1 Lucas-Phelps islands

- Lucas 1972
- overlapping generations
- agents work at date $t$ consume at date $t + 1$
- preferences

$$E \left[ C_{i,t+1} - \frac{1}{2} N_{i,t}^2 \right]$$
agents work, accumulate money, spend, die

\[ Y_{i,t} = N_{i,t} \]
\[ M_{i,t+1} = P_{i,t} Y_{i,t} e^{\varepsilon t+1} \]
\[ P_{j,t+1} C_{i,t+1} = M_{i,t+1} \]

- \( e^{\varepsilon t+1} - 1 \) is a proportional subsidy from the government (money injection)

\[ M_{t+1} = M_t e^{\varepsilon t+1} \]

- at date \( t + 1 \) agent \( i \) consumes the output of agent \( j \)
• continuum of islands, \( i \in [0, 1] \)

• unit mass of agents on each

• old agents receive proportional transfer of money from govt’

• they travel to one island where they spend all their money

• prices \( P_{i,t} \) determined in Walrasian equilibrium

• young agents decide their labor supply based on \( P_{i,t} \)
• old agents in island $i$ are representative sample

• but different mass $\exp\{u_{i,t}\}$ in island $i$

• nominal demand demand in island $i$ is

$$e^{u_{i,t}} \int_0^1 M_{i,t} di = e^{u_{i,t}} M_t$$

• $u_{i,t}$ normal with

$$\int_0^1 e^{u_{i,t}} di = 1$$
• normal idiosyncratic demand shock: $u_{i,t} \sim N\left(0, \sigma_u^2\right)$

• normal monetary shock: $\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$

• total nominal demand is

$$D_{i,t} = e^{u_{i,t}+\varepsilon_t}M_{t-1}$$

in logs

$$d_{i,t} = m_{t-1} + u_{i,t} + \varepsilon_t$$

*Market clearing*

$$P_{i,t}N_{i,t} = D_{i,t}$$
Information structure

- all agents observe \( \{M_{t-1}, M_{t-2}, \ldots\} \)

- old agents observe \( \varepsilon_t, P_{j,t} \)

- young agents only observe \( P_{i,t} \)

observing \( P_{i,t} \) and \( M_{t-1} \)—and knowing their own \( N_{i,t} \)—young agents can infer the sum

\[
u_{i,t} + \varepsilon_t = p_{i,t} + n_{i,t} - m_{t-1}\]
Labor supply

agents solve

$$\max_{N_{i,t}, C_{i,t+1}} E \left[ C_{i,t+1} - \frac{1}{2} N_{i,t}^2 \mid P_{i,t}, M_{t-1}, M_{t-2}, \ldots \right]$$

s.t. $$P_{j,t+1} C_{i,t+1} = P_{i,t} N_{i,t} e^{\varepsilon t+1}$$

substitute $$C_{i,t+1}$$ and obtain FOC:

$$E \left[ \frac{P_{i,t}}{P_{j,t+1}} e^{\varepsilon t+1} - N_{i,t} \mid P_{i,t}, M_{t-1}, M_{t-2}, \ldots \right] = 0$$

interpretation

$$N_{i,t} = \underbrace{E[\frac{P_{i,t}}{P_{j,t+1}} e^{\varepsilon t+1} \mid P_{i,t}, M_{t-1}, M_{t-2}, \ldots]}_{\text{labor supply}} + \underbrace{E[\text{exp.infl.}]}_{\text{exp.infl.}}$$
Equilibrium prices

Look for stationary equilibrium where

\[ \frac{P_{i,t}}{M_{t-1}} = G(u_{i,t}, \varepsilon_t) \]

Decompose

\[ N_{i,t} = E \left[ \frac{P_{i,t}}{P_{j,t+1}} e^{\varepsilon_{t+1}} \mid P_{i,t}, M_{t-1}, \ldots \right] = \]
\[ = E \left[ \frac{P_{i,t}}{M_t} \mid P_{i,t}, M_{t-1}, \ldots \right] E \left[ \frac{M_t}{P_{j,t+1}} e^{\varepsilon_{t+1}} \mid P_{i,t}, M_{t-1}, \ldots \right] \]
In stationary equilibrium second piece is a constant

\[ \xi = E \left[ \frac{e^{\xi_{t+1}}}{G(u_{j,t+1}, \xi_{t+1})} \right] \]

In the first piece, only information on \( M_t \) in the first piece is in \( P_{i,t} \) and \( M_{t-1} \).

So we have

\[ N_{i,t} = \xi E \left[ \frac{P_{i,t}}{M_t} \mid P_{i,t}, M_{t-1} \right] \]
From equilibrium condition we obtain

\[ e^{u_{i,t}} M_t = P_{i,t} N_{i,t} = P_{i,t} \xi E \left[ \frac{P_{i,t}}{M_t} \mid P_{i,t}, M_{t-1} \right] \]

in logs

\[ m_t - p_{i,t} + u_{i,t} = (\ldots) - E \left[ m_t - p_{i,t} \mid p_{i,t}, m_{t-1} \right] \]

(constant terms in (\ldots), depend on variances)

We obtain

\[ p_{i,t} = \bar{p} + \frac{1}{2} \left( m_t + u_{i,t} \right) + \frac{1}{2} E \left[ m_t \mid p_{i,t}, m_{t-1} \right] \]

We will see that this can be rewritten as

\[ p_{i,t} - m_{t-1} = \bar{p} + \frac{1}{2} \left( \varepsilon_t + u_{i,t} \right) + \frac{1}{2} E \left[ \varepsilon_t \mid \varepsilon_t + u_{i,t} \right] \]

confirming our conjecture that \( P_{i,t} / M_t \) is only a function of \( \varepsilon_t \) and \( u_{i,t} \)
Agents observe $m_{t-1}$ and nominal demand in their island

$$m_{t-1} + \varepsilon_t + u_{i,t}$$

Define

$$\bar{E}_t [m_t] = \int_0^1 E \left[ m_t | m_{t-1}, \varepsilon_t + u_{i,t} \right] \, di$$

Then, averaging, we have

$$p_t = \bar{p} + \frac{1}{2} m_t + \frac{1}{2} \bar{E}_t [m_t]$$
Imperfect information

\[ \overline{E}_t [m_t] \neq m_t \]

in particular

\[ E \left[ m_t | m_{t-1}, \varepsilon_t + u_{i,t} \right] = m_{t-1} + \beta (\varepsilon_t + u_{it}) \]

where

\[ \beta = \frac{\sigma^2_\varepsilon}{\sigma^2_\varepsilon + \sigma^2_u} \]

so

\[ \overline{E}_t [m_t] = m_{t-1} + \beta \varepsilon_t \neq m_{t-1} + \varepsilon_t \]
We have

\[ p_t = \bar{p} + m_{t-1} + \frac{1}{2} (1 + \beta) \varepsilon_t \]

and output is

\[ y_t = m_t - p_t \]
\[ = \bar{y} + \frac{1}{2} (1 - \beta) \varepsilon_t \]

- larger \( \sigma^2_\varepsilon / \sigma^2_\mu \) implies smaller real effects of monetary policy
- Phillips curve depends on the monetary regime
• OLS inflation on output gap

\[ \pi_t = \frac{1 + \beta}{1 - \beta} (y_t - \bar{y}) - \frac{1}{2} (\beta - 1) \varepsilon_{t-1} \]

• as \( \sigma^2_\varepsilon / \sigma^2_u \to \infty \) we have \( \beta \to 1 \) and vertical Phillips curve
Wrapping up:

- imperfect information affects transmission of monetary shocks

- in particular, explains sluggish response of prices to $m_t$: short-run non-neutrality

- crucial formal step: agents can be wrong on average $m_t \neq \bar{E}_t [m_t]$

- policy regime affects inference and thus effects of shocks
2 Higher order expectations

- price setting firms with quadratic costs

\[
\sum_{t=0}^{\infty} \beta^t E \left[ \frac{P_{i,t} Y_{i,t}}{P_t} - \frac{1}{2} (Y_{i,t})^2 \right]
\]

facing demand function

\[
Y_{i,t} = \left( \frac{P_{i,t}}{P_t} \right)^{-\sigma} \frac{M_t}{P_t}
\]

- have to set \( P_{i,t} \) with imperfect knowledge of \( M_t \)
• optimality condition

\[ E_{i,t} \left[ (\sigma - 1) \frac{1}{P_t} \left( \frac{P_{i,t}}{P_t} \right)^{-\sigma} M_t \right] = E_{i,t} \left[ \sigma \frac{1}{P_{i,t}} \left( \frac{P_{i,t}}{P_t} \right)^{-2\sigma} \left( \frac{M_t}{P_t} \right)^2 \right] \]

• if everything log-normal (or in approximation)

\[ E_{i,t} \left[ -\sigma p_{i,t} - (1 - \sigma) p_t + m_t - p_t \right] = \]
\[ = E_{i,t} \left[ -(1 + 2\sigma) p_{i,t} + 2\sigma p_t + 2(m_t - p_t) \right] \]
• we get

\[ p_{i,t} = \frac{\sigma}{1 + \sigma} E_{i,t}[p_t] + \frac{1}{1 + \sigma} E_{i,t}[m_t] \]

optimal price weighted average of expected price of other price setters

\[ \xi = \frac{1}{1 + \sigma} \]

\[ p_{i,t} = (1 - \xi) E_{i,t}[p_t] + \xi E_{i,t}[m_t] \]

• higher \( \sigma \) higher weight on other price setters prices: *strategic complementarity*
• now specify the information structure

• simple static case

\[ m_t = u_t \]

• private signal

\[ z_{i,t} = u_t + v_{i,t} \]

• Undetermined coefficients

\[ p_t = \phi m_t \]
• substituting gives

\[ \begin{align*}
    p_{i,t} &= ((1 - \xi) \phi + \xi) E_{i,t} [m_t] \\
    &= ((1 - \xi) \phi + \xi) \beta z_{i,t}
\end{align*} \]

• aggregating gives

\[ p_t = ((1 - \xi) \phi + \xi) \beta m_t \]

• fixed point

\[ \phi = ((1 - \xi) \phi + \xi) \beta \]

• solution

\[ \phi = \frac{\xi \beta}{1 - (1 - \xi) \beta} \]
- the response of prices to a monetary shock now depends on how informative is the signal and on the degree of strategic complementarity

  - less informative signal→smaller price response (bigger quantity response)

  - more strategic complementarity→smaller price response (bigger quantity response)

- Alternative interpretation: use notation

\[
\bar{E}_t [X_t] = \int E_{i,t} [X_t] \, di \\
\bar{E}_t^{(j)} [X_t] = \int E_{i,t}^{(j-1)} [X_t] \, di \text{ (higher order exp)} \\
p_t = (1 - \xi) \bar{E}_t [p_t] + \xi \bar{E} [m_t]
\]
\[ p_t = \sum_{j=0}^{\infty} \xi (1 - \xi)^j E_t^{(j)} [m_t] \]
\[ = \xi \beta \sum (1 - \xi)^j \beta^j m_t \]

• (obviously gives the same \( \phi \))
2.1 Dynamics

- process for money

\[ m_t = m_{t-1} + u_t \]

- price setters observe

\[ z_{i,t} = m_t + v_{i,t} \]

- need to form expectations about \( m_t \) and \( p_t \)

- Conjecture: state variables

\[
\begin{bmatrix}
  m_t \\
  p_t
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 \\
  \phi_m & \phi_p
\end{bmatrix}
\begin{bmatrix}
  m_{t-1} \\
  p_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
  u_t \\
  \phi_u u_t
\end{bmatrix}
\]
• Kalman filter

\[ E_{i,t} \begin{bmatrix} m_t \\ p_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi_m & \phi_p \end{bmatrix} E_{i,t-1} \begin{bmatrix} m_{t-1} \\ p_{t-1} \end{bmatrix} + K (z_{i,t} - E_{i,t-1}m_t) \]

• Integrating across agents we get

\[ \begin{bmatrix} m_{t|t} \\ p_{t|t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi_m & \phi_p \end{bmatrix} \begin{bmatrix} m_{t-1|t-1} \\ p_{t-1|t-1} \end{bmatrix} + K [m_t - m_{t|t-1}] \]

• Now use the optimality condition

\[ p_t = (1 - \xi) p_{t|t} + \xi m_{t|t} \]
(as in usual method of undetermined coefficients) to get

$$p_t = \begin{bmatrix} \xi & 1 - \xi \end{bmatrix} \begin{bmatrix} m_{t|t} \\ p_{t|t} \end{bmatrix} =$$

$$= \xi m_{t-1|t-1} + (1 - \xi) \left( \phi_m m_{t-1|t-1} + \phi_p p_{t-1|t-1} \right) +$$

$$+ \begin{bmatrix} \xi & 1 - \xi \end{bmatrix} K \left[ m_{t-1} + u_t - m_{t-1|t-1} \right]$$

- here is where we use Woodford’s trick: use

$$p_{t-1|t-1} = \frac{p_{t-1} - \xi m_{t-1|t-1}}{1 - \xi}$$

- now we have $p_t$ in terms of the state variables $p_{t-1}$ and $m_{t-1}$:

$$p_t = \phi_m m_{t-1} + \phi_p p_{t-1} + \phi_u u_t = \phi_p p_{t-1} + \left( \xi \begin{bmatrix} 1 - \xi \end{bmatrix} K m_{t-1} + u_t \right)$$

$$p_t = \phi_m m_{t-1} + \phi_p p_{t-1} + \phi_u u_t = \phi_p p_{t-1} + \left( \xi \begin{bmatrix} 1 - \xi \end{bmatrix} K m_{t-1} + u_t \right)$$

(1)
if the following condition is satisfied

\[ \xi + (1 - \xi) \left( \phi_m - \phi_p \xi / (1 - \xi) \right) - \left( \xi \ 1 - \xi \right) K = 0 \quad (2) \]

this condition makes the term with \( m_{t-1|t-1} \) disappear.

- Condition (2) pins down \( \phi_p \) (which, fortunately, is not pinned down by matching coefficients in (1)!!)

- So we have

\[ \phi_m = \phi_u = \phi = \left( \xi \ 1 - \xi \right) K \]

and from (2), after some algebra,

\[ \phi_p = 1 - \phi \]
Now we have a map: $\phi \rightarrow$ Kalman gains $K \rightarrow \phi'$

...and we need to find a fixed point of it

- Implications for price and output dynamics

$$m_t = m_{t-1} + u_t$$
$$p_t = (1 - \phi)p_{t-1} + \phi(m_{t-1} + u_t)$$

and

$$y_t = m_t - p_t = (1 - \phi)(m_{t-1} + u_t) - (1 - \phi)p_{t-1}$$
$$= (1 - \phi)(y_{t-1} + u_t)$$

- The parameter $\phi$ determines both the impact effect of the shock $u_t$ on prices and the persistence
• Computational experiments (see matlab codes): higher $\sigma^2_v$ and lower $\xi$ increase $\phi$

• (See the paper for a closed form expression for $\phi$ as a function of $\sigma^2_u/\sigma^2_v$ in equations 3.6 and 3.7 with $\phi = \hat{k}$)
2.2 Tools: Kalman filter

A simple intro to the Kalman filter. State space representation

\[ X_t = AX_{t-1} + U_t. \]

Information set \( \{Y_t, Y_{t-1}, \ldots\} \) where

\[ Y_t = FX_t + V_t. \]

We want to derive the steady state dynamics of

\[ X_{t|t} \equiv E_t [X_t]. \]

Assumption: \( U_t \) and \( W_t \) are mutually independent, each of them is an i.i.d. Gaussian vector with mean zero and variance-covariance matrices \( \Sigma_U \) and \( \Sigma_W \). (can be extended to \( U_t \) and \( W_t \) correlated)
Updating rule

\[ X_{t|t} = X_{t|t-1} + K \left( Y_t - Y_{t|t-1} \right) \]
\[ = AX_{t-1|t-1} + K \left( Y_t - Y_{t|t-1} \right) \]
\[ = (I - KF) AX_{t-1|t-1} + KY_t \]

(alternative common approach focuses on one-step-ahead forecast \( X_{t+1|t} \) here we focus on \( X_{t|t} \))

Define

\[ P = Var_{t-1} [X_t] \]

then \( K \) satisfies

\[ K = PF' \left( FPF' + \Sigma_V \right)^{-1} \]

(from orthogonality condition)
We need to find expressions for the matrix $P$. Bayesian updating for the variances gives

$$\hat{P} \equiv Var_t [X_t] = P - PF'(FPF' + \Sigma_V)^{-1} FP,$$

and the dynamics of $X_t$ imply that

$$Var_t [X_{t+1}] = A\hat{P}A' + \Sigma_U =$$

$$= A \left[ P - PF'(FPF' + \Sigma_V)^{-1} FP \right] A' + \Sigma_U$$

so imposing steady state for learning we have

$$P = A \left[ P - PF'(FPF' + \Sigma_W)^{-1} FP \right] A' + \Sigma_U$$

(Riccati equation for $P$)

**Example:** State law of motion:

$$x_t = x_{t-1} + \epsilon_t$$
Observation equation:

\[ y_t = x_t + \eta_t \]

Now

\[
k = \frac{P}{P + \sigma^2_\eta} = \frac{1/\sigma^2_\eta}{1/P + 1/\sigma^2_\eta}
\]

and \( P \) satisfies

\[
P = \frac{P}{P + \sigma^2_\eta} \sigma^2_\eta + \sigma^2_{\varepsilon}
\]

simple quadratic equation

\[
P = \frac{1}{2} \sigma^2_{\varepsilon} + \frac{1}{2} \sqrt{4 \sigma^2_{\varepsilon} \sigma^2_\eta + (\sigma^2_{\varepsilon})^2}
\]

Dynamics

\[
x_{t|t} = (1 - k) x_{t-1|t-1} + k y_t
\]