LEcTURe 23

LAtT TimE: 

• Broadcast channel

• Capacity region for degraded broadcast channel

• Distributed Source Coding,

• Slepian-Wolf Theorem and Random binning

Lecture outline

• Using random binning in broadcast channels

• Marton’s region

• Fading Channels
Review

- Degraded broadcast channel

\[
R_2 \leq I(U; Y_2) \\
R_1 \leq I(X; Y_1|U)
\]

- Distributed source coding
  - Use large enough number of bins to distinguish the typical sequences.

\[
R_1 > H(U|V) \\
R_2 > H(V|U) \\
R_1 + R_2 > H(U, V)
\]
Using Random Binning in Channels

Consider a deterministic point-to-point channel $Y = f(X)$ where $f(.)$ is deterministic and one-to-one. We try to transmit $R$ bits per symbol.

- Assign all the possible $Y$ sequences into $2^{nR}$ bins. The receiver will always decode as the bin number upon receiving $Y$.

- We want to send $M = i$, need to control $Y$ to be in the $i^{th}$ bin. As long as there is a typical $y$ in the $i^{th}$ bin, we transmit the corresponding $x$.

- To guarantee there is at least one typical sequence in each bin, the number of bins should be much smaller than $|A_\epsilon(Y)|$, $2^{n(H(Y)-\epsilon)}$ suffice.
What about a random channel $P_{Y|X}$?

- Introduce an auxiliary random variable $U$ as the desired output.

- Observing $Y$, we can determine $2^{nI(U;Y)}$ different $U$ sequences.

- Notice we can talk about both the discrete and the continuous cases

- Point-to-point case: $X = U$, trivial
Capacity Region for Deterministic Broadcast Channel

**Theorem** Fix an input distribution, any rate pair \((R_1, R_2)\) that satisfies

\[
\begin{align*}
R_1 &< H(Y_1) \\
R_2 &< H(Y_2) \\
R_1 + R_2 &< H(Y_1, Y_2)
\end{align*}
\]

can be achieved.

- Divide the output space \(\mathcal{Y}_1^n \times \mathcal{Y}_2^n\) into \(2^{nR_1} \times 2^{nR_2}\) bins,

- For an input message \(M_1 \times M_2 \in \{1, \ldots, 2^{nR_1}\} \times \{1, \ldots, 2^{nR_2}\}\), find a typical sequence \(y_1 \times y_2\) in the corresponding bin. This is the desired output

- Transmit \(x\) to produce \(y_1, y_2\).

- \(R_1 < H(Y_1)\) ensures that exists typical \(y_1\) per bin

- \(R_1 + R_2 < H(Y_1, Y_2)\) ensures that exists a typical \(y_1, y_2\) per product bin
General Broadcast Channel

Now if channel has randomness $P_{Y_1,Y_2|X}$.

- Introduce auxiliary random variables $U, V$.
- For fixed distribution of $U, V$, from the channel output, the number of distinguishable sequences $u$ and $v$ are $2^{nI(U;Y_1)}$ and $2^{nI(V;Y_2)}$.

**Theorem** (Marton 75’) The broadcast channel capacity region is given by

\begin{align*}
R_1 & \leq I(U;Y_1) \\
R_2 & \leq I(V;Y_2) \\
R_1 + R_2 & \leq I(U;Y_1) + I(V;Y_2) - I(U;V)
\end{align*}

for a fixed distribution $P_{U,V,X}$. 
Outline of Proof

- generate $2^{nI(U;Y_1)}$ typical sequence $u$'s and throw into $2^{nR_1}$ bins, generate $2^{n(I(V;Y_2)}$ typical $v$'s throw into $2^{nR_2}$ bins.

- upon receiving $Y_1$, $u$ can be uniquely determined. similar to $v$.

- There are $2^{n(I(U;Y_1)+I(V;Y_2))}$ possible $(u,v)$ pairs. Each pair being jointly typical with probability $2^{-nI(U;V)}$.

- If $R_1+R_2 \leq I(U;Y_1)+I(V;Y_2)-I(U;V)$, then there exists a jointly typical pair $(u,v)$ in each product bin. This is the desired received sequences.

- To transmit that bin number, simply transmit $x$ that is jointly typical with $(u,v)$. 
Wireless Channels

Key difference from AWGN channel Multi-path Fading

Flat fading model

\[ Y_i = H_i X_i + W_i \]

where \( W_i \) is the AWGN with variance \( \sigma_W^2 \).

- \( \{H_i\} \) is a random process, for which the marginal distribution is modelled as unit variance Rayleigh or Ricean

- The rate at which \( \{H_i\} \) changes over time depends on
  - the speed that receiver moves,
  - the environment
  - the carrier frequency
  - the symbol period
The wireless challenge

• How do we define a capacity for this random channel
  – the assumptions on the availability of CSI
  – the assumptions on the channel time-variation

• How well the different assumptions apply to practical channels

• How do the optimal signaling change with the channel assumptions

• How do these apply when we have a network

Example TCP
First simple example

- Assume $H$ remains constant
- Assume $H$ is perfectly known at the receiver
- Assume the transmitter does not know $H$, so the input distribution can not depend on $H$.

The capacity

$$C(H) = \max_{P_X} I(X; Y)$$

where

$$P_{Y|X} = N(HX, \sigma_w^2)$$

$$I(X; Y) = h(Y) - h(Y|X)$$
$$= h(Y) - h(W)$$

- To maximize $h(Y)$, let $X$ be Gaussian
- Resulting $Y \sim N(0, |H|^2\sigma_X^2 + \sigma_w^2)$

$$C(H) = \frac{1}{2} \log \left( 1 + \frac{|H|^2\sigma_X^2}{\sigma_w^2} \right)$$

The capacity is a function of $H$. 
Outage Capacity and Ergodic Capacity

- From the transmitter point of view ($H$ unknown), what is the maximum rate that can be transmitted with 0 probability?
- Shannon capacity for fixed fading channel is 0.
- Considering the distribution of $H$, $C(H)$ is a random variable.

**Definition** $a\%$ outage capacity is the data rate that can be supported with $a\%$, i.e.,

$$P(C(H) < R) \leq a\%$$

**Definition** The ergodic capacity

$$C = E[C(H)]$$
Ergodic Capacity

\[ C = E \left[ \frac{1}{2} \log \left( 1 + |H|^2 \frac{P}{N} \right) \right] \]

• To achieve the ergodic capacity, we need to code long enough that the statistics of \( H \) start to kick in.

• Do we want channel to be time-varying or static?

• Interleaving, delay, rate, burstiness trade-off.

• Diversity issues.

• Evaluating the capacity is hard.

• Two asymptotic results
  – as \( P/N \to 0 \), low SNR

  \[ C \to E[|H|^2] \frac{1}{2} \frac{P}{N} \]

  the same limit as the AWGN channel

  – as \( P/N \to \infty \), high SNR

  \[ C \to \frac{1}{2} E[\log |H|^2] + \frac{1}{2} \log \frac{P}{N} \]

Different from the AWGN channel by a constant.