<td>D</td>
<td>4</td>
<td>7</td>
<td>10</td>
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<td>4</td>
<td>5</td>
<td>6</td>
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</table>
## Monte Carlo Simulation Example

### Summary of Simulation Runs for Example Project

<table>
<thead>
<tr>
<th>Run Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>Critical Path</th>
<th>Completion Time</th>
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<tr>
<td>1</td>
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<td>2.2</td>
<td>8.8</td>
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<td>8.8</td>
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<td>5.0</td>
<td>4.9</td>
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<td>7.2</td>
<td>7.2</td>
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<td>A-C-F-G</td>
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<td>3.1</td>
<td>4.3</td>
<td>A-C-F-G</td>
<td>19.7</td>
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</tbody>
</table>
Project Duration Distribution

Frequency

Project Length

10
9
8
7
6
5
4
3
2
1
0

17 18 19 20 21 22 23 24 25 26 27 28 29
Probability Computations from Monte Carlo results

\[ P(T \leq T^*) = \frac{\text{Number of Times Project Finished in Less Than or Equal to } T^*}{\text{Total Number of Replications}} \]

\[ P(T \geq T^*) = \frac{\text{Number of Times Project Finished in More Than or Equal to } T^*}{\text{Total Number of Replications}} \]

ETC.
Critcality Index for an Activity

Definition:
Proportion of Runs in which the Activity is in the Critical Path
Criticality Index for a Path

Definition I (‘Naïve Definition’):
Proportion of Runs in which the Activity is in the Critical Path (see Slide # 60)
Criticality Index for a Path

Definition I ("Naïve Definition):
Proportion of Runs in which the Activity is in the Critical Path (see Slide # 60)

<table>
<thead>
<tr>
<th>Run Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>Critical Path</th>
<th>Completion Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.3</td>
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<td>7.6</td>
<td>5.7</td>
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<td>5.2</td>
<td>A-C-F-G</td>
<td>22.3</td>
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<td>4.6</td>
<td>5.5</td>
<td>A-C-F-G</td>
<td>24.2</td>
</tr>
<tr>
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<td>6.2</td>
<td>4.4</td>
<td>8.9</td>
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<td>3.0</td>
<td>4.0</td>
<td>A-C-F-G</td>
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<td>1.1</td>
<td>7.4</td>
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<td>5.9</td>
<td>A-C-F-G</td>
<td>18.9</td>
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<td>4.3</td>
<td>7.1</td>
<td>3.1</td>
<td>4.3</td>
<td>A-C-F-G</td>
<td>19.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PATH</th>
<th>Frequency as a Critical Path</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-C-F-G</td>
<td>8</td>
<td>80%</td>
</tr>
<tr>
<td>A-D-F-G</td>
<td>2</td>
<td>20%</td>
</tr>
<tr>
<td>E-F-G</td>
<td>0</td>
<td>0%</td>
</tr>
</tbody>
</table>

10
Criticality Index for a Path

Definition I:
Proportion of Runs in which the Activity is in the Critical Path (see Slide # 60)

<table>
<thead>
<tr>
<th>PATH</th>
<th>Frequency as a Critical Path</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-C-F-G</td>
<td>8</td>
<td>80%</td>
</tr>
<tr>
<td>A-D-F-G</td>
<td>2</td>
<td>20%</td>
</tr>
<tr>
<td>E-F-G</td>
<td>0</td>
<td>0%</td>
</tr>
</tbody>
</table>

![Bar graph showing frequency as a critical path for different paths]

- A-C-F-G: 80%
- A-D-F-G: 20%
- E-F-G: 0%
Criticality Index for a Path

Definition II:
How much slack exists in that path.

Less Slack → Higher criticality
More Slack  → Lower criticality

\[ \lambda = \frac{\alpha_{\text{max}} - \beta}{\alpha_{\text{max}} - \alpha_{\text{min}}} \times (100\%) \]

Ranges from 0% to 100%

\[ \alpha_{\text{min}} = \text{minimum total float} \]
\[ \alpha_{\text{max}} = \text{maximum total float} \]
\[ \beta = \text{total float or slack in current path} \]

Using the index, we can rank project paths from most critical to least critical.
Criticality Index for a Path

Definition II:

See Example In Excel File

(Path Criticality Slide)
References

- Pritsker, Sigal, and Hammesfahr (1994). SLAM II: Network Models for Decision Support
- http://www.enr.com/