Section 1

Probability Spaces, Properties of Probability.

A pair $(\Omega, \mathcal{A})$ is a measurable space if $\mathcal{A}$ is a $\sigma$-algebra of subsets of $\Omega$. A collection $A$ of subsets of $\Omega$ is an algebra (ring) if:

1. $\Omega \in A$.
2. $C, B \in A \implies C \cap B, C \cup B \in A$.
3. $B \in A \implies \Omega \setminus B \in A$.
4. $A$ is a $\sigma$-algebra, if in addition, $C_i \in A, \forall i \geq 1 \implies \bigcup_{i \geq 1} C_i \in A$.

$(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space if $\mathbb{P}$ is a probability measure on $\mathcal{A}$, i.e.

1. $\mathbb{P}(\Omega) = 1$.
2. $\mathbb{P}(A) \geq 0, A \in \mathcal{A}$.
3. $\mathbb{P}$ is countably additive: $A_i \in \mathcal{A}, \forall i \geq 1, A_i \cap A_j = \emptyset \forall i \neq j \implies \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

An equivalent formulation of Property 3 is:

3'. $\mathbb{P}$ is a finitely additive measure and

$$B_n \supseteq B_{n+1}, \bigcap_{n \geq 1} B_n = B \implies \mathbb{P}(B) = \lim_{n} \mathbb{P}(B_n).$$

**Lemma 1** Properties 3 and 3' are equivalent.
Proof.

3 ⟹ 3': Let $C_n = B_n \setminus B_{n+1}$, then $B_n = B \bigcup \left( \bigcup_{k \geq n} C_k \right)$ - all disjoint.

By 3, $\mathbb{P}(B_n) = \mathbb{P}(B) + \sum_{k \geq n} \mathbb{P}(C_k) \to \mathbb{P}(B)$ when $n \to \infty$.

$3' \implies 3: \bigcup_{i \geq 1} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \cup \left( \bigcup_{i \geq n} A_i \right)$.

$\mathbb{P}\left( \bigcup_{i \geq 1} A_i \right) = \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n) + \mathbb{P}(B_n)$ where $B_n = \bigcup_{i \geq n} A_i$.

Since $B_n \supseteq B_{n+1}$ we have $\mathbb{P}(B_n) - \mathbb{P}\left( \bigcap_{n \geq 1} B_n \right) = \mathbb{P}(\emptyset) = 0$ because $A_i$’s are disjoint. □

Given algebra $A$, let $\mathcal{A} = \sigma(A)$ be a $\sigma$-algebra generated by $A$, i.e. intersection of all $\sigma$-algebras that contain $A$. It is easy to see that intersection of all such $\sigma$-algebras is itself a $\sigma$-algebra. Indeed, consider a sequence $A_i$ for $i \geq 1$ such that each $A_i$ belongs to all $\sigma$-algebras that contains $A$. Then $\bigcup_{i \geq 1} A_i$ belongs to all these $\sigma$-algebras and therefore to their intersection.

Let us recall an important result from measure theory.

**Theorem 1 (Caratheodory extension)** If $A$ is an algebra of sets and $\mu : A \to \mathbb{R}$ is a non-negative countably additive function on $A$, then $\mu$ can be extended to a measure on $\sigma$-algebra $\sigma(A)$. If $\mu$ is $\sigma$-finite, then this extension is unique. ($\sigma$-finite means that $\Omega = \bigcup_{i \geq 1} A_i$ for disjoint sequence $A_i$ and $\mu(A_i) < \infty$.)

**Example.** Let $A$ be an algebra of sets $\bigcup_{i \leq n} [a_i, b_i)$ where all $[a_i, b_i)$ are disjoint and $n \geq 1$. Let

$$\lambda\left( \bigcup_{i \leq n} [a_i, b_i) \right) = \sum_{i=1}^{n} |b_i - a_i|.$$  

One can prove that $\lambda$ is countably additive on $A$ and therefore can be extended to a Lebesgue measure $\lambda$ on a sigma algebra $\sigma(A)$ of Borel-measurable sets. □

**Lemma 2 (Approximation property)** If $A$ is an algebra of sets then for any $B \in \sigma(A)$ there exists a sequence $B_n \in A$ such that $\mathbb{P}(B \triangle B_n) \to 0$.

**Remark.** Here $\triangle$ denotes symmetric difference $(B \cup B_n) \setminus (B \cap B_n)$. Lemma states that any $B$ in $\sigma(A)$ can be approximated by elements of $A$.

**Proof.** Let

$$\mathcal{D} = \{ B \in \sigma(A) : \exists B_n \in A, \mathbb{P}(B \triangle B_n) \to 0 \}. $$

We will prove that $\mathcal{D}$ is a $\sigma$-algebra and since $A \subseteq \mathcal{D}$ this will imply that $\sigma(A) \subseteq \mathcal{D}$. One can easily check that

$$d(B, C) := \mathbb{P}(B \triangle C)$$

is a metric. It is also easy to check that

1. $d(BC, DE) \leq d(B, D) + d(C, E)$,
2. $|\mathbb{P}(B) - \mathbb{P}(C)| \leq d(B, C)$,
3. $d(B^c, C^c) = d(B, C)$.
Consider $D_1, \ldots, D_n \in \mathcal{D}$. If a sequence $C_{ij} \in \mathcal{A}$ for $j \geq 1$ approximates $D_i$, 

$$
\mathbb{P}(C_{ij} \Delta D_i) \to 0, j \to \infty
$$

then by properties 1 - 3, $C_j^n := \bigcup_{i \leq n} C_{ij}$ approximates $D^n := \bigcup_{i \leq n} D_i$, which means that $D^n \in \mathcal{D}$. Let $D = \bigcup_{i \geq 1} D_i$. Then 

$$
\mathbb{P}(D) = \mathbb{P}(D^n) + \mathbb{P}(D \setminus D^n)
$$

and obviously $\mathbb{P}(D \setminus D^n) \to 0$ as $n \to \infty$. Therefore, $D \in \mathcal{D}$ and $\mathcal{D}$ is a $\sigma$-algebra. 

\[ \square \]