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# Eisenstein Polynomials over Function Fields

Edoardo Dotti · Giacomo Micheli

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**Abstract** In this paper we compute the density of monic and non-monic Eisenstein polynomials of fixed degree having entries in an integrally closed subring of a function field over a finite field. This gives a function field analogue of results by A. Dubickas (2003) and by R. Heyman and I. Shparlinski (2013).

**Keywords** Function fields, Density, Polynomials, Riemann-Roch spaces.

## 1 Introduction

Let us start with the definition of *Eisenstein polynomial* and *natural density*:

**Definition 1** Let  $R$  be an integral domain. A polynomial  $f(X) = \sum_{i=0}^n a_i x^i \in R[X]$  is said to be Eisenstein if there exists a prime ideal  $\mathfrak{p} \subseteq R$  for which

- $a_i \in \mathfrak{p}$  for all  $i \in \{0, \dots, n-1\}$ ,
- $a_0 \notin \mathfrak{p}^2$ ,
- $a_n \notin \mathfrak{p}$ .

**Definition 2** A subset  $A$  of  $\mathbb{Z}^n$  is said to have *density*  $a$  if

$$a = \lim_{B \rightarrow \infty} \frac{|A \cap [-B, B]^n|}{(2B)^n}.$$

A classical result from the literature is that any Eisenstein polynomial is irreducible. In addition, observe that any polynomial of degree at most  $d$  and coefficients over  $\mathbb{Z}$  can be regarded as an element of  $\mathbb{Z}^{d+1}$ , while any monic polynomial of degree  $d$  can be regarded as an element of  $\mathbb{Z}^d$ . Recently, it has been of interest the explicit computation of the natural density of both degree  $d$  Eisenstein polynomials and monic Eisenstein polynomials over  $\mathbb{Z}$ , see for example [3, 1].

As was first proved by Dubickas in [1], the natural density of monic Eisenstein polynomials over  $\mathbb{Z}$  of fixed degree  $d$  is

$$1 - \prod_{p \text{ prime}} \left(1 - \frac{p-1}{p^{d+1}}\right). \quad (1)$$

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Heyman and Shparlinski extended the results of Dubickas to non-monic Eisenstein polynomials and computed the error term of the densities [3, Theorem 1, Theorem 2].

In this paper we would like to establish a function field analogue of these results that will include all the cases in which  $R$  is selected as an integrally closed subring of a function field of a curve over a finite field.

The general case that we will analyse needs an appropriate definition of density which makes use of Moore-Smith convergence for directed sets, as described in [5]. For the moment, let us fix the notation for the basic structures we are going to deal with, which is essentially the same as in [7].

Let  $q$  be a prime power and  $\mathbb{F}_q$  be the finite field of order  $q$ . Let  $F$  be a function field having full constant field  $\mathbb{F}_q$ . Let  $\mathbb{P}_F$  be the set of places of  $F$  and  $\mathcal{S}$  a non-empty proper subset of  $\mathbb{P}_F$ . Let us denote by  $\mathcal{O}_P$  the valuation ring at a place  $P$  of  $F$ . Let  $H = \bigcap_{P \in \mathcal{S}} \mathcal{O}_P$  be the holomorphy ring associated to  $\mathcal{S}$  [7, Definition 3.2.2]. As it is well known,  $H$  is a Dedekind Domain therefore any prime ideal is also maximal. In addition the maximal ideals of  $H$  correspond exactly to the places in  $\mathcal{S}$ , see [7, Proposition 3.2.9]. Therefore, if  $P$  is a place of  $F$  which lies in  $\mathcal{S}$  there exists a unique maximal ideal  $P_H \subseteq H$  corresponding to  $P$  for which  $P \cap H = P_H$ . In order not to heavier the notation, we will denote  $P_H$  again by  $P$ . Let  $\mathcal{D}$  be the set of positive divisors of  $\text{Div}(F)$  having support outside the holomorphy set  $\mathcal{S}$ . It is easy to observe

$$H = \bigcup_{D \in \mathcal{D}} \mathcal{L}(D)$$

and that  $\mathcal{D}$  is a directed set.

Let now  $A \subseteq H^m$ , we define the *upper* and *lower density* of  $A$  as

$$\overline{\mathbb{D}}(A) = \limsup_{D \in \mathcal{D}} \frac{|A \cap \mathcal{L}(D)^m|}{q^{m\ell(D)}},$$

$$\underline{\mathbb{D}}(A) = \liminf_{D \in \mathcal{D}} \frac{|A \cap \mathcal{L}(D)^m|}{q^{m\ell(D)}},$$

where the limit is defined using Moore-Smith convergence over the directed set  $\mathcal{D}$  (see [4, Chapter 2]). The *density* of  $A$  is then defined if  $\underline{\mathbb{D}}(A) = \overline{\mathbb{D}}(A) =: \mathbb{D}(A)$ .

As already observed in [5], if we specialize our definition of density to the case of the univariate polynomial ring over a finite field we get the usual definition of density for  $\mathbb{F}_q[x]$ , see for example [2, 8].

In addition, the final formulas for the density we get are analogous to the ones over the rational integers obtained in [1, 3].

The reader should notice that the results presented in this paper are also achievable by extending [6, Lemma 20] to holomorphy rings, using the definition of density previously described. Nevertheless the effect of this would be to involve tools from the theory of  $p$ -adic completions, whereas our approach only needs Riemann-Roch theorem and other elementary tools.

The paper is structured as follows: in the next subsection we specify the notation we are going to use for the rest of the paper, in section 2 we compute the density of monic Eisenstein polynomials, in section 3 we apply a similar strategy to compute the density of non-monic Eisenstein polynomials.

## 1.1 Notation

Throughout this paper, when  $Y$  is a set and  $m$  is a positive integer, we will denote by  $Y^m$  the cartesian product of  $m$ -copies of  $Y$ . To avoid confusion, the square of an ideal  $Q$  will then be denoted by  $\widehat{Q} = Q \cdot Q$ . Furthermore notice that in the whole paper we consider polynomials of degree  $d > 1$ . To easier the notation, we fix an enumeration  $\{Q_1, Q_2, \dots, Q_i, \dots\}$  of the places of  $\mathcal{S}$ . Since we will deal with the density of both monic and non-monic Eisenstein polynomials, we have to distinguish the notation, which we clarify in the following two paragraphs.

### Notation for monic Eisenstein polynomials

With a small abuse of notation we identify  $H^d$  with the set of all monic polynomials of degree  $d$  having entries over  $H$ . In particular, if  $(h_0, \dots, h_{d-1}) \in H^d$  then  $h_i$  denotes the coefficient of the monomial of degree  $i$ . Furthermore, we denote by  $\mathcal{E} \subset H^d$  the set of monic Eisenstein polynomials of degree  $d$  and by  $\mathcal{N}$  its complement in  $H^d$ . We denote by  $\mathcal{E}_i$  the set of monic polynomials which are Eisenstein with respect to  $Q_i$ :

$$\mathcal{E}_i = \{(h_0, \dots, h_{d-1}) \in H^d : h_j \in Q_i \forall j \in \{0, \dots, d-1\} \text{ and } h_0 \notin \widehat{Q}_i\}.$$

We denote by  $\mathcal{N}_i$  the complement of  $\mathcal{E}_i$ .

### Notation for non-monic Eisenstein polynomials

Analogously, we identify the set of all polynomials of degree  $d$  having entries over  $H$  with  $H^{d+1}$ . Let  $\mathcal{E}^+ \subseteq H^{d+1}$  be the set of Eisenstein polynomials of degree  $d$  and  $\mathcal{N}^+$  be its complement in  $H^{d+1}$ . We denote by  $\mathcal{E}_i^+$  the set of polynomials which are Eisenstein with respect to  $Q_i$ :

$$\mathcal{E}_i^+ = \{(h_0, \dots, h_d) \in H^{d+1} : h_j \in Q_i \forall j \in \{0, \dots, d-1\}, h_0 \notin \widehat{Q}_i \text{ and } h_d \notin Q_i\}.$$

We denote by  $\mathcal{N}_i^+$  the complement of  $\mathcal{E}_i^+$ .

## 2 Monic Eisenstein Polynomials

In this section we compute the density of monic Eisenstein polynomials via approximating the complement of  $\mathcal{E}$  (i.e.  $\mathcal{N}$ ) with  $\overline{\mathcal{N}}_t = \bigcap_{i=1}^t \mathcal{N}_i$ . First we show that we can explicitly compute the density of  $\overline{\mathcal{N}}_t$  (Proposition 1). Then, we give a criterion to check whether the approximation is “sharp”: i.e. whether the limit of the densities of  $\overline{\mathcal{N}}_t$  converges to the density of  $\mathcal{N}$  (Lemma 1). Finally, we verify that the conditions under which the approximation is sharp are verified (Theorem 1).

**Proposition 1** *The density of  $\overline{\mathcal{N}}_t$  is*

$$\mathbb{D}(\overline{\mathcal{N}}_t) = \prod_{i=1}^t \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1)\deg(Q_i)}} \right).$$

*Proof* Consider the map

$$\tilde{\phi} : H^d \rightarrow \left( H / (\widehat{Q}_1 \cdots \widehat{Q}_t) \right)^d,$$

which is defined componentwise by the reduction modulo the ideal  $(\widehat{Q}_1 \cdots \widehat{Q}_t)$ . Observe also that  $\left( H / (\widehat{Q}_1 \cdots \widehat{Q}_t) \right)^d \simeq \prod_{i=1}^t \left( H / \widehat{Q}_i \right)^d$ , by the Chinese Remainder Theorem.

Consider now a divisor  $D \in \mathcal{D}$ . In order to compute the density of  $\overline{\mathcal{N}}_t$  it is enough to count how many elements there are in  $\overline{\mathcal{N}}_t \cap \mathcal{L}(D)^d$ , when the degree of  $D$  is large.

We start by showing that  $\mathcal{L}(D)^d$  maps surjectively onto  $\left( H / (\widehat{Q}_1 \cdots \widehat{Q}_t) \right)^d$  when the degree of  $D$  is large enough.

For this consider the  $\mathbb{F}_q$  linear map  $\phi : \mathcal{L}(D) \rightarrow \left( H / (\widehat{Q}_1 \cdots \widehat{Q}_t) \right)$ . We have  $\ker(\phi) = \mathcal{L}(D) \cap (\widehat{Q}_1 \cdots \widehat{Q}_t)$ , which represents the elements of  $\mathcal{L}(D)$  having at least a double root at each  $Q_i$ . Hence  $\ker(\phi) = \mathcal{L}(D - 2 \sum_{i=1}^t Q_i)$ .

By Riemann's theorem [7, Theorem 1.4.17], if the degree of  $D$  is large enough, the dimension of the kernel as an  $\mathbb{F}_q$  vector space is

$$\ell \left( D - 2 \sum_{i=1}^t Q_i \right) = \deg \left( D - 2 \sum_{i=1}^t Q_i \right) + 1 - g = \deg(D) - 2 \sum_{i=1}^t \deg(Q_i) + 1 - g, \quad (2)$$

where  $g$  denotes the genus of the function field.

By the same theorem  $\ell(D) = \deg(D) + 1 - g$ . Hence we obtain

$$\dim_{\mathbb{F}_q} (\mathcal{L}(D)/\ker(\phi)) = \ell(D) - \ell \left( D - 2 \sum_{i=1}^t Q_i \right) = 2 \sum_{i=1}^t \deg(Q_i).$$

On the other hand, by the Chinese Remainder Theorem,

$$\dim_{\mathbb{F}_q} \left( H/(\widehat{Q}_1 \cdots \widehat{Q}_t) \right) = \dim_{\mathbb{F}_q} \left( H/\widehat{Q}_1 \times \cdots \times H/\widehat{Q}_t \right) = 2 \sum_{i=1}^t \deg(Q_i).$$

Therefore when the degree of  $D$  is large enough  $\phi$  is surjective, thus  $\tilde{\phi}$  is surjective.

Let  $\psi_i : \left( H/(\widehat{Q}_1 \cdots \widehat{Q}_t) \right)^d \longrightarrow \left( H/\widehat{Q}_i \right)^d$ . We have the following situation:

$$\mathcal{L}(D)^d \xrightarrow{\tilde{\phi}} \left( H/(\widehat{Q}_1 \cdots \widehat{Q}_t) \right)^d \xrightarrow{\psi} \prod_{i=1}^t \left( H/\widehat{Q}_i \right)^d,$$

where  $\psi = (\psi_1, \dots, \psi_t)$ . Notice that the check for  $f \in H^d$  not to be Eisenstein with respect to  $Q_i$  can be performed by looking at the reduction modulo  $\widehat{Q}_i$ . Therefore  $f \in \overline{\mathcal{N}}_t \cap \mathcal{L}(D)^d$  if and only if  $\psi_i \circ \tilde{\phi}(f) \notin \left( (Q_i/\widehat{Q}_i) \setminus \{0\} \right) \times \left( Q_i/\widehat{Q}_i \right)^{d-1} =: E_i$  for all  $i \in \{1, \dots, t\}$ .

It follows that  $\overline{\mathcal{N}}_t \cap \mathcal{L}(D)^d = \tilde{\phi}^{-1} \left( \psi^{-1} \left( \prod_{i=1}^t \left( (H/\widehat{Q}_i)^d \setminus E_i \right) \right) \right) \cap \mathcal{L}(D)^d$ . Hence

$$\begin{aligned} |\overline{\mathcal{N}}_t \cap \mathcal{L}(D)^d| &= |\ker(\tilde{\phi})| \cdot \prod_{i=1}^t \left| \left( H/\widehat{Q}_i \right)^d \setminus E_i \right| \\ &= q^{d(\deg(D) - 2 \sum_{i=1}^t \deg(Q_i) + 1 - g)} \cdot \prod_{i=1}^t \left| \left( H/\widehat{Q}_i \right)^d \setminus E_i \right|, \end{aligned}$$

where the last equality follows from (2). Now it remains to compute

$$\begin{aligned} \left| \left( H/\widehat{Q}_i \right)^d \setminus E_i \right| &= q^{2d \deg(Q_i)} - \left| \left( (Q_i/\widehat{Q}_i) \setminus \{0\} \right) \times \left( Q_i/\widehat{Q}_i \right)^{d-1} \right| \\ &= q^{2d \deg(Q_i)} - \left( q^{\deg(Q_i)} - 1 \right) \cdot q^{(d-1) \deg(Q_i)} \\ &= q^{2d \deg(Q_i)} \left( 1 - q^{-d \deg(Q_i)} + q^{-(d+1) \deg(Q_i)} \right). \end{aligned}$$

Therefore for  $D$  of degree large enough

$$\begin{aligned} \frac{|\overline{\mathcal{N}}_t \cap \mathcal{L}(D)^d|}{|\mathcal{L}(D)^d|} &= \frac{q^{d(\deg(D) - 2 \sum_{i=1}^t \deg(Q_i) + 1 - g)}}{q^{d(\deg(D) + 1 - g)}} \cdot \prod_{i=1}^t q^{2d \deg(Q_i)} \left( 1 - q^{-d \deg(Q_i)} + q^{-(d+1) \deg(Q_i)} \right) \\ &= \prod_{i=1}^t \left( 1 - q^{-d \deg(Q_i)} + q^{-(d+1) \deg(Q_i)} \right) = \prod_{i=1}^t \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1) \deg(Q_i)}} \right). \end{aligned}$$

Hence

$$\mathbb{D}(\overline{\mathcal{N}}_t) = \lim_{D \in \mathcal{D}} \frac{|\overline{\mathcal{N}}_t \cap \mathcal{L}(D)^d|}{|\mathcal{L}(D)^d|} = \prod_{i=1}^t \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1) \deg(Q_i)}} \right).$$

**Lemma 1** *Let  $n \in \mathbb{N}$ ,  $A \subseteq H^n$ . Let  $\{A_t\}_{t \in \mathbb{N}}$  be a family of subsets of  $H^n$  such that  $A_{t+1} \subseteq A_t$  and  $\bigcap_{t \in \mathbb{N}} A_t = A$ . Assume also that  $\mathbb{D}(A_t)$  exists for all  $t$ . If  $\lim_{t \rightarrow \infty} \mathbb{D}(A_t \setminus A) = 0$ , then  $\mathbb{D}(A) = \lim_{t \rightarrow \infty} \mathbb{D}(A_t)$ .*

*Proof* We start from the equality  $|A_t \cap \mathcal{L}(D)^n| = |A \cap \mathcal{L}(D)^n| + |(A_t \setminus A) \cap \mathcal{L}(D)^n|$ , from which it follows that

$$\begin{aligned} \liminf_{D \in \mathcal{D}} \frac{|A \cap \mathcal{L}(D)^n|}{|\mathcal{L}(D)^n|} &= \liminf_{D \in \mathcal{D}} \left( \frac{|A_t \cap \mathcal{L}(D)^n|}{|\mathcal{L}(D)^n|} - \frac{|(A_t \setminus A) \cap \mathcal{L}(D)^n|}{|\mathcal{L}(D)^n|} \right) \\ &\geq \liminf_{D \in \mathcal{D}} \frac{|A_t \cap \mathcal{L}(D)^n|}{|\mathcal{L}(D)^n|} - \limsup_{D \in \mathcal{D}} \frac{|(A_t \setminus A) \cap \mathcal{L}(D)^n|}{|\mathcal{L}(D)^n|}. \end{aligned}$$

Therefore we have  $\mathbb{D}(A_t) - \overline{\mathbb{D}}(A_t \setminus A) \leq \mathbb{D}(A)$ . Since  $\mathbb{D}(A_t)$  exists for all  $t$  we get

$$\mathbb{D}(A_t) - \overline{\mathbb{D}}(A_t \setminus A) \leq \mathbb{D}(A).$$

Now notice that  $\lim_{t \rightarrow \infty} \mathbb{D}(A_t)$  exists since  $\mathbb{D}(A_t)$  is decreasing and bounded from below. By taking the limit in  $t$ , the last expression then becomes

$$\lim_{t \rightarrow \infty} \mathbb{D}(A_t) - \lim_{t \rightarrow \infty} \overline{\mathbb{D}}(A_t \setminus A) \leq \mathbb{D}(A).$$

Since  $\lim_{t \rightarrow \infty} \overline{\mathbb{D}}(A_t \setminus A) = 0$  by assumption, it follows that  $\lim_{t \rightarrow \infty} \mathbb{D}(A_t) \leq \mathbb{D}(A)$ .

On the other hand  $A \subseteq A_t$  which implies  $\overline{\mathbb{D}}(A) \leq \mathbb{D}(A_t)$ . In particular  $\overline{\mathbb{D}}(A) \leq \lim_{t \rightarrow \infty} \mathbb{D}(A_t)$ . Combining all together we get

$$\lim_{t \rightarrow \infty} \mathbb{D}(A_t) \leq \mathbb{D}(A) \leq \overline{\mathbb{D}}(A) \leq \lim_{t \rightarrow \infty} \mathbb{D}(A_t),$$

therefore the claim follows.

**Theorem 1** *The density of the set of monic Eisenstein polynomials with coefficients over  $H$  is*

$$\mathbb{D}(\mathcal{E}) = 1 - \prod_{Q \in \mathcal{S}} \left( 1 - \frac{q^{\deg(Q)} - 1}{q^{(d+1)\deg(Q)}} \right).$$

*Proof* We make use of Lemma 1 for the family  $\{\overline{\mathcal{N}}_t\}_{t \in \mathbb{N}}$ . Hence we want to show that  $\lim_{t \rightarrow \infty} \overline{\mathbb{D}}(\overline{\mathcal{N}}_t \setminus \mathcal{N}) = 0$ .

First notice that

- $\overline{\mathcal{N}}_t \setminus \mathcal{N} = \bigcup_{r>t} \mathcal{E}_r \subseteq \bigcup_{r>t} Q_r^d$ ,
- $Q_r \cap \mathcal{L}(D) = \mathcal{L}(D - Q_r) = 0$ , if  $\deg(D) - \deg(Q_r) < 0$ .

Now we get

$$\begin{aligned} \overline{\mathbb{D}}(\overline{\mathcal{N}}_t \setminus \mathcal{N}) &= \limsup_{D \in \mathcal{D}} \frac{|(\overline{\mathcal{N}}_t \setminus \mathcal{N}) \cap \mathcal{L}(D)^d|}{|\mathcal{L}(D)^d|} \leq \limsup_{D \in \mathcal{D}} \left| \bigcup_{\substack{r>t \\ \deg(Q_r) \leq \deg(D)}} Q_r^d \cap \mathcal{L}(D)^d \right| q^{-d\ell(D)} \\ &= \limsup_{D \in \mathcal{D}} \left| \bigcup_{\substack{r>t \\ \deg(Q_r) \leq \deg(D)}} \mathcal{L}(D - Q_r)^d \right| q^{-d\ell(D)} \leq \limsup_{D \in \mathcal{D}} \sum_{\substack{r>t \\ \deg(Q_r) \leq \deg(D)}} \frac{q^{d\ell(D - Q_r)}}{q^{d\ell(D)}}. \end{aligned} \quad (3)$$

Observe now that if  $\deg(D - Q_r) \geq 0$  we have that  $\ell(D - Q_r) \leq \deg(D - Q_r) + 1$  [7, Eq. 1.21] and also that  $\ell(D) \geq \deg(D) + 1 - g$  by Riemann's theorem. Hence we have that (3) is less than or equal to

$$\limsup_{D \in \mathcal{D}} \sum_{\substack{r>t \\ \deg(Q_r) \leq \deg(D)}} \frac{q^{d(1+\deg(D) - \deg(Q_r))}}{q^{d(\deg(D) + 1 - g)}} \leq \sum_{r>t} q^{d(g - \deg(Q_r))} = q^{dg} \sum_{r>t} q^{-d\deg(Q_r)}.$$

We now notice that  $\sum_{r>t} q^{-d\deg(Q_r)}$  is the tail of a subseries of the Zeta function, which is absolutely convergent for  $d > 1$ . Letting  $t$  going to infinity the tail converges to 0, thus  $\lim_{t \rightarrow \infty} \overline{\mathbb{D}}(\overline{\mathcal{N}}_t \setminus \mathcal{N}) = 0$ .

$\mathcal{N}) = 0$ . We are now able to apply Lemma 1 with  $n = d$ ,  $A_t = \overline{\mathcal{N}}_t$  and  $A = \mathcal{N}$

$$\mathbb{D}(\mathcal{N}) = \lim_{t \rightarrow \infty} \mathbb{D}(\overline{\mathcal{N}}_t) = \lim_{t \rightarrow \infty} \prod_{i=1}^t \left( 1 - \frac{q^{\deg(Q_i)} - 1}{q^{(d+1)\deg(Q_i)}} \right) = \prod_{Q \in \mathcal{S}} \left( 1 - \frac{q^{\deg(Q)} - 1}{q^{(d+1)\deg(Q)}} \right).$$

We conclude by taking the complement

$$\mathbb{D}(\mathcal{E}) = 1 - \mathbb{D}(\mathcal{N}) = 1 - \prod_{Q \in \mathcal{S}} \left( 1 - \frac{q^{\deg(Q)} - 1}{q^{(d+1)\deg(Q)}} \right).$$

### 3 Non-Monic Eisenstein Polynomials

In this section we compute the density of Eisenstein polynomials applying the same strategy as that of section 2. For this let  $\overline{\mathcal{N}}_t^+ = \bigcap_{i=1}^t \mathcal{N}_i^+$ .

**Proposition 2** *The density of  $\overline{\mathcal{N}}_t^+$  is*

$$\mathbb{D}(\overline{\mathcal{N}}_t^+) = \prod_{i=1}^t \left( 1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}} \right).$$

*Proof* Consider a divisor  $D \in \mathcal{D}$ . With the same reasoning of the monic case one can show that  $\mathcal{L}(D)^{d+1}$  maps surjectively onto  $(H/(\widehat{Q}_1 \cdots \widehat{Q}_t))^{d+1}$  when the degree of  $D$  is large enough.

Let  $\psi_i : (H/(\widehat{Q}_1 \cdots \widehat{Q}_t))^{d+1} \rightarrow (H/\widehat{Q}_i)^{d+1}$  as before. The situation is now the following:

$$\mathcal{L}(D)^{d+1} \xrightarrow{\tilde{\phi}} (H/(\widehat{Q}_1 \cdots \widehat{Q}_t))^{d+1} \xrightarrow{\psi} \prod_{i=1}^t (H/\widehat{Q}_i)^{d+1},$$

where  $\psi = (\psi_1, \dots, \psi_t)$ .

Analogously to the case of monic polynomials we notice that we can verify that  $f \in H$  is not Eisenstein with respect to  $Q_i$  by looking at the reduction modulo  $\widehat{Q}_i$ . Hence  $f \in \overline{\mathcal{N}}_t^+ \cap \mathcal{L}(D)^{d+1}$  if and only if  $\psi_i \circ \tilde{\phi}(f) \notin ((Q_i/\widehat{Q}_i) \setminus \{0\}) \times (Q_i/\widehat{Q}_i)^{d-1} \times ((H/\widehat{Q}_i) \setminus (Q_i/\widehat{Q}_i)) =: E_i^+$  for all  $i \in \{1, \dots, t\}$ .

Hence we get

$$\begin{aligned} |\overline{\mathcal{N}}_t^+ \cap \mathcal{L}(D)^{d+1}| &= |\ker(\tilde{\phi})| \cdot \prod_{i=1}^t |(H/\widehat{Q}_i)^{d+1} \setminus E_i^+| \\ &= q^{(d+1)(\deg(D) - 2 \sum_{i=1}^t \deg(Q_i) + 1 - g)} \cdot \prod_{i=1}^t |(H/\widehat{Q}_i)^{d+1} \setminus E_i^+|, \end{aligned}$$

where

$$\begin{aligned} |(H/\widehat{Q}_i)^{d+1} \setminus E_i^+| &= q^{2(d+1)\deg(Q_i)} - |((Q_i/\widehat{Q}_i) \setminus \{0\}) \times (Q_i/\widehat{Q}_i)^{d-1} \times ((H/\widehat{Q}_i) \setminus (Q_i/\widehat{Q}_i))| \\ &= q^{2(d+1)\deg(Q_i)} - \left( (q^{\deg(Q_i)} - 1) q^{(d-1)\deg(Q_i)} (q^{2\deg(Q_i)} - q^{\deg(Q_i)}) \right) \\ &= q^{2(d+1)\deg(Q_i)} \left( 1 - \frac{q^{2\deg(Q_i)} - 2q^{\deg(Q_i)} + 1}{q^{(d+2)\deg(Q_i)}} \right) \\ &= q^{2(d+1)\deg(Q_i)} \left( 1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}} \right). \end{aligned}$$

Therefore for  $D$  of degree large enough

$$\begin{aligned} \frac{|\overline{\mathcal{N}}_t^+ \cap \mathcal{L}(D)^{d+1}|}{|\mathcal{L}(D)^{d+1}|} &= \frac{q^{(d+1)(\deg(D)-2\sum_{i=1}^t \deg(Q_i)+1-g)}}{q^{(d+1)(\deg(D)+1-g)}} \cdot \prod_{i=1}^t q^{2(d+1)\deg(Q_i)} \left(1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}}\right) \\ &= \prod_{i=1}^t \left(1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}}\right). \end{aligned}$$

Hence

$$\mathbb{D}(\overline{\mathcal{N}}_t^+) = \lim_{D \in \mathcal{D}} \frac{|\overline{\mathcal{N}}_t^+ \cap \mathcal{L}(D)^{d+1}|}{|\mathcal{L}(D)^{d+1}|} = \prod_{i=1}^t \left(1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}}\right).$$

**Theorem 2** *The density of the set of Eisenstein polynomials with coefficients over  $H$  is*

$$\mathbb{D}(\mathcal{E}^+) = 1 - \prod_{Q \in \mathcal{S}} \left(1 - \frac{(q^{\deg(Q)} - 1)^2}{q^{(d+2)\deg(Q)}}\right).$$

*Proof* Again by Lemma 1 we have to show that  $\lim_{t \rightarrow \infty} \mathbb{D}(\overline{\mathcal{N}}_t^+ \setminus \mathcal{N}^+) = 0$ . Observe that  $\mathcal{E}_r^+ \cap \mathcal{L}(D)^{d+1} \subseteq Q_r^d \times \mathcal{L}(D)$ . We get

$$\begin{aligned} \mathbb{D}(\overline{\mathcal{N}}_t^+ \setminus \mathcal{N}^+) &= \limsup_{D \in \mathcal{D}} \frac{|\overline{\mathcal{N}}_t^+ \setminus \mathcal{N}^+ \cap \mathcal{L}(D)^{d+1}|}{|\mathcal{L}(D)^{d+1}|} \\ &\leq \limsup_{D \in \mathcal{D}} \left| \bigcup_{\substack{r > t \\ \deg(Q_r) \leq \deg(D)}} \mathcal{E}_r^+ \cap \mathcal{L}(D)^{d+1} \right| q^{-(d+1)\ell(D)} \\ &\leq \limsup_{D \in \mathcal{D}} \left| \bigcup_{\substack{r > t \\ \deg(Q_r) \leq \deg(D)}} (Q_r^d \times \mathcal{L}(D)) \cap \mathcal{L}(D)^{d+1} \right| q^{-(d+1)\ell(D)} \\ &\leq \limsup_{D \in \mathcal{D}} \sum_{\substack{r > t \\ \deg(Q_r) \leq \deg(D)}} \frac{|(Q_r^d \times \mathcal{L}(D)) \cap \mathcal{L}(D)^{d+1}|}{q^{(d+1)\ell(D)}} \\ &= \limsup_{D \in \mathcal{D}} \sum_{\substack{r > t \\ \deg(Q_r) \leq \deg(D)}} \frac{|Q_r \cap \mathcal{L}(D)|^d |\mathcal{L}(D)|}{q^{(d+1)\ell(D)}} \\ &= \limsup_{D \in \mathcal{D}} \sum_{\substack{r > t \\ \deg(Q_r) \leq \deg(D)}} \frac{|Q_r \cap \mathcal{L}(D)|^d}{q^{d\ell(D)}} \end{aligned}$$

which is the expression appearing in inequality (3). Hence for  $t$  going to infinity we obtain  $\mathbb{D}(\overline{\mathcal{N}}_t^+ \setminus \mathcal{N}^+) = 0$ .

We now apply Lemma 1 with  $n = d + 1$ ,  $A_t = \overline{\mathcal{N}}_t^+$  and  $A = \mathcal{N}^+$  obtaining

$$\mathbb{D}(\mathcal{N}^+) = \lim_{t \rightarrow \infty} \mathbb{D}(\overline{\mathcal{N}}_t^+) = \lim_{t \rightarrow \infty} \prod_{i=1}^t \left(1 - \frac{(q^{\deg(Q_i)} - 1)^2}{q^{(d+2)\deg(Q_i)}}\right) = \prod_{Q \in \mathcal{S}} \left(1 - \frac{(q^{\deg(Q)} - 1)^2}{q^{(d+2)\deg(Q)}}\right).$$

We now take the complement

$$\mathbb{D}(\mathcal{E}^+) = 1 - \mathbb{D}(\mathcal{N}^+) = 1 - \prod_{Q \in \mathcal{S}} \left(1 - \frac{(q^{\deg(Q)} - 1)^2}{q^{(d+2)\deg(Q)}}\right).$$



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