UNIMODAL, LOG-CONCAVE AND PÓLYA FREQUENCY SEQUENCES IN COMBINATORICS

by

Francesco Brenti

Laurea, Università di Firenze
(1982)

Submitted to the Department of Mathematics in partial
fulfillment of the requirements for the degree of

Doctor of Philosophy

at the
Massachusetts Institute of Technology
June, 1988

©Francesco Brenti, 1988

The author hereby grants to M.I.T. permission to reproduce and
to distribute copies of this thesis document in whole or in part

Signature redacted

Signature of Author _____

Department of Mathematics
April 29, 1988

Signature redacted

Certified by _____

Signature redacted Professor Richard P. Stanley
Thesis Supervisor

Accepted by _____

Professor Sigurdur Helgason, Chairman
Departmental Graduate Committee
Department of Mathematics

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY

AUG 19 1988

LIBRARIES

Archives

Unimodal, Log-concave and Pólya frequency sequences in Combinatorics

by

Francesco Brenti

Submitted to the Department of Mathematics on April
29, 1988 in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in Mathematics.

Abstract

A sequence of real numbers $\{a_n\}_{n=0,1,2,\dots}$ is said to be a *Pólya frequency sequence of order r* (or, a *PF_r sequence*, for short) if, for all $1 \leq t \leq r$, the determinants of all the minors of order t of the infinite matrix $(a_{j-i})_{i,j=0,1,2,\dots}$ (where $a_k \stackrel{\text{def}}{=} 0$ if $k < 0$) are nonnegative. Clearly $PF_2 \Rightarrow \text{log-concave}$ and $PF \Rightarrow PF_2 \Rightarrow \text{unimodal}$. It is known that a sequence $\{a_0, a_1, \dots, a_d, 0, 0, \dots\}$ is *PF* if and only if the polynomial $\sum_{i=0}^d a_i x^i$ has nonnegative coefficients and only real zeros.

Many sequences in Combinatorics are known to be unimodal, *PF₂* or *PF*. The object of this thesis is to develop general methods which generate new unimodal, *PF₂* or *PF* sequences from existing ones, and to apply such methods to prove the unimodality, *PF₂* or *PF* property of sequences of combinatorial interest.

In the first part of this thesis we consider the vector space V_d of all polynomials having real coefficients and degree $\leq d$. If $\{v_i(x)\}_{i=0,\dots,d}$ is an ordered basis of V_d then we define a subset $PF[v_i] \subseteq V_d$ by $PF[v_i] \stackrel{\text{def}}{=} \{\sum_{i=0}^d a_i v_i(x) \in V_d : \{a_0, a_1, \dots, a_d, 0, 0, \dots\} \text{ is a } PF \text{ sequence}\}$ and we define $PF_2[v_i]$ and $PF_1[v_i]$ analogously. We then investigate which inclusions hold between these subsets if $\{v_i\}_{i=0,\dots,d}$ and $\{u_i\}_{i=0,\dots,d}$ are two different bases of V_d . Using the theory of total positivity we find many interesting results in this direction. For example,

Theorem: *The following strict inclusions hold in V_d :*

$$PF \left[\left(\binom{x}{i} \right) \right] \subset PF[\langle x \rangle_i] \subset PF[x^i] \subset PF[(x)_i],$$

$$PF \left[\binom{x+d-i}{d} \right] \subset PF \left[\binom{x}{i} \right] \subset PF[(x)_i],$$

where $(x)_i \stackrel{\text{def}}{=} \binom{x}{i} (i!)$, $\langle x \rangle_i \stackrel{\text{def}}{=} (-1)^i (-x)_i$ and $\left(\binom{x}{i} \right) \stackrel{\text{def}}{=} \langle x \rangle_i / (i!)$.

Theorem: *The following strict inclusions hold in V_d :*

$$PF_2 \left[\left(\binom{x}{i} \right) \right] \subset PF_2[\langle x \rangle_i] \cap PF_2 \left[\binom{x}{i} \right],$$

$$PF_2 \left[\binom{x+d-i}{d} \right] \subset PF_2 \left[\binom{x}{i} \right] \subset PF_2[(x)_i],$$

$$PF[(x)_i] \subset PF_2 \left[\binom{x}{i} \right], \quad PF[\langle x \rangle_i] \subset PF_2 \left[\left(\binom{x}{i} \right) \right].$$

Theorem: Let $A(x) \in V_d$. Then the following are equivalent:

- (i) $A(x) \in PF \left[\binom{x+d-i}{d} \right];$
- (ii) $A(x) \in PF \left[\binom{x}{i} \right] \cap PF_1 \left[\binom{x+d-i}{d} \right];$
- (iii) the sequence $\{A(n)\}_{n=0,1,2,\dots}$ is a *PF sequence*.

In the second part of this thesis we apply the results found in the first one to combinatorial problems. In particular we verify the Poset Conjecture for labeled disjoint unions of chains, naturally labeled Gaussian posets and naturally or column strict labeled Ferrers posets thus proving a generalization of a conjecture of R. Stanley. Other applications to enumerative Combinatorics include polynomials arising from symmetric functions and from Jack polynomials, zeta polynomials of partially ordered sets, chromatic polynomials of graph, functions of a finite set into itself, associated Lah numbers, Stirling polynomials, Ward and Jordan numbers. We also introduce a new class of digraphs which we call *PF digraphs* and show their relevance to many of the combinatorial problems considered in this thesis.

Thesis Supervisor: Richard P. Stanley.
Title: Professor of Applied Mathematics.

**Unimodal, Log-Concave and Pólya
Frequency Sequences in Combinatorics**

a mio padre

*“[Mathematics] which has
enriched my life with so
many joys.”*

C. F. GAUSS
(Letter to S. Germain)

Contents

<i>Aknowledgments</i>	15
<i>Foreword</i>	17
Chapter 1 The Poset Conjecture	19
1.1 Introduction	19
1.2 Statement of the Poset Conjecture	19
1.3 Partial results on the Poset Conjecture	23
1.4 A second look at the Poset Conjecture	26
 Chapter 2 A general theory	 29
2.1 Introduction	29
2.2 Polynomials with only real zeros and total positivity	30
2.3 The six fundamental bases and the transition matrices between them	35
2.4 Linear Transformations that preserve the PF property	41
2.5 Linear Transformations that preserve the PF_2 property	47
2.6 Summary of results and open problems	53
 Chapter 3 Ramifications of the general theory	 57
3.1 Introduction	57
3.2 Further properties of the six fundamental bases	58
3.3 The special case of polynomials with no constant term	61
3.4 Eulerian, Lagrange, and Krawtchouk polynomials	63
3.5 Three general problems	69

<i>Chapter 4</i>	Polynomials in $\text{PF}\left[\binom{x+d-i}{d}\right]$	75
4.1	Introduction	75
4.2	Elementary properties of $\text{PF}\left[\binom{x+d-i}{d}\right]$	77
4.3	A Fundamental Theorem	79
4.4	Consequences of the Fundamental Theorem	83
4.5	The characterization of PF-sequences	85
4.6	The characterization of $\text{PF}\left[\binom{x+d-i}{d}\right]$	90
4.7	Products of polynomials in $\text{PF}\left[\binom{x+d-i}{d}\right]$	93
 <i>Chapter 5</i>	 Applications to the Poset Conjecture	 101
5.1	Introduction	101
5.2	Ferrers Posets	102
5.3	Column strict labeled Ferrers Posets and the solution of a conjecture by R. Stanley	106
5.4	A class of naturally labeled Ferrers Posets	109
5.5	Disjoint unions of chains	111
5.6	Gaussian Posets	112
5.7	Necessary and sufficient conditions for the Poset Conjecture	115
 <i>Chapter 6</i>	 Applications to Enumerative Combinatorics	 119
6.1	Introduction	119
6.2	PF sequences arising from symmetric functions and Jack polynomials	120
6.3	Zeta polynomials of partially ordered sets	125
6.4	Functions of a finite set into itself	129
6.5	Associated Lah numbers	131
6.6	Stirling permutations	134
6.7	Associated Stirling numbers	139
6.8	Colorings of graphs	141

<i>Chapter 7</i>	Pólya frequency digraphs	145
7.1	Introduction	145
7.2	PF digraphs and the distributive lattice conjecture	146
7.3	A general result	150
7.4	The connection with the theory of symmetric functions	153
7.5	The inversion Theorem	157
7.6	PF ₂ digraphs	158
7.7	Semitransitive digraphs	163
<i>Bibliography</i>		167
<i>Appendix</i>		173
<i>Tables</i>		175
<i>Notations</i>		179
<i>Index</i>		182

Acknowledgments

In this moment of completion of my student life my first thought goes to three teachers that have deeply influenced and inspired me. First, I wish to thank my high school teacher Professor Loretta Felli of the IV Liceo Scientifico of Firenze, Italy, for spending with me countless after-class hours teaching me extra-curricular Mathematics, and patiently answering my many questions. Then, I wish to thank my undergraduate advisor, Professor Guido Zappa of the Università di Firenze, Italy, for his courses that introduced me to the wonders of Algebra and for skillfully guiding me through my very first research experience. Last, and most of all, I wish to thank my advisor, Professor Richard Stanley, his papers, courses, seminars and conversations have filled me with excitement and inspiration and I am fortunate to have been one of his students.

There are many other people to whom I am grateful and mathematically indebted. I especially wish to thank Professor Gian-Carlo Rota for the many hours spent with me talking of and about Mathematics, and Professor Daniel Kleitman for teaching me how to read combinatorics papers. I also wish to thank Professors Anders Björner, Persi Diaconis, Takayuki Hibi, Ira Gessel and David Jackson for many helpful conversations.

Financially, I wish to thank the Italian Ministero della Pubblica Istruzione, the M.I.T. Mathematics Department, Professor Richard Stanley, and my father for some extra help that made my life so much easier.

I wish to thank my parents for their love, and for always encouraging me to follow my own path, though it led very far from where it had begun.

I wish to thank my dear friends Massimo, Trubi, Lorella and especially Tiziano for their support, encouragement and enthusiasm.

Last, but not least, I wish to thank my darling wife, Annamaria, not only for a wonderful job in preparing this document with the $\text{\LaTeX}$ typesetting system, but for her love, patience and understanding, and for sharing with me all the bad and good times during these four trying years.

Francesco Brenti
Cambridge, Massachusetts
April 25, 1988

Foreword

I have tried to be as self-contained as possible. However, I assume on the part of the reader a knowledge of modern enumerative combinatorics as can be found, e. g. , in [63]. Any terminology or notation that does not appear in the indices at the end of this work will be found in the index of [63].

In sections 5.2, 5.3, 6.2, 7.4, 7.5 and 7.7 I use the theory of symmetric functions. For this theory I follow the terminology and notations of [38], Chapter I.

I collect here some notations that will be adhered to throughout this thesis. A list of notations of more than transient use appears at the end of this work.

Z	the integers
N	the nonnegative integers
P	the positive integers
Q, R, C	the rational, real and complex numbers
$[p]$	the set $\{1, 2, \dots, p\}$, ($p \in \mathbf{P}$)
$[a, b]$	the set $\{x \in \mathbf{R}, a \leq x \leq b\}$ ($a, b \in \mathbf{R}$)
$\lfloor a \rfloor$	the largest integer $\leq a$ ($a \in \mathbf{R}$)
$\lceil a \rceil$	the smallest integer $\geq a$ ($a \in \mathbf{R}$)
$ S $	the cardinality of the set S
$\uplus$	disjoint union of sets
$A[x_1, \dots, x_n]$	the ring of polynomials in the variables $x_1, \dots, x_n$ with coefficients in the integral domain A
$A(x_1, \dots, x_n)$	the ring of rational functions in the variables $x_1, \dots, x_n$ with coefficients in the integral domain A
$A[[x_1, \dots, x_n]]$	the ring of formal power series in the variables $x_1, \dots, x_n$ with coefficients in the integral domain A
δ_{ij}	the Kronecker delta: $\delta_{ij} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

Each result (theorem, corollary, proposition, or lemma) is numbered consecutively. So, for example, Theorem 2.3.3 is the third result in the third section of chapter 2 (i. e. in section 2.3). I denote with a $\square$ the end of a proof. A $\square$ appearing at the end of the statement of a result signifies that the result should be obvious to the reader at that stage of reading. A result whose statement is neither followed by a proof nor by a $\square$ is a result whose proof goes beyond the scope of this work. A reference to a proof will always be given in such cases.

Chapter 1

The Poset Conjecture

1.1 Introduction

Many of the results that are in this thesis have been discovered (as often happens in Mathematics) in my attempts to understand and solve a specific, particular problem. In this chapter I will state what this problem is, I will derive some elementary results about it, some equivalent formulations of it and I will review some special cases in which the conjecture is known to be true. I hope that, by giving to the reader the state of knowledge that I had when I started working on the problem, he will be more motivated in the reading of the general theory that I will develop in the next three chapters.

In this chapter I will assume that the reader is familiar with all the basic concepts of modern enumerative combinatorics (as can be found e. g. in [63]) and with the theory of (P, ω) -partitions as developed in [57] or [58].

1.2 Statement of the Poset Conjecture

Let P be a (finite) partially ordered set (or, poset, for short). Recall that a *labeling* of P is a bijection $\omega : P \rightarrow [p]$ where $p = |P|$. The pair (P, ω) is called a *labeled poset*. The labeling ω is called *natural* if $x, y \in P$ and $x \leq y$ implies $\omega(x) \leq \omega(y)$.

If (P, ω) is a labeled poset and $s \in \mathbf{N}$ then a (P, ω) -*partition* with largest

part $\leq s$ is an order reversing map $\sigma : P \rightarrow [s]$ such that $x < y$ and $\omega(x) > \omega(y)$ implies $\sigma(x) > \sigma(y)$ (where, by convention, $[0] \stackrel{\text{def}}{=} \emptyset$).

We denote with $\Omega(P, \omega; s)$ (respectively, $e_s(P, \omega)$) the number of (P, ω) -partitions (respectively, surjective (P, ω) -partitions) with largest part $\leq s$. So, for example, $e_s(P, \omega) = 0$ if $s > |P|$ and $\Omega(P, \omega; 0) = e_0(P, \omega) = 0$. Now, by an elementary counting argument, we easily see that

$$\Omega(P, \omega; x) = \sum_{s=1}^{|P|} e_s(P, \omega) \binom{x}{s} \quad (1)$$

for all $x \in \mathbb{N}$. This shows that $\Omega(P, \omega; x)$ is a polynomial function of x of degree $|P|$. The polynomial $\Omega(P, \omega; x) \in \mathbb{Q}[x]$ is called the *order polynomial* of (P, ω) and is a fundamental invariant of the labeled poset (P, ω) .

Since $\Omega(P, \omega; x)$ is a polynomial of degree $|P|$ there follows from a well known result about rational generating functions (see e. g. [63], Cor.4.3.1) that there exists a polynomial $W(P, \omega; z) \in \mathbb{R}[z]$ of degree $\leq |P|$ such that

$$\sum_{n \geq 0} \Omega(P, \omega; n) z^n = \frac{W(P, \omega; z)}{(1 - z)^{|P|+1}}, \quad (2)$$

as formal power series in z . We are now already in a position to state the poset conjecture:

Conjecture 1 (The Poset Conjecture) *For all labeled posets (P, ω) the polynomial $W(P, \omega; z)$ defined by (2) has only real zeros.*

In the case that ω is a natural labeling of P Conjecture 1 was first stated in 1978 in [42] (in the form of Conjecture 2, see p. 21) and later also in [65]. The poset conjecture in the present form has been first conjectured by Stanley in [64]. We should remark that the polynomial $W(P, \omega; z)$ has an important and well known combinatorial interpretation. However, before we can state this interpretation we need some additional definitions. Let P be a poset of cardinality p , a *linear extension* of P is an order preserving bijection $\tau : P \rightarrow [p]$ (i. e. a natural labeling of P). If (P, ω) is a labeled poset and τ is a linear

extension of P then we let $d(\tau, \omega) \stackrel{\text{def}}{=} |\{i \in [p-1] : \omega(\tau^{-1}(i)) > \omega(\tau^{-1}(i+1))\}|$.

Using the theory of (P, ω) -partitions it is possible to show (see e. g. [57]) that

$$W(P, \omega; z) = \sum_{\tau} z^{d(\tau, \omega)+1} \quad (3)$$

where the sum is over all linear extensions τ of P . This non trivial result implies in particular that $W(P, \omega; z) \in \mathbf{N}(z)$ and we will use this fact often in the sequel.

Now, if we substitute the expression for $\Omega(P, \omega; n)$ given in (1) into (2) we obtain

$$\begin{aligned} W(P, \omega; z) &= (1-z)^{|P|+1} \sum_{n \geq 0} \sum_{s=1}^{|P|} e_s(P, \omega) \binom{n}{s} z^n \\ &= (1-z)^{|P|+1} \sum_{s=1}^{|P|} e_s(P, \omega) \frac{z^s}{(1-z)^{s+1}} \\ &= (1-z)^{|P|} E(P, \omega; \frac{z}{1-z}), \end{aligned} \quad (4)$$

where $E(P, \omega; z) \stackrel{\text{def}}{=} \sum_{s=1}^{|P|} e_s(P, \omega) z^s$. Therefore, we may equivalently state the poset conjecture in the following form:

Conjecture 2 (The Poset Conjecture) *For any labeled poset (P, ω) the polynomial $E(P, \omega; z)$ defined above has only real zeros*

Now the reader is probably wondering: “Why should we be interested in knowing whether or not those polynomials have only real zeros?”. The answer lies in the following classic result whose proof can be found e. g. in [29]. Recall that a sequence $\{a_0, a_1, \dots, a_d\}$ (of real numbers) is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i = 1, \dots, d-1$, is said to be *unimodal* if there exists an index $0 \leq j \leq d$ such that $a_i \leq a_{i+1}$ for $i = 0, \dots, j-1$ and $a_i \geq a_{i+1}$ for $i = j, \dots, d-1$, and is said to have no *internal zeros* if there are not three indices $0 \leq i < j < k \leq d$ such that $a_i, a_k \neq 0$ and $a_j = 0$. We say that a polynomial $\sum_{i=0}^d a_i x^i$ is *log-concave* (respectively, *unimodal*, with no *internal zeros*) if the sequence $\{a_0, a_1, \dots, a_d\}$ has the corresponding property.

Theorem 1.2.1 *Let $\sum_{i=0}^d a_i x^i$ be a polynomial with nonnegative coefficients and*

with only real zeros. Then the sequence $\{a_0, a_1, \dots, a_d\}$ is log-concave with no internal zeros; in particular it is unimodal.

This result allows us immediately to derive from Conjectures 1 and 2 some weaker (easier?) but almost just as interesting conjectures:

Conjecture 1.1 *The polynomial $W(P, \omega; z)$ is log-concave with no internal zeros.*

Conjecture 1.2 *The polynomial $W(P, \omega; z)$ is unimodal.*

Conjecture 2.1 *The polynomial $E(P, \omega; z)$ is log-concave with no internal zeros.*

Conjecture 2.2 *The polynomial $E(P, \omega; z)$ is unimodal.*

It should be noted that while equation (4) immediately implies that conjectures 1 and 2 are equivalent it does *not* imply the equivalence of Conjectures 1.1 and 2.1 or that of 1.2 and 2.2. As a matter of fact, we will show in section 2.5 that Conjecture 1.1 actually implies Conjecture 2.1, (see Theorem 2.5.8).

All the conjectures above stated can be specialized also in another direction, namely to the case of ω being a natural labeling. In this case the combinatorial interpretation of the polynomials $E(P; z)$ and $W(P; z)$ (we will usually write just $E(P; z)$, $W(P; z)$, etc., instead of $E(P, \omega; z)$, $W(P, \omega; z)$, etc., if ω is a natural labeling) becomes particularly nice and simple. In fact, it is easy to see that, for $s \in \mathbb{N}$, $e_s(P)$ is the number of surjective order reversing (or, equivalently, order preserving) maps $P \rightarrow \mathbf{s}$, (where $\mathbf{s}$ denotes a chain with s elements) while the coefficient of z^s in $W(P; z)$ equals the number of linear extensions of P with exactly s descents. Now, it is very easy to see that if $\sigma : P \rightarrow \mathbf{s}$ is a surjective order preserving map then $\sigma^{-1}([1]) \subset \sigma^{-1}([2]) \subset \dots \subset \sigma^{-1}([s-1])$ is a chain of $s-1$ order ideals in $J(P)$ (the distributive lattice of order ideals of P) and that this correspondence is a bijection. Therefore, we may also think of $e_s(P)$ as the number of chains of length s from $\emptyset$ to P in $J(P)$. Since, by a well known classical result (see, e. g., [5]), every finite distributive

lattice is of the form $J(P)$ for some poset P , we may restate Conjecture 2, in the case of a natural labeling, as follows:

Conjecture 3 (The distributive lattice conjecture) *Let D be a (finite) distributive lattice and let, for $s \in \mathbb{N}$, $c_s(D)$ be the number of chains of length s from $\hat{0}$ to $\hat{1}$ in D . Then the polynomial $C(D, z) \stackrel{\text{def}}{=} \sum_{s=0}^{|P|} c_s(D) z^s$ has only real zeros.*

In chapter 7 we will see how the point of view of Conjecture 3 is probably the most useful one in trying to solve the poset conjecture for natural labelings.

Finally, let us remark that even in the case that ω is a natural labeling Conjectures 1.1, 1.2, 2.1 and 2.2 are also still open.

1.3 Partial results on the Poset Conjecture

In this section we present some partial results on the poset conjecture. Of course, one of the first things that come to mind in trying to resolve the conjecture is to see if some of the familiar operations on posets preserve its validity.

It is immediately clear, for example, that the conjecture holds for a labeled poset (P, ω) if and only if it holds for its dual $(P^*, \bar{\omega})$ (where $\bar{\omega}(x) \stackrel{\text{def}}{=} |P| + 1 - \omega(x)$ for $x \in P$) because $\Omega(P, \omega; z) = \Omega(P^*, \bar{\omega}; z)$ by the combinatorial definition of the order polynomial. On the other hand, we can easily convince ourselves that even if the conjecture holds for two labeled posets (P, ω_1) and (Q, ω_2) it is almost impossible to understand if it also holds for their direct product $(P \times Q, \omega_1 \times \omega_2)$ or for the power $(P, \omega_1)^{(Q, \omega_2)}$, (see [63], §3.2 for the definitions of these operations).

The only other operations for which we can have any reasonable hope of success are the ordinal sum $\oplus$ and the disjoint union $+$. As it turns out the first operation does indeed preserve the poset conjecture as we will now show.

Let (P, ω_1) and (Q, ω_2) be two labeled posets. If we denote with $\leq_P$ and $\leq_Q$ the partial orderings on P and Q then recall that the *ordinal sum* $P \oplus Q$ is the poset that has $P \uplus Q$ as underlying set partially ordered by letting $x \leq y$ if either $x, y \in P$ and $x \leq_P y$; or $x, y \in Q$ and $x \leq_Q y$; or $x \in P, y \in Q$. We denote with $\omega_1 \oplus \omega_2$ the labeling of $P \oplus Q$ defined by

$$(\omega_1 \oplus \omega_2)(x) \stackrel{\text{def}}{=} \begin{cases} \omega_1(x) & \text{if } x \in P, \\ \omega_2(x) + |P| & \text{if } x \in Q. \end{cases}$$

Note that $\omega_1 \oplus \omega_2$ is natural if and only if both ω_1 and ω_2 are natural.

Proposition 1.3.1 *Let (P, ω_1) and (Q, ω_2) be as above. Then*

$$\begin{aligned} zW(P \oplus Q, \omega_1 \oplus \omega_2; z) &= zW(Q \oplus P, \omega_2 \oplus \omega_1; z) \\ &= W(P, \omega_1; z)W(Q, \omega_2; z) \end{aligned}$$

In particular, $W(P \oplus Q, \omega_1 \oplus \omega_2; z)$ has only real zeros if and only if both $W(P, \omega_1; z)$ and $W(Q, \omega_2; z)$ have only real zeros.

Proof: From the definitions of $P \oplus Q$ and $\omega_1 \oplus \omega_2$ it follows easily by direct combinatorial reasoning that

$$e_s(P \oplus Q, \omega) = \sum_{i=0}^s e_{s-i}(Q, \omega_2)(e_i(P, \omega_1) + e_{i+1}(P, \omega_1)),$$

for all $s \geq 0$, where $\omega = \omega_1 \oplus \omega_2$. Hence we have

$$\begin{aligned} zE_{(P \oplus Q, \omega)}(z) &= \sum_{s \geq 0} e_s(P \oplus Q, \omega) z^{s+1} \\ &= \sum_{s \geq 0} \sum_{i=0}^s e_{s-i}(Q, \omega_2)(e_i(P, \omega_1) + e_{i+1}(P, \omega_1)) z^{s+1} \\ &= \left(\sum_{j \geq 0} e_j(Q, \omega_2) z^j \right) \left(\sum_{i \geq 0} (e_i(P, \omega_1) + e_{i+1}(P, \omega_1)) z^{i+1} \right) \\ &= E_{(Q, \omega_2)}(z) E_{(P, \omega_1)}(z) (1 + z). \end{aligned} \tag{5}$$

Hence, using (4), we get

$$\begin{aligned} zW_{(P \oplus Q, \omega)}(z) &= z(1 - z)^{|P|+|Q|} E_{(P \oplus Q, \omega)}\left(\frac{z}{1 - z}\right) \\ &= (1 - z)^{|P|+|Q|} E_{(P, \omega_1)}\left(\frac{z}{1 - z}\right) E_{(Q, \omega_2)}\left(\frac{z}{1 - z}\right) \\ &= W_{(P, \omega_1)}(z) W_{(Q, \omega_2)}(z), \end{aligned}$$

as desired. $\square$

We will see later how Proposition 1.3.1 and equation (5) actually tell us that also conjectures 1.1 and 2.1 are preserved by the ordinal sum operation. We could show this now by a direct argument but since it will follow immediately from a much more general and powerful machinery we prefer to do it later.

Let us now turn our attention to the disjoint union operation. Let (P, ω_1) and (Q, ω_2) be two labeled posets. If we denote by $\leq_P$ and $\leq_Q$ the partial orderings on P and Q then recall that the *disjoint union* $P + Q$ is the poset that has $P \uplus Q$ as underlying set partially ordered by letting $x \leq y$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$. We define a labeling of $P + Q$ by $\omega_1 + \omega_2 \stackrel{\text{def}}{=} \omega_1 \oplus \omega_2$. Again note that $\omega_1 + \omega_2$ is natural if and only if both ω_1 and ω_2 are natural. Now, by the definition of the order polynomial and by a direct counting argument we easily see that

$$\Omega(P + Q, \omega_1 + \omega_2; z) = \Omega(P, \omega_1; z) \Omega(Q, \omega_2; z).$$

Using this and equation (2) it is possible to show that

$$W(P + Q, \omega_1 + \omega_2; z) = \frac{(1 - z)^{p+q+1}}{q!} \left[\left(\frac{d}{dx} \right)^q \left[\frac{W(P, \omega_1; x) W(P, \omega_1; \frac{z}{x}) x^q}{(1 - x)^{p+1}} \right] \right]_{|x=z}$$

where $p = |P|$ and $q = |Q|$. The proof of this formula will be given in section 4.7 (see Theorem 4.7.4) in a more general setting. However, the preceding formula is complicated enough to make it difficult to understand whether it preserves the poset conjecture or not. On the other hand, we have found no counterexamples, so that the question is still open. This matter will be further discussed in much greater detail and length in section 4.7.

I finish this section by presenting one general class of posets and labelings, for which the poset conjecture is true. The following result first appeared (though expressed in different form) in [56].

Theorem 1.3.2 *Let P be a disjoint union of chains. Then the polynomial $W(P; z)$ has only real zeros.*

Note that if P is the disjoint union of n points then $W(P; z)$ equals the n -th Eulerian polynomial. Theorem 1.3.2 is therefore a generalization of the well known fact that Eulerian polynomials have only real zeros (see, e. g. , [63], §1.3 for the definition of the Eulerian polynomials). We will encounter many more generalizations of this fact later on (see, e. g. , section 6.6). In section 5.5 we will prove that Theorem 1.3.2 actually holds for any labeling of P (whether natural or not). For this reason, we defer the proof of Theorem 1.3.2 to chapter 5.

I should mention that not much numerical evidence has been gathered so far for the poset conjecture. In fact, it has only been checked for posets having cardinality ≤ 5 and even then only for natural labelings. However, in this thesis we will show that the poset conjecture is true for many interesting classes of labeled posets. So this thesis gives additional evidence for the truth of the poset conjecture. However, my main intention in this first chapter is just to motivate the reader into the reading of the Mathematics that I discovered in trying to solve these conjectures.

It is my firm belief that conjectures and open problems in Mathematics are not so much interesting and important “per se” but because they are symptoms that our knowledge is not complete in some area. Their greatest value is not whether they are true or false but that they stimulate and lead us into deeper knowledge. This point of view has been my guiding principle throughout the work that led to this thesis.

1.4 A second look at the Poset Conjecture

In section 1.2 we have defined the polynomial $W(P, \omega; z)$, for any labeled poset (P, ω) , by

$$\sum_{n \geq 0} \Omega(P, \omega; n) z^n = \frac{W(P, \omega; z)}{(1 - z)^{p+1}} \quad (6)$$

where $p = |P|$. If we now expand the order polynomial of (P, ω) in terms of the basis $\left\{ \binom{x+p-i}{p} \right\}_{i=0, \dots, p}$ we obtain

$$\Omega(P, \omega; x) = \sum_{i=0}^p w_i(P, \omega) \binom{x+p-i}{p}$$

where $w_i(P, \omega) \in \mathbf{Q}$, $i = 0, \dots, p$. Substituting this expression in (6) and then interchanging the order of summation we get

$$\begin{aligned} \frac{W(P, \omega; z)}{(1-z)^{p+1}} &= \sum_{i=0}^p w_i(P, \omega) \sum_{n \geq 0} \binom{n+p-i}{p} z^n \\ &= \sum_{i=0}^p w_i(P, \omega) \frac{z^i}{(1-z)^{p+1}} \end{aligned}$$

using the binomial theorem (see, e. g. [63], p.16). Hence $W(P, \omega; z) = \sum_{i=0}^p w_i(P, \omega) z^i$, so by the combinatorial interpretation of the coefficients of $W(P, \omega; z)$ we have that, in fact, $w_i(P, \omega) \in \mathbf{N}$ for all $i = 0, \dots, p$. Let now V_p be the vector space of all real polynomials of degree $\leq p$ and $\mathcal{L}$ the invertible linear transformation $V_p \rightarrow V_p$ defined by

$$\mathcal{L} \left(\binom{x+p-i}{p} \right) \stackrel{\text{def}}{=} x^i \tag{7}$$

for $i = 0, \dots, p$. Then the above computation shows that (6) is equivalent to the formula

$$\mathcal{L}(\Omega(P, \omega; z)) = W(P, \omega; z).$$

We may therefore restate the poset conjecture in yet another equivalent form:

Conjecture 4 (The Poset Conjecture) *For any labeled poset (P, ω) the polynomial $\mathcal{L}(\Omega(P, \omega; z))$, where $\mathcal{L}$ is defined by (7), has only real zeros.*

This formulation of the poset conjecture immediately suggests many different possible lines of attack. In particular we may ask: “What properties of a polynomial $A(x)$ assure that $\mathcal{L}(A(x))$ has only real zeros?”. For example, is it sufficient that $A(x)$ itself has only real zeros? Is it necessary? Since many more things are known about $\Omega(P, \omega; z)$ than about $W(P, \omega; z)$ (for example, it is known that $\Omega(P, \omega; z)$ does not always have only real zeros, its coefficients

are, in general, neither positive nor alternating etc...) an answer to the above questions would already be quite interesting. One can also ask the analogous questions about other properties of polynomials that we are interested in, like unimodality or log-concavity of the coefficients. An answer to those questions would also be very interesting. Finally, as always in Mathematics, we may be tempted to generalize even more and ask the same questions for linear transformations other than $\mathcal{L}$. For example, if the polynomial $\sum_{i=0}^p a_i x^i$ has only real zeros is it true that $\sum_{i=0}^p a_i \langle x \rangle_i$ (where $\langle x \rangle_i \stackrel{\text{def}}{=} x(x+1)\dots(x+i-1)$) does also? Is the converse true? If I express a polynomial with respect to two different bases of V_p , say $\sum_{i=0}^p a_i \binom{x}{i} = \sum_{i=0}^p c_i \left(\binom{x}{i} \right)$, and if the sequence $a_0, a_1, \dots, a_p$ is log-concave, does it follow that $c_0, c_1, \dots, c_p$ is also? As it turns out, one can give answers to many of the above questions and these answers, in a surprisingly large number of cases, turn out to be “yes”. In the next two chapters we will look closely at the above questions and we will answer many of them. We will concentrate particularly on six bases of V_p that are very often used in Combinatorics and on the 30 linear transformations that send them into each other. The results of this study will give us a rich, elegant and powerful theory which we will then apply in chapters 5 and 6 to obtain new results about the original conjectures that motivated it as well as about many other combinatorial situations.

Chapter 2

A general theory

2.1 Introduction

In this chapter, motivated by the considerations in the preceding one, we will analyze how a polynomial behaves with respect to properties like unimodality, or log-concavity of the coefficients or having only real zeros, under a change of basis. We will consider what are maybe the six most commonly used bases of V_p in Combinatorics which we will call the 6 fundamental bases. These 6 bases will be introduced in section 2.3 where the transition matrices between them and some of their basic properties will be derived. In sections 2.4 and 2.5 we will prove the results that form the core of our theory and of the chapter. These will be collected and completed in section 2.6 where we will also discuss their significance in Combinatorics as well as some questions that remain open in the theory and some possible lines of further investigation. Many of the proofs of the main results in this chapter depend heavily on the theory of total positivity. This theory will be introduced in the next section where we will also derive some basic results in it that will be used later, we will then explain the basic connection between this theory and polynomials with only real zeros and hence also with the problems that we are studying in this chapter.

2.2 Polynomials with only real zeros and total positivity

The poset conjecture asserts that a certain combinatorially defined polynomial associated with a given labeled poset has only real zeros. It is therefore natural to look for properties that are sufficient, or necessary, for a polynomial to have only real zeros and for operations on polynomials that preserve this property. We have already seen in Theorem 1.2.1 a necessary (but not sufficient) condition for a polynomial to have only real zeros. It turns out that the necessary conditions of Theorem 1.2.1 can be strengthened to become also sufficient. This result (Theorem 2.2.4 below) is of fundamental importance to us and will be used routinely throughout this work *without* explicit mention. Before we can state it, however, we need some definitions from the theory of total positivity. By an *infinite matrix* we will mean a matrix $M = (M_{ij})_{i,j \in \mathbb{N}}$ whose rows and columns are indexed by nonnegative integers. A matrix $M = (M_{ij})_{i,j \in \mathbb{N}}$ (finite or infinite) is said to be *sign regular of order r* (or an *SR_r matrix*, for short) if, for all $1 \leq t \leq r$, all the minors of order t of M have determinants of the same sign (i. e. either all ≥ 0 or all ≤ 0). Denoting with $\varepsilon_t (= \pm 1)$ the common sign of these determinants, the vector $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ is called the *sign sequence* of M . A matrix M as above is said to be *totally positive of order r* (or a *TP_r matrix*, for short) if it is an *SR_r matrix* with sign sequence $(\underbrace{1, 1, \dots, 1}_r)$. Furthermore, M is said to be *totally positive of infinite order* (or a *TP_r matrix*, for short) if it is *TP_r* for all $r \geq 1$.

The theory of sign regular matrices and especially of totally positive matrices is rich with deep and highly non-trivial results and has been studied extensively. Many of these results, together with powerful generalizations, can be found in the fundamental book [33] which also contains an excellent bibliography. For a

more recent (and shorter) survey of the subject of totally positive matrices the reader can consult [3]. For the moment, however, we will just need the following fundamental result about sign regular matrices. Recall that a sequence $\{a_i\}_{i \in \mathbb{N}}$ is said to be *eventually vanishing* if $a_i \neq 0$ for only a finite number of indices $i \in \mathbb{N}$.

Theorem 2.2.1 *Let B and C be two infinite matrices with sign sequences $(\varepsilon_1, \dots, \varepsilon_r)$ and $(\eta_1, \dots, \eta_r)$, respectively. Suppose that either all rows of B or all columns of C are eventually vanishing and let $A \stackrel{\text{def}}{=} BC$. Then A is an SR_r matrix with sign sequence $(\varepsilon_1 \eta_1, \dots, \varepsilon_r \eta_r)$.*

This theorem is a special case of Lemma 1.1 on page 99 of [33] and a proof can be found there. However, it can also be easily proved directly using the classical Cauchy-Binet formula and we therefore omit its proof.

We now come to the crucial definitions for us. A sequence (of real numbers) $\{a_i\}_{i \in \mathbb{N}}$ is called a *sign regular sequence of order r* (or an SR_r sequence) if the matrix $N = (N_{ij})_{i,j \in \mathbb{N}}$ defined by $N_{ij} \stackrel{\text{def}}{=} a_{i+j}$ for all $i, j \in \mathbb{N}$ is an SR_r matrix. A sequence (of real numbers) $\{a_i\}_{i \in \mathbb{N}}$ is called a *Pólya frequency sequence of order r* (or a PF_r sequence) if the matrix $M = (M_{ij})_{i,j \in \mathbb{N}}$ defined by $M_{ij} \stackrel{\text{def}}{=} a_{j-i}$ for all $i, j \in \mathbb{N}$ (where $a_k \stackrel{\text{def}}{=} 0$ if $k < 0$) is a TP_r matrix. A sequence $\{a_i\}_{i \in \mathbb{N}}$ as above is called a *Pólya frequency sequence* (or, a PF sequence) if it is a PF_r sequence for all $r \geq 1$. Note that a PF_r sequence is in particular an SR_r sequence with sign sequence $(\varepsilon_1, \dots, \varepsilon_r)$ where $\varepsilon_i = (-1)^{\binom{i}{2}}$ for $i = 1, \dots, r$ but the converse is in general false (except when $r = 2$). We will also make the following conventions: we will call a finite sequence $\{a_1, a_2, \dots, a_d\}$ (of real numbers) a PF_r (respectively, PF) sequence if the infinite sequence $\{a_1, a_2, \dots, a_d, 0, 0, \dots\}$ has the corresponding property; we will call a formal power series $\sum_{n \geq 0} a_n z^n \in \mathbb{R}[[z]]$ a PF_r (respectively PF) formal power series, if the sequence $\{a_0, a_1, a_2, \dots\}$ of its coefficients has the corresponding property. From Theorem 2.2.1 above we can immediately deduce the following fundamental property of PF_r formal power

series.

Theorem 2.2.2 *Let $B(z)$ and $C(z)$ be two PF_r formal power series and let $A(z) \stackrel{\text{def}}{=} B(z)C(z)$. Then $A(z)$ is also PF_r . $\square$*

We also note here the following elementary property of PF_r sequences that will yield in Chapter 4 some unexpected and non trivial results.

Proposition 2.2.3 *Let $\{a_n\}_{n \in \mathbb{N}}$ be a PF_r sequence. Then, for each $k \in \mathbb{P}$, $\{a_{nk}\}_{n \in \mathbb{N}}$ is a PF_r sequence.*

Proof: Define two matrices $A = (a_{ij})_{i,j \in \mathbb{N}}$ and $B = (B_{ij})_{i,j \in \mathbb{N}}$ by $a_{ij} \stackrel{\text{def}}{=} a_{j-i}$ and $b_{ij} \stackrel{\text{def}}{=} a_{kj-ki}$ (where $a_t \stackrel{\text{def}}{=} 0$ if $t < 0$). Then every minor of B is also a minor of A and the proof follows. $\square$

The concepts of PF_r and PF formal power series play a crucial and decisive role in this work. While this will become more and more apparent as we go on we may even now get a glimpse of the connections that they have to unimodality and log-concavity questions in Combinatorics, by noting, for example, that a PF_1 sequence is just a nonnegative sequence (and in Combinatorics those are the only ones that are “meaningful”), and that a PF_2 sequence is in particular log-concave (though the converse need not hold, for example, the sequence $\{a_0, a_1, \dots, a_4\} = \{1, 1, 0, 0, 1\}$ is log-concave but not PF_2 since $\begin{vmatrix} a_2 & a_4 \\ a_1 & a_3 \end{vmatrix} = -1$). Furthermore, it is immediate from the definitions that linear polynomials with nonnegative coefficients are PF polynomials. Using Theorem 2.2.2 repeatedly we then deduce that polynomials of the form $\prod_{i=1}^d (a_i + b_i z)$ where $a_i, b_i \geq 0$ for all $i = 1, \dots, d$ are also PF . In other words, every polynomial having nonnegative coefficients and only real zeros is a PF polynomial. (Note how this, in particular, proves Theorem 1.2.1). This is already quite interesting because it provides many more necessary conditions than Theorem 1.2.1 for a polynomial to have only real zeros and we were looking for such conditions. What is even more interesting, however, is that the converse statement is also true. In fact, the following theorem holds:

Theorem 2.2.4 *Let $A(x) \stackrel{\text{def}}{=} \sum_{i=0}^d a_i x^i$ be a polynomial with nonnegative coefficients. Then $A(x)$ has only real zeros if and only if $\{a_0, \dots, a_d\}$ is a PF sequence.*

Theorem 2.2.4 (and Theorem 4.5.3 from which it follows) provide the crucial and basic link between the Combinatorics of polynomials with only real zeros and the theory of PF sequences. We will derive Theorems 2.2.4 and 4.5.3 from a deep result in the theory of PF sequences in section 4.5. Theorem 2.2.4 has been first proved in [18]; it is implicit in [33] though it's not stated explicitly there and also appears in [23] and [3], p.216. The reader should note that Theorem 2.2.4 implicitly assumes that all constant polynomials have only real zeros, we will keep this natural convention throughout this work.

The preceding result immediately presents some challenges to the combinatorialist. Many polynomials having a combinatorial interpretation are known (and many more are conjectured to) have only real zeros. This being the case one cannot resist the temptation to believe that, in all those cases, all the determinants supposed to be nonnegative in the definition of a PF sequence actually “count something”. The big question, of course, is: “What do they count?”. Some very interesting results in this direction have already been obtained by Gessel and Viennot in [22] and later in [23]. We will examine and put in a general perspective these results in chapter 7.

As stated in the Introduction, our purpose in this chapter is to find linear transformations of V_p that send polynomials that are unimodal, log-concave or have only real zeros into polynomials that have at least one of those properties. Because of the relationship of these concepts with those of a PF_r , SR_r and PF sequence we may naturally also ask if there are linear transformations that preserve some of those properties. In view of Theorem 2.2.2, for example, one is tempted to conjecture that SR_r matrices send SR_r sequences into SR_r sequences or that TP_r matrices send PF_r sequences into PF_r sequences. Un-

fortunately, it is easy to see that both statements are false. For example, the sequence $(1, 0, 0)$ is PF and the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is TP but $(2, 1, 1)$ is not even PF_2 . Nonetheless, there is at least one general class of linear transformations that send SR_r sequences into SR_r sequences.

In order to describe it, however, it will be convenient to introduce one more piece of notation. If $M = (M_{ij})_{i,j \in \mathbb{N}}$ is an infinite matrix then we let

$$M_i(z) \stackrel{\text{def}}{=} \sum_{j \geq 0} M_{ij} z^j$$

for all $i \geq 0$, and we call $M_i(z)$ the i -th row generating function of M .

Theorem 2.2.5 *Let L, M, N be three infinite matrices and let $\{q_i\}_{i \in \mathbb{N}}$ be an eventually vanishing SR_r sequence. Suppose that M and N are TP_r and that the following “semigroup property” is satisfied*

$$L_{i+j}(z) = M_i(z)N_j(z),$$

for all $i, j \in \mathbb{N}$. Define a sequence $\{c_t\}_{t \in \mathbb{N}}$ by

$$c_t = \sum_{i \geq 0} L_{t,i} q_i, \tag{1}$$

for all $t \in \mathbb{N}$, then $\{c_t\}_{t \in \mathbb{N}}$ is an SR_r sequence with the same sign sequence of $\{q_i\}_{i \in \mathbb{N}}$.

Proof: We have, using (1), for $i, j \geq 0$,

$$\begin{aligned} c_{i+j} &= \sum_{n \geq 0} L_{i+j,n} q_n \\ &= \sum_{n \geq 0} \sum_{r=0}^n M_{i,r} N_{j,n-r} q_n \\ &= \sum_{r \geq 0} \left(\sum_{n \geq r} N_{j,n-r} q_n \right) M_{i,r} \\ &= \sum_{r \geq 0} \sum_{n \geq 0} N_{j,n} q_{n+r} M_{i,r}. \end{aligned}$$

This shows that c_{i+j} is the element in the i -th row and j -th column of the matrix $M Q N^t$ (where $Q \stackrel{\text{def}}{=} (q_{i+j})_{i,j \geq 0}$) and the thesis follows from Theorem 2.2.1. $\square$

The preceding result is, to the best of my knowledge, new. It is, however, an easy generalization of a special case of Theorem 5.4 on p. 130 of [33]. As might be expected, Theorem 2.2.5 will prove to be an extremely useful tool for us. In fact, all the results in section 2.5 will follow from a special case of it.

2.3 The six fundamental bases and the transition matrices between them

In this and the following sections in this chapter we will derive a general theory of unimodal, log-concave and PF polynomials. So far we have not stated rigorously the exact objects that we will study; we do this now.

Let $\{p_i(x)\}_{i=0,\dots,d}$ be an ordered basis of V_d (recall that V_d is the vector space of all polynomials with real coefficients and degree $\leq d$). We will denote by $PF[p_i(x)]$ (respectively $PF_2[p_i(x)]$, $UNI[p_i(x)]$, $PF_1[p_i(x)]$) the subset of V_d consisting of all polynomials $A(x) = \sum_{i=0}^d a_i p_i(x)$ such that the sequence $\{a_0, \dots, a_d\}$ is PF (respectively PF_2 , unimodal and PF_1 , PF_1). We will call $PF[p_i(x)]$ (respectively $PF_2[p_i(x)]$, $UNI[p_i(x)]$, $PF_1[p_i(x)]$) the PF -set (respectively PF_2 -set, *Unimodal-set*, PF_1 -set) of the basis $\{p_i(x)\}_{i=0,\dots,d}$. We will also denote by $\mathcal{R}$ the subset of V_d consisting of all the polynomials in V_d that have only real zeros (recall that we include in $\mathcal{R}$ also constant polynomials). If $P(x) \in \mathcal{R}$ then we will denote its smallest zero by $\lambda(P)$ and its largest zero by $\Lambda(P)$. Theorem 2.2.4 can be succinctly stated as follows.

Theorem 2.3.1 $\mathcal{R} \cap PF_1[x^i] = PF[x^i]$. $\square$

We may also easily express the poset conjecture with this notation (see Conjecture 4):

Conjecture 5 (The Poset Conjecture) *For any labeled poset (P, ω) we have that $\Omega(P, \omega; x) \in PF \left[\binom{x+p-i}{p} \right]$ where $\Omega(P, \omega; x)$ is the order polynomial of (P, ω) and $p = |P|$.*

We can of course similarly translate all the other equivalent statements of the poset conjecture and consequences of it stated in chapter 1 in this notation. Furthermore, by the preceding section it is clear that the following result holds.

Proposition 2.3.2 *Let $\{p_i(x)\}_{i=0,\dots,d}$ be an ordered basis of V_d . Then*

$$PF[p_i(x)] \subset PF_2[p_i(x)] \subset UNI[p_i(x)] \subset PF_1[p_i(x)]. \square$$

The advantages of the notation that we have just introduced are not only that we can concisely express our conjectures and results but also that they enable us to formulate precisely and in a unified language the questions raised at the end of chapter 1. In fact, it is easy to recognize that those questions are just questions about which inclusions hold among the eight subsets associated to two different bases of V_d . There are of course infinitely many bases of V_d and therefore infinitely many questions and even to examine only those bases that are actually used in practice would probably be an impossible task. In this chapter and in part of the next one we will concentrate on six bases of V_d that are probably the most used in Mathematics and particularly in Combinatorics. I will call them the “six fundamental bases” of V_d and they are listed below:

- (a) $\{x^i\}_{i=0,\dots,d}$;
- (b) $\left\{ \binom{x}{i} \right\}_{i=0,\dots,d}$;
- (c) $\{(x)_i\}_{i=0,\dots,d}$;
- (d) $\left\{ \left(\binom{x}{i} \right) \right\}_{i=0,\dots,d}$;
- (e) $\{\langle x \rangle_i\}_{i=0,\dots,d}$;
- (f) $\left\{ \binom{x+d-i}{d} \right\}_{i=0,\dots,d}$,

where $(x)_i \stackrel{\text{def}}{=} (i!) \binom{x}{i}$, $\langle x \rangle_i \stackrel{\text{def}}{=} (-1)^i (-x)_i$, and $\left(\binom{x}{i} \right) \stackrel{\text{def}}{=} \langle x \rangle_i / (i!)$. These six bases give us a total of 24 different subsets of V_d and our goal would be to find

all the inclusions that hold between any two of them. In this chapter, using the results from total positivity of the last section, we will be able to solve our problem almost completely. In order to do this it will be convenient first to find the transition matrices between all pairs of fundamental bases. These matrices and their combinatorial significance and interpretation are of course well known. However, for completeness of exposition and in order to fix some standard notation for them, that will be adhered to throughout this work, we rapidly derive them in the remainder of this section.

Theorem 2.3.3 *Let*

$$P(x) = \sum_{i=0}^d \omega_i \binom{x+d-i}{d} = \sum_{i=0}^d a_i \binom{x}{i} = \sum_{i=0}^d c_i \left(\binom{x}{i} \right)$$

be a polynomial of degree $\leq d$. Then the following formulas hold:

$$\sum_{n \geq 0} P(n)z^n = \frac{W(z)}{(1-z)^{d+1}}, \quad (2)$$

$$W(z) = (1-z)^d A\left(\frac{z}{1-z}\right), \quad (3)$$

$$W(z) = (1-z)^d \left[zC\left(\frac{1}{1-z}\right) + (1-z)C(0) \right], \quad (4)$$

$$(1+z)A(z) = zC(1+z) + C(0), \quad (5)$$

where $A(z) \stackrel{\text{def}}{=} \sum_{i=0}^d a_i z^i$, $C(z) \stackrel{\text{def}}{=} \sum_{i=0}^d c_i z^i$ and $W(z) \stackrel{\text{def}}{=} \sum_{i=0}^d \omega_i z^i$. In particular we have:

- (i) $W(z)$ has only real zeros if and only if $A(z)$ does;
- (ii) $A(z) - A(0)$ has only real zeros if and only if $C(z) - C(0)$ does.

Proof: All formulas are obtained easily by straightforward computations. To prove (2) let $P(n) = \sum_{i=0}^d \omega_i \binom{n+d-i}{d}$, then we have

$$\begin{aligned} \sum_{n \geq 0} P(n)z^n &= \sum_{i=0}^d \omega_i \sum_{n \geq 0} \binom{n+d-i}{d} z^n \\ &= \frac{\sum_{i=0}^d \omega_i z^i}{(1-z)^{d+1}}, \end{aligned}$$

by the binomial theorem (see e. g. , [63], p.16). The proof of (3) is analogous, we have

$$\begin{aligned}
\sum_{n \geq 0} P(n)z^n &= \sum_{i=0}^d a_i \sum_{n \geq 0} \binom{n}{i} z^n \\
&= \sum_{i=0}^d a_i \frac{z^i}{(1-z)^{i+1}} \\
&= \frac{\sum_{i=0}^d a_i \left(\frac{z}{1-z}\right)^i}{(1-z)}
\end{aligned}$$

which, by comparison with (2), establishes (3). In the same way we prove (4). In fact,

$$\begin{aligned}
\sum_{n \geq 0} P(n)z^n &= \sum_{i=0}^d c_i \sum_{n \geq 0} \left(\binom{n}{i} \right) z^n \\
&= \sum_{i=1}^d c_i \frac{z}{(1-z)^{i+1}} + \frac{c_0}{(1-z)} \\
&= \frac{z \sum_{i=1}^d c_i (1-z)^{d-i} + c_0 (1-z)^d}{(1-z)^{d+1}},
\end{aligned}$$

and comparing this with (2) yields (4). Finally, (5) follows from (3) and (4). Regarding the last two assertions of the theorem, (i) follows immediately from (3), so we only have to prove (ii). But from our definitions we have that $A(0) = P(0) = C(0)$, so from (5) we deduce that

$$(1+z)(A(z) - A(0)) = z(C(1+z) - C(0))$$

and (ii) follows. $\square$

Using the preceding result we can easily find the transition matrices between any two of the fundamental bases. Each one of these matrices can be compactly expressed in terms of a few basic matrices that have a well known combinatorial meaning. We list these matrices, together with the notation that we will use for them, here: (note that in what follows “i” indexes rows and “j” indexes

columns, so all the six matrices below are lower triangular)

$$\begin{aligned}
B_d &\stackrel{\text{def}}{=} \left(\binom{i}{j} \right)_{0 \leq i, j \leq d} \text{ (matrix of binomial coefficients)} \\
F_d &\stackrel{\text{def}}{=} \text{diag}(1, 1!, 2!, \dots, d!) \text{ (diagonal matrix of factorials)} \\
S_d &\stackrel{\text{def}}{=} (S(i, j))_{0 \leq i, j \leq d} \text{ (matrix of Stirling numbers of the second kind)} \\
s_d &\stackrel{\text{def}}{=} (s(i, j))_{0 \leq i, j \leq d} \text{ (matrix of Stirling numbers of the first kind)} \\
C_d &\stackrel{\text{def}}{=} (c(i, j))_{0 \leq i, j \leq d} \text{ (matrix of signless Stirling numbers of the first kind)} \\
M_d &\stackrel{\text{def}}{=} 1 \oplus B_{d-1} \text{ (where } 1 \stackrel{\text{def}}{=} (1 \times 1)\text{-identity matrix)}.
\end{aligned}$$

We will usually omit the subscript d when there is no danger of confusion. Also, if $A = (A_{ij})_{0 \leq i, j \leq d}$ is a $(d+1) \times (d+1)$ matrix, then we will denote by $A^\#$ the matrix defined by $A_{ij}^\# \stackrel{\text{def}}{=} A_{d-i, d-j}$ for $0 \leq i, j \leq d$. The matrix $A^\#$ is sometimes called the *conversion* of A (see, e. g. [3] p. 171). It is immediate from its definition that $(A^\#)^{-1} = (A^{-1})^\#$ and that $(A^t)^\# = (A^\#)^t$. We can now rapidly derive formulas for all the transition matrices between the six fundamental bases.

If $\{p_i(x)\}_{i=0, \dots, d}$ and $\{q_i(x)\}_{i=0, \dots, d}$ are two ordered bases of V_d we will denote by $L(p_i(x), q_i(x))$ the real matrix that takes the vector $(q_0(x), \dots, q_d(x))^t$ into $(p_0(x), \dots, p_d(x))^t$. Then, clearly

$$L(p_i(x), q_i(x))^{-1} = L(q_i(x), p_i(x)) \quad (6)$$

and

$$L(p_i(x), q_i(x))L(q_i(x), r_i(x)) = L(p_i(x), r_i(x)) \quad (7)$$

for all ordered bases $\{p_i(x)\}_{i=0, \dots, d}$, $\{q_i(x)\}_{i=0, \dots, d}$ and $\{r_i(x)\}_{i=0, \dots, d}$ of V_d . Now it is well known (see, e. g. , [63], §4.3), that

$$\begin{aligned}
L\left(\left(x\right)_i, x^i\right) &= s, \\
L\left(\left\langle x\right\rangle_i, x^i\right) &= C.
\end{aligned} \quad (8)$$

Since, clearly, $L\left(\left(\begin{smallmatrix} x \\ i \end{smallmatrix}\right), (x)_i\right) = L\left(\left(\begin{smallmatrix} x \\ i \end{smallmatrix}\right), \left\langle x\right\rangle_i\right) = F^{-1}$, we deduce

$$L\left(\left(\begin{smallmatrix} x \\ i \end{smallmatrix}\right), x^i\right) = F^{-1}s, \quad L\left(\left(\begin{smallmatrix} x \\ i \end{smallmatrix}\right), \left\langle x\right\rangle_i\right) = F^{-1}C. \quad (9)$$

Also, from Theorem 2.3.3 we easily deduce, in particular, that

$$L\left(\begin{pmatrix} x+d-i \\ d \end{pmatrix}, \begin{pmatrix} x \\ i \end{pmatrix}\right) = B^\# , \quad L\left(\begin{pmatrix} x \\ i \end{pmatrix}, \begin{pmatrix} x \\ i \end{pmatrix}\right) = M \quad (10)$$

Therefore, from (7) and (8) we have

$$L\left(\begin{pmatrix} x+d-i \\ d \end{pmatrix}, x^i\right) = B^\# F^{-1} s \quad (11)$$

From (8), (9) and (11) we can now, using (6) and (7), easily compute all the remaining transition matrices. The results are shown in Table 1. Note that, since

$$L\left(\langle x \rangle_i, x^i\right) L\left(x^i, (x)_i\right) = L\left(\langle x \rangle_i, (x)_i\right),$$

we obtain, by (8) and (10), the relation

$$C S = F M F^{-1}. \quad (12)$$

Therefore Table 1 can be written in many different but equivalent ways; we have chosen the one that best suits our future needs and that seems the more explicit. We would like to remark that Table 1 is indeed quite explicit and easy to compute from the six basic matrices. In fact, letting $D_d \stackrel{\text{def}}{=} \text{diag}(1, -1, 1, \dots, (-1)^d)$, we have the well known relation (see, e. g. [63], §2.1)

$$B^{-1} = D B D.$$

By the definition of M this also implies that

$$M^{-1} = D M D.$$

Finally, by (12), we also have the formula

$$C^{-1} = S F M^{-1} F^{-1}.$$

Therefore all the matrices appearing in Table 1 can be computed without great trouble starting from tables of Stirling numbers and binomial coefficients. The transition matrices for the value $d = 2$ appears in Table 2.

We conclude this section by noting one easy but important consequence of the transition matrices obtained in Table 1.

Proposition 2.3.4 *The following inclusions hold:*

$$\begin{aligned} PF_1[\langle x \rangle_i] &= PF_1 \left[\left(\binom{x}{i} \right) \right] \subset PF_1[x^i] \subset PF_1 \left[\binom{x}{i} \right], \\ PF_1 \left[\binom{x+d-i}{d} \right] &\subset PF_1 \left[\binom{x}{i} \right] = PF_1[(x)_i]. \square \end{aligned}$$

We will see in section 2.6 (and it can also be checked directly from Table 1) that no other inclusions hold between the PF_1 -sets of the six fundamental bases except those stated in the preceding proposition.

2.4 Linear Transformations that preserve the PF property

In this section we prove some key results that will enable us, in section 2.6, to determine which inclusions hold among the PF -sets of the six fundamental bases. Our first result is not new and a proof of it can be found in [46], vol. II. The proof that we present here is a simplification of the one in [46].

Theorem 2.4.1 *Let $P(x) = \sum_{i=0}^d a_i x^i$ be a polynomial having only real zeros. Then the polynomial $Q(x) = \sum_{i=0}^d \frac{a_i}{i!} x^i$ also has only real zeros.*

Proof: (Pólya and Szegő, problem 65, Part V, Vol.II).

Since $P(x)$ has only real zeros it follows that $x^d P\left(\frac{1}{x}\right)$ also does. So we may write

$$\sum_{i=0}^d a_{d-i} x^i = a_{d-r} \prod_{i=1}^r (x - \alpha_i), \quad (13)$$

where $\alpha_1, \dots, \alpha_r \in \mathbf{R}$ and $0 \leq r \leq d$. Now, (13) implies that

$$\begin{aligned} a_{d-r} \left[\prod_{i=1}^r \left(\frac{d}{dx} - \alpha_i \right) \right] (x^d) &= \sum_{i=0}^d a_{d-i} (d)_i x^{d-i} \\ &= d! Q(x). \end{aligned} \quad (14)$$

It is well known and easy to see that if $T(x)$ is a real polynomial having only real zeros and $\alpha \in \mathbf{R}$ then $T'(x) - \alpha T(x)$ also has only real zeros. Therefore the polynomial on the LHS of (14) has only real zeros and hence $Q(x)$ does also, as desired. $\square$

Our second result is the following.

Theorem 2.4.2 *Let $Q(x) = \sum_{i=0}^d a_i(x)_i$ be a polynomial having only real non-positive zeros. Then $P(x) \stackrel{\text{def}}{=} \sum_{i=0}^d a_i x^i$ has only simple, real, nonpositive zeros.*

Proof: We proceed by induction on d , the case $d = 1$ being clearly true. Let $\beta_1, \dots, \beta_d \leq 0$ be the zeros of $Q(x)$ and let $a_d \prod_{i=1}^{d-1} (x - \beta_i) = \sum_{i=0}^{d-1} b_i(x)_i$. Then we have

$$\begin{aligned} \sum_{i=0}^d a_i(x)_i &= Q(x) \\ &= a_d \prod_{i=1}^d (x - \beta_i) \\ &= (x - \beta_d) \sum_{i=0}^{d-1} b_i(x)_i \\ &= \sum_{i=0}^{d-1} b_i[(x)_{i+1} + i(x)_i] - \sum_{i=0}^{d-1} \beta_d b_i(x)_i \\ &= \sum_{i=0}^d [b_{i-1} + b_i(i - \beta_d)](x)_i, \end{aligned}$$

where, of course, $b_{-1} = b_d = 0$. Hence we have

$$a_i = b_{i-1} + b_i(i - \beta_d),$$

for $i = 0, \dots, d$. Therefore,

$$\begin{aligned} P(x) &= \sum_{i=0}^d a_i x^i \\ &= \sum_{i=0}^d [b_{i-1} + b_i(i - \beta_d)] x^i \\ &= (x - \beta_d) R(x) + x R'(x), \end{aligned} \tag{15}$$

where $R(x) \stackrel{\text{def}}{=} \sum_{i=0}^{d-1} b_i x^i$. Now let

$$T(x) \stackrel{\text{def}}{=} e^{-x} x^{-\beta_d} R(-x), \tag{16}$$

for $x \geq 0$. Then, using (15), we get

$$T'(x) = e^{-x} x^{-\beta_d-1} P(-x), \quad (17)$$

for $x > 0$. Now, by the induction hypothesis, $R(x)$ has only simple real non-positive zeros. So we may write $R(x) = a_d \prod_{i=1}^{d-1} (x - \xi_i)$ where $\xi_1 < \xi_2 < \dots < \xi_{d-1} \leq 0$. Now, by (16), we have that $T(-\xi_i) = 0$ for $i = 1, \dots, d-1$ and $\lim_{x \rightarrow +\infty} T(x) = 0$. By Rolle's Theorem this implies that $T'(x)$ has a zero in $(-\xi_{i+1}, -\xi_i)$ for $i = 1, \dots, d-2$ and a zero in $(-\xi_1, +\infty)$. This, by (17), implies that $P(x)$ has a zero in (ξ_i, ξ_{i+1}) for $i = 1, \dots, d-2$ and a zero in $(-\infty, \xi_1)$. Finally, $P(x)$ has a zero in $[\xi_{d-1}, 0]$ because if $P(0) \neq 0$ then, by (15), $R(0) \neq 0$ and $\beta_d \neq 0$, hence $\xi_{d-1} < 0$ and $T(-\xi_{d-1}) = T(0) = 0$. Therefore, by Rolle's Theorem, $T'(x) = 0$ for some $x \in (0, -\xi_{d-1})$ and hence $P(x)$ has a zero in $(\xi_{d-1}, 0)$. This proves that $P(x)$ has only simple real nonpositive zeros. This concludes the induction step and hence the proof. $\square$

Our next result is in a sense “dual” to the last theorem although I haven't been able to give a unified proof of both of them.

Theorem 2.4.3 *Let $\sum_{i=0}^d c_i x^i$ be a polynomial with nonnegative coefficients and having only real zeros. Then the polynomial $\sum_{i=0}^d c_i \langle x \rangle_i$ has only real, simple, nonpositive zeros and the distance between any two of them is ≥ 1 .*

Proof: We proceed by induction on d . The result is clearly true if $d = 1$. So suppose that it is true for polynomials of degree $\leq d$, ($d \geq 1$). Let $\sum_{i=0}^{d+1} a_i x^i$ be a real polynomial of degree $d+1$ and suppose that $a_i \geq 0$ for $i = 0, \dots, d+1$ and that

$$\sum_{i=0}^{d+1} a_i x^i = a_{d+1} \prod_{i=1}^{d+1} (x + \beta_i), \quad (18)$$

where $\beta_1, \dots, \beta_{d+1} \geq 0$. Let $\sum_{i=0}^d c_i x^i \stackrel{\text{def}}{=} a_{d+1} \prod_{i=1}^d (x + \beta_i)$. Then we have, from (18),

$$\sum_{i=0}^{d+1} a_i x^i = (x + \beta_{d+1}) \sum_{i=0}^d c_i x^i$$

$$\begin{aligned}
&= \sum_{i=0}^d c_i x^{i+1} + \beta_{d+1} \sum_{i=0}^d c_i x^i \\
&= \sum_{i=0}^{d+1} (c_{i-1} + \beta_{d+1} c_i) x^i,
\end{aligned}$$

where $c_{-1} = c_{d+1} = 0$. So we have that

$$a_i = c_{i-1} + \beta_{d+1} c_i, \quad i = 0, \dots, d+1. \quad (19)$$

Now let $P(x) \stackrel{\text{def}}{=} \sum_{i=0}^d c_i \langle x \rangle_i$, $Q(x) \stackrel{\text{def}}{=} \sum_{i=0}^{d+1} a_i \langle x \rangle_i$. Using (19) we then have

$$\begin{aligned}
Q(x) &= \sum_{i=0}^{d+1} a_i \langle x \rangle_i \\
&= \sum_{i=0}^{d+1} (c_{i-1} + \beta_{d+1} c_i) \langle x \rangle_i \\
&= \sum_{i=1}^{d+1} c_{i-1} \langle x \rangle_i + \beta_{d+1} \sum_{i=0}^d c_i \langle x \rangle_i \\
&= x P(x+1) + \beta_{d+1} P(x).
\end{aligned} \quad (20)$$

Now, by the induction hypothesis, we have that $P(x) = a_{d+1} \prod_{i=1}^d (x - \delta_i)$ where $\delta_d < \delta_{d-1} < \dots < \delta_1 \leq 0$ and $\delta_i - \delta_{i+1} \geq 1$ for $i = 1, \dots, d-1$. Then from (20) there follows that

$$Q(0) \geq 0,$$

$$(-1)^i Q(\delta_i) \geq 0, \quad i = 1, \dots, d$$

$$(-1)^i Q(\delta_i - 1) \geq 0, \quad i = 1, \dots, d$$

$$\lim_{x \rightarrow -\infty} \text{sgn}(Q(x)) = (-1)^{d+1}.$$

This shows that $Q(x)$ has a real zero in $[\delta_{i+1}, \delta_i - 1]$ for $i = 1, \dots, d-1$, a real zero in $[\delta_1, 0]$ and a real zero in $(-\infty, \delta_d - 1]$. Hence $Q(x)$ has only simple nonpositive zeros, say $\xi_{d+1} < \xi_d < \dots < \xi_1 \leq 0$, and they satisfy,

$$\delta_i \leq \xi_i, \quad \xi_{i+1} \leq \delta_i - 1,$$

for $i = 1, \dots, d$. Therefore we have

$$\xi_i - \xi_{i+1} \geq \delta_i + 1 - \delta_i = 1, \quad i = 1, \dots, d.$$

This completes the induction step and hence the proof. $\square$

We conclude this section with a result that won't be needed until chapter 4, but that is similar in style and techniques to the ones already presented in this section. Recall that if f and g are two polynomials having only real zeros then we say that g *interlaces* f if $\deg(g) = \deg(f) - 1$ and we have that

$$\eta_1 \leq \xi_1 \leq \eta_2 \leq \dots \leq \eta_{d-1} \leq \xi_{d-1} \leq \eta_d$$

where $\eta_1, \eta_2, \dots, \eta_d$ are the zeroes of f and $\xi_1, \xi_2, \dots, \xi_{d-1}$ are those of g .

Theorem 2.4.4 *Let $f(x)$ and $g(x)$ be two polynomials of degrees d and $d - 1$ respectively. Suppose that $f(x), g(x) \in PF[x^i]$ and that g interlaces f . Define a polynomial $h(x)$ by*

$$h(x) \stackrel{\text{def}}{=} (x - \beta)f(x) - x(x - \alpha)g(x), \quad (21)$$

where $\alpha > \max(\beta, 0)$. Then, if $h(x)$ has nonnegative coefficients, $h(x)$ has only real zeros.

Proof: We may clearly assume that both f and g have only simple zeros and that they have no zeros in common. Let $\beta_1 < \dots < \beta_d \leq 0$ be the zeros of f and $\alpha_1 < \dots < \alpha_{d-1}$ be those of g . Then $\beta_i < \alpha_i < \beta_{i+1}$ for $i = 1, \dots, d - 1$ and, by (21), we have that

$$h(x) = -c \prod_{j=1}^{d+1} (x - \alpha_j) + l(x - \beta) \prod_{j=1}^d (x - \beta_j),$$

where $\alpha_d \stackrel{\text{def}}{=} 0$, $\alpha_{d+1} \stackrel{\text{def}}{=} \alpha$ and c and l are, respectively, the leading coefficients of g and f . Let now $r \stackrel{\text{def}}{=} \max\{i \in [d] : \alpha_i \leq \beta\}$ and set $r \stackrel{\text{def}}{=} 0$ if $\{\{i \in [d] : \alpha_i \leq \beta\} = \emptyset\}$ (i. e. if $\beta < \alpha_1$). Then it is easy to check that

$$\text{sgn}(h(\alpha_i)) = \begin{cases} (-1)^{d+1-i} & \text{if } 1 \leq i \leq r - 1, \\ (-1)^{d-i} & \text{if } r + 1 \leq i \leq d - 1, \end{cases}$$

$$\text{sgn}(h(\beta_i)) = (-1)^{d+1-i}, \quad i = 1, \dots, d - 1,$$

and, if $1 \leq r \leq d-1$,

$$\operatorname{sgn}(h(\alpha_r)) = \begin{cases} (-1)^{d+1-r} & \text{if } \alpha_r < \beta, \\ 0 & \text{if } \alpha_r = \beta. \end{cases}$$

Hence $h(\alpha_i)h(\beta_{i+1}) < 0$ for $i = 1, \dots, r-1$ and $h(\beta_i)h(\alpha_i) < 0$ for $i = r+1, \dots, d-1$. Therefore $h(x)$ has a real zero in (α_i, β_{i+1}) for $i = 1, \dots, r-1$ and in (β_i, α_i) if $i = r+1, \dots, d-1$. Also, $h(x)$ has a real zero in $[\beta_d, \alpha_d]$ because since $h(x)$ has nonnegative coefficients we have $h(\alpha_d) = h(0) \geq 0$ and if $h(0) > 0$ the $\beta_d < \alpha_d = 0$ and hence $h(\beta_d) < 0$. Finally, if $1 \leq r \leq d-1$, then $h(x)$ has a real zero in $[\alpha_r, \beta_{r+1}]$ because if $\alpha_r = \beta$ then $h(\alpha_r) = 0$ and if $\alpha_r < \beta$ then $h(\alpha_r)h(\beta_{r+1}) < 0$. Therefore $h(x)$ has at least d real zeros and since $\deg(h(x)) \leq d+1$ it follows that $h(x)$ has only real zeros, as desired. $\square$

As an immediate corollary of the preceding theorem we obtain the following result.

Theorem 2.4.5 *Let $f(x)$ be a polynomial such that $f(x) \in PF[x^i]$. Define a polynomial $h(x)$ by*

$$h(x) \stackrel{\text{def}}{=} (\gamma x - \beta)f(x) - x(x - \alpha)f'(x), \quad (22)$$

where $\alpha > 0$, $\gamma \geq 0$ and $\alpha\gamma > \beta$. Then $h(x) \in PF[x^i]$ if and only if $h(x) \in PF_1[x^i]$.

Proof: It is well known and easy to see that if a polynomial $f(x)$ has only real zeros then $f'(x)$ also has only real zeros and $f'(x)$ interlaces $f(x)$, hence the thesis follows from Theorem 2.4.4 if $\gamma > 0$. If $\gamma = 0$ and $h(x) \in PF_1[x^i]$ then from (22) it easily follows that $f(x)$ and $h(x)$ are two nonnegative constants and hence $h(x) \in PF[x^i]$ also in this case. $\square$

The reader should note that equation (22) defines a linear map $\mathbf{R}^{d+1} \rightarrow \mathbf{R}^{d+2}$ with the property that the image of a PF sequence is PF if and only if it is PF_1 . Linear maps of the kind given by equation (22) will occur so frequently in later chapters that it is worthwhile to restate Theorem 2.4.5 in terms of sequences

instead than in terms of polynomials and derivatives.

Theorem 2.4.6 *Let $\{a_0, \dots, a_d\}$ be a PF sequence and let $\alpha, \beta, \gamma \in \mathbf{R}$ be such that $\alpha > 0$, $\gamma \geq 0$ and $\gamma\alpha > \beta$. Define a sequence $\{b_0, \dots, b_{d+1}\}$ by*

$$b_i \stackrel{\text{def}}{=} (\alpha i - \beta)a_i + (\gamma - i + 1)a_{i-1}, \quad (23)$$

for $i = 0, \dots, d+1$ (where $a_i \stackrel{\text{def}}{=} 0$ if $i < 0$ or $i > d$). Then $\{b_0, \dots, b_{d+1}\}$ is PF if and only if it is PF_1 . $\square$

We should remark that our insistence in proving Theorem 2.4.5 (and hence Theorem 2.4.6) also in the degenerate case $\gamma = 0$ does not come from an excessive “love” of maximum possible generality (I do, in fact, believe that Theorem 2.4.5 can still be generalized in many ways) but from mere necessity. In chapter 4 we will use Theorem 2.4.5 repeatedly (in fact, chapter 4 will consist almost entirely of applications of it) and it will be crucial for our applications that Theorem 2.4.5 continues to hold also in the case that $\gamma = 0$. As a matter of fact, the case $\gamma = 0$ will be the fundamental induction step in many of our applications of Theorem 2.4.5 (see, e. g. , Theorems 4.3.1 and 4.3.4).

2.5 Linear Transformations that preserve the PF_2 property

In the study of the questions mentioned in the Introduction it became more and more apparent that the log-concavity of a sequence is almost never preserved while the slightly stronger concept of a PF_2 sequence is much more well behaved. Before going on, therefore, we prefer to clarify the exact relationship between the two concepts.

Proposition 2.5.1 *Let $\{b_1, \dots, b_n\}$ be a sequence of real numbers. Then $\{b_1, \dots, b_n\}$ is a PF_2 sequence if and only if it is PF_1 , log-concave and with no internal zeros.*

Proof: Suppose $\{b_1, \dots, b_n\}$ is PF_2 . Then $b_i^2 - b_{i+1} b_{i-1} = \begin{vmatrix} b_i & b_{i+1} \\ b_{i-1} & b_i \end{vmatrix} \geq 0$ for $i = 2, \dots, n-1$ and hence $\{b_1, \dots, b_n\}$ is log-concave. Also, if $b_r \neq 0$ and if $b_t = 0$ where $t \geq r+1$ then we have

$$-b_{t+1} b_r = \begin{vmatrix} b_{r+1} & b_{t+1} \\ b_r & b_t \end{vmatrix} \geq 0,$$

and since $b_r > 0$ this implies $b_{t+1} \leq 0$ and hence $b_{t+1} = 0$. This shows that $\{b_1, \dots, b_n\}$ has no internal zeros.

Conversely, suppose $\{b_1, \dots, b_n\}$ is PF_1 , log-concave and with no internal zeros. Then we may assume that $b_i \neq 0$ for all $i = 1, \dots, n$. We have to show that

$$\begin{vmatrix} b_r & b_{r+p} \\ b_t & b_{t+p} \end{vmatrix} \geq 0, \quad (24)$$

for all $1 \leq t \leq r$ and $p \geq 1$. We begin by showing that

$$\begin{vmatrix} b_r & b_{r+1} \\ b_t & b_{t+1} \end{vmatrix} \geq 0, \quad (25)$$

for all $1 \leq t < r$. We prove (25) by induction on r . If $r = t+1$ then (25) is true by the log-concavity of $\{b_1, \dots, b_n\}$. Suppose now that (25) is true for $r > t$, then we have

$$\begin{aligned} b_r \begin{vmatrix} b_{r+1} & b_{r+2} \\ b_t & b_{t+1} \end{vmatrix} &= b_r b_{r+1} b_{t+1} - b_r b_{r+2} b_t \geq \\ &\geq (b_{r+1}^2 - b_r b_{r+2}) b_t \geq 0 \end{aligned}$$

where we have used our induction hypothesis (namely (25)) in the first inequality and log-concavity in the second one. Since $b_r > 0$ this shows that (25) holds for $r+1$ and hence for all $r > t \geq 1$. We now prove (24) by induction on $p \geq 1$, the case $p = 1$ being what we have just proved. So suppose (24) holds for $p \geq 1$. Then we have

$$\begin{aligned} b_{t+p} \begin{vmatrix} b_r & b_{r+p+1} \\ b_t & b_{t+p+1} \end{vmatrix} &= b_{t+p} b_r b_{t+p+1} - b_{t+p} b_{r+p+1} b_t \geq \\ &\geq (b_{r+p} b_{t+p+1} - b_{t+p} b_{r+p+1}) b_t \geq 0, \end{aligned}$$

where we have used the induction hypothesis, namely (24), in the first inequality and (25) (with $r + p$ and $t + p$ in place of r and t , respectively) in the second one. Since $b_{t+p} > 0$ this shows that (24) also holds for $p + 1$ and hence holds for all $p \geq 1$, as desired. $\square$

Note that the preceding proposition implies in particular that a (nonnegative) log-concave sequence is unimodal if and only if it is PF_2 . This is one of the reasons why we prefer to consider PF_2 sequences over log-concave ones. In fact, had we considered log-concave sequences we would not have obtained a chain of subsets in Proposition 2.3.2.

A surprising feature of the results of this section is that they all follow from the following special case of Theorem 2.2.5.

Theorem 2.5.2 *Let L, M, N be three infinite matrices and let $\{q_i\}_{i \in \mathbb{N}}$ be an eventually vanishing PF_2 sequence. Suppose that M and N are TP_2 and that L, M, N satisfy the “semigroup property”*

$$L_{i+j}(z) = M_i(z) N_j(z)$$

for all $i, j \in \mathbb{N}$. Define a sequence $\{c_t\}_{t \in \mathbb{N}}$ by

$$c_t = \sum_{i \geq 0} L_{t,i} q_i,$$

for all $t \in \mathbb{N}$. Then $\{c_t\}_{t \in \mathbb{N}}$ is a PF_2 sequence.

Proof: It is easily verified that a sequence is PF_2 if and only if it is SR_2 with sign sequence $(1, -1)$. Hence the thesis follows from Theorem 2.2.5. $\square$

The main reason why PF_2 sequences are so much better behaved than log-concave sequences is exactly that the preceding result holds for the former but (as we will see) not for the latter. Our first application of the above theorem is the following.

Theorem 2.5.3 *Let $\{q_i\}_{i \geq 0}$ be an eventually vanishing PF_2 sequence. Define*

another sequence $\{c_t\}_{t \geq 0}$ by

$$c_t = \sum_{n \geq 0} \binom{n}{t} q_n,$$

for all $t \geq 0$. Then $\{c_t\}_{t \geq 0}$ is a PF_2 sequence.

Proof: Define an infinite matrix N by $N \stackrel{\text{def}}{=} I_1 \oplus B^t$, where I_1 is the 1×1 identity matrix and B is the (infinite) matrix of binomial coefficients as defined in section 2.3. Then we have

$$(B^t)_i(z) = \frac{z^i}{(1-z)^{i+1}}, \quad N_i(z) = \frac{z^i}{(1-z)^i}, \quad (26)$$

for all $i \geq 0$. From this it follows that

$$(B^t)_{i+j}(z) = N_i(z)(B^t)_j(z),$$

for all $i, j \geq 0$. Now, it is easy to see that B is a TP_2 matrix (in fact, B is actually a TP matrix) (see, e. g., [33], or [22] for a beautiful combinatorial proof of this fact). Therefore B^t and hence also $I_1 \oplus B^t$ are TP_2 . So the three matrices B^t , N , B^t satisfy all the hypotheses of Theorem 2.5.2 and the thesis follows. $\square$

As an immediate consequence of this last theorem we obtain the following result.

Theorem 2.5.4 *Let*

$$Q(x) = \sum_{i=0}^d a_i \binom{x}{i} = \sum_{i=0}^d c_i \left(\binom{x}{i} \right) \quad (27)$$

be a polynomial of degree d . Then, if $\{c_0, c_1, \dots, c_d\}$ is a PF_2 sequence, so is $\{a_0, a_1, \dots, a_d\}$.

Proof: An easy computation shows that (27) is equivalent to

$$a_i = \begin{cases} \sum_{j=1}^d \binom{j-1}{i-1} c_j, & \text{if } 1 \leq i \leq d, \\ c_0, & \text{if } i = 0. \end{cases}$$

This in turn is equivalent to

$$a_i = \sum_{j \geq 0} N_{i,j} c_j,$$

for all $i \geq 0$, where $N = (N_{i,j})_{i,j \geq 0}$ has the same meaning as in the proof of Theorem 2.5.3 and where $a_i = c_i = 0$ if $i > d$. From (26) we then have that

$$N_{i+j}(z) = N_i(z) N_j(z),$$

for all $i, j \geq 0$. Also, we know that N is a TP_2 matrix. This shows that all the hypotheses of Theorem 2.5.2 are satisfied and the thesis follows. $\square$

The next result is elementary but nonetheless deserves to be mentioned.

Proposition 2.5.5 *Let $\{a_i\}_{i=0,\dots,d}$ be a PF_2 sequence. Then the sequence $\{\frac{a_i}{i!}\}_{i=0,\dots,d}$ is PF_2 . $\square$*

The converse of the preceding proposition is not true, however, the following result holds.

Theorem 2.5.6 *Let $\{\alpha_i\}_{i=0,\dots,d}$ be a PF sequence. Then the sequence $\{(i!) \alpha_i\}_{i=0,\dots,d}$ is PF_2 .*

Proof: Since $\{\alpha_0, \alpha_1, \dots, \alpha_d\}$ is a PF sequence there follows from Theorem 2.3.1 that the polynomial $\sum_{i=0}^d \alpha_i x^i$ has only real zeros. By Newton's inequalities (see, e. g. [12], p. 270 or [29], p.104) this implies that

$$\alpha_j^2 \geq \alpha_{j-1} \alpha_{j+1} \left(\frac{j+1}{j} \right) \left(\frac{d-j+1}{d-j} \right),$$

for $1 \leq j \leq d-1$. There follows that

$$\begin{aligned} (j! \alpha_j)^2 &\geq (j-1)! \alpha_{j-1} (j+1)! \alpha_{j+1} \left(\frac{d-j+1}{d-j} \right) \\ &\geq (j-1)! \alpha_{j-1} (j+1)! \alpha_{j+1}, \end{aligned}$$

for $1 \leq j \leq d-1$. This shows that the sequence $\{j! \alpha_j\}_{j=0,\dots,d}$ is log-concave. Since $\{\alpha_0, \alpha_1, \dots, \alpha_d\}$ is a PF sequence there follows in particular that it has no internal zeros. Hence the same is true for $\{j! \alpha_j\}_{j=0,\dots,d}$ and therefore, by Proposition 2.5.1, we can conclude that $\{\alpha_0, 1! \alpha_1, 2! \alpha_2, \dots, d! \alpha_d\}$ is a PF_2 sequence, as desired. $\square$

The next result is a “transpose” version of Theorem 2.5.3. Though it will not be needed until chapter 3 it is an easy consequence of Theorem 2.5.2 and

we therefore prove it here. It also appears, essentially, in [33], Theorem 7.3, p. 142.

Theorem 2.5.7 *Let $\{a_0, \dots, a_d\}$ be a PF_2 sequence. Define a polynomial $A(x)$ by $A(x) \stackrel{\text{def}}{=} \sum_{i=0}^d a_i \binom{x}{i}$. Then the sequence $\{A(n)\}_{n \in \mathbb{N}}$ is PF_2 .*

Proof: Let B have the same meaning as in the proof of Theorem 2.5.3. We know that B is a TP_2 matrix. Furthermore, we clearly have that $B_i(z) = (1+z)^i$, for $i \in \mathbb{N}$. This shows that taking $L = N = M = B$ will satisfy all the hypotheses of Theorem 2.5.2 and the thesis follows. $\square$

Our last result of this section is the following.

Theorem 2.5.8 *Let*

$$P(x) = \sum_{i=0}^d \omega_i \binom{x+d-i}{d} = \sum_{i=0}^d a_i \binom{x}{i} \quad (28)$$

be a polynomial of degree d . Then, if $\{\omega_0, \omega_1, \dots, \omega_d\}$ is a PF_2 sequence, so is $\{a_0, a_1, \dots, a_d\}$.

Proof: By Theorem 2.3.3 we know that (28) is equivalent to

$$a_j = \sum_{i=0}^j \binom{d-i}{d-j} \omega_i, \quad j = 0, \dots, d.$$

Hence we have, for $j = 0, \dots, d$,

$$\begin{aligned} a_{d-j} &= \sum_{i=0}^{d-j} \binom{d-i}{j} \omega_i \\ &= \sum_{i=j}^d \binom{i}{j} \omega_{d-i}. \end{aligned}$$

Now, it is clear that a sequence $\{b_0, \dots, b_d\}$ is PF_2 if and only if the sequence $\{b_d, \dots, b_1, b_0\}$ is PF_2 , so the thesis follows from Theorem 2.5.3. $\square$

Note that Theorem 2.5.8 shows that, in fact, Conjecture 1.1 implies Conjecture 2.1. The reader should also note that in all the results in this section except Proposition 2.5.5 the hypothesis that the given sequence be PF_2 is essential. If the sequence is just log-concave then the results are false. For example if $\{q_t\}_{t \geq 0} = \{4, 1, 0, 0, 1, 1, 0, 0, \dots\}$ in Theorem 2.5.3 then $\{c_t\}_{t \geq 0} =$

$\{7, 10, 16, 14, 6, 1, 0, 0, \dots\}$ which is not even log-concave. Analogously we have that $\binom{x+3}{3} + 3\binom{x}{3} = 1 + 3\binom{x}{1} + 3\binom{x}{2} + 4\binom{x}{3}$ but $\{1, 3, 3, 4\}$ is not log-concave (cfr. Theorem 2.5.8). Also, $1 + \binom{x}{3} = 1 + \binom{x}{1} + 2\binom{x}{2} + \binom{x}{3}$ but $\{1, 1, 2, 1\}$ is not log-concave (cfr. Theorem 2.5.4). All these examples, of course, also show that Theorem 2.5.2 fails for log-concave sequences.

2.6 Summary of results and open problems

In this final section of the chapter we show how, using the results of the preceding three sections, we can almost completely solve the problem that we set out to study in section 2.3.

Our first result combines Theorems 2.4.1, 2.4.2 and 2.4.3 and describes the inclusions that hold among the PF -sets of the six fundamental bases.

Theorem 2.6.1 *The following strict inclusions hold:*

$$PF\left[\left(\binom{x}{i}\right)\right] \subset PF[\langle x \rangle_i] \subset PF[x^i] \subset PF[(x)_i],$$

$$PF\left[\binom{x+d-i}{d}\right] \subset PF\left[\binom{x}{i}\right] \subset PF[(x)_i].$$

Proof: The inclusions $PF\left[\left(\binom{x}{i}\right)\right] \subseteq PF[\langle x \rangle_i]$ and $PF\left[\binom{x}{i}\right] \subseteq PF[(x)_i]$ follow immediately from Theorem 2.4.1, while the inclusion $PF\left[\binom{x+d-i}{d}\right] \subseteq PF\left[\binom{x}{i}\right]$ follows from (3) of Theorem 2.3.3 and the fact that $PF_1\left[\binom{x+d-i}{d}\right] \subseteq PF_1\left[\binom{x}{i}\right]$ (see Proposition 2.3.4). To prove that $PF[\langle x \rangle_i] \subseteq PF[x^i]$ let $P(x) \stackrel{\text{def}}{=} \sum_{i=0}^d c_i \langle x \rangle_i \in PF[\langle x \rangle_i]$. Then $\sum_{i=0}^d c_i x^i$ has only real zeros, but, by Theorem 2.4.3, this implies that $P(x)$ has only real nonpositive zeros and since the leading coefficient of $P(x)$ is positive there follows that $P(x) \in PF[x^i]$, as desired. Finally, to prove that $PF[x^i] \subseteq PF[(x)_i]$ take $Q(x) \stackrel{\text{def}}{=} \sum_{i=0}^d a_i (x)_i \in PF[x^i]$. Then $Q(x)$ has only real nonpositive zeros; by Theorem 2.4.2 this implies that

$\sum_{i=0}^d a_i x^i$ also has only real nonpositive zeros. Therefore $\{a_0, a_1, \dots, a_d\}$ is a PF sequence and hence $Q(x) \in PF[(x)_i]$, as desired. That all the inclusions are strict is easy to see. $\square$

Our second result combines Theorems 2.5.4 and 2.5.8 and Proposition 2.5.5 and gives the inclusions that hold among the PF_2 -sets of the six fundamental bases.

Theorem 2.6.2 *The following strict inclusions hold:*

$$PF_2 \left[\left(\binom{x}{i} \right) \right] \subset PF_2[(x)_i] \cap PF_2 \left[\binom{x}{i} \right],$$

$$PF_2 \left[\binom{x+d-i}{d} \right] \subset PF_2 \left[\binom{x}{i} \right] \subset PF_2[(x)_i].$$

Proof: The inclusions $PF_2 \left[\left(\binom{x}{i} \right) \right] \subseteq PF_2 \left[\binom{x}{i} \right]$ and $PF_2 \left[\binom{x+d-i}{d} \right] \subseteq PF_2 \left[\binom{x}{i} \right]$ are just restatements of Theorems 2.5.4 and 2.5.8, respectively, while $PF_2 \left[\left(\binom{x}{i} \right) \right] \subseteq PF_2[(x)_i]$ and $PF_2 \left[\binom{x}{i} \right] \subseteq PF_2[(x)_i]$ are immediate consequences of Proposition 2.5.5. That all the inclusions are strict is easy to see. $\square$

Finally, our third result states which inclusions hold between the PF -sets and the PF_2 -sets of the six fundamental bases.

Theorem 2.6.3 *The following strict inclusions hold:*

$$PF[(x)_i] \subset PF_2 \left[\binom{x}{i} \right], \quad PF[(x)_i] \subset PF_2 \left[\left(\binom{x}{i} \right) \right].$$

Proof: Both inclusions are just a restatement of Theorem 2.5.6, strictness is trivial. $\square$

Using the counterexamples in the Appendix the reader can now see (though with some patience) that the preceding three theorems, together with Propositions 2.3.2 and 2.3.4, give (together with transitivity of the inclusion relation) all the inclusions that hold among the 24 subsets associated with the six fundamental bases, with at most two exceptions. The two (related) exceptions are still open and are the following:

- (i) Is it true that $PF_2 \left[\left(\binom{x}{i} \right) \right] \subseteq PF_2[x^i]$?

(ii) Is it true that $PF_2 \left[\left(\binom{x}{i} \right) \right] \subseteq UNI[x^i]$?

The answer is yes in both cases if $d \leq 5$. I personally feel that the above inclusions are either both true or both false. It is easy to see that if the answer to question (i) is yes then this can't be proved by a direct application of Theorem 2.5.2 (recall that all our results on PF_2 -preserving linear transformations followed from Theorem 2.5.2). Therefore, a positive solution of question (i) would probably bring with it a new general result on PF_2 -preserving linear transformations of the kind of Theorem 2.5.2 and this would of course be very interesting.

However, whatever the answer to the above two open questions will turn out to be, some interesting features clearly emerge from the study that we have carried out. First and foremost is, of course, the observation that *there are* inclusions that hold among the 24 subsets other than the obvious ones of Propositions 2.3.2 and 2.3.4. This makes one think that there are probably many other similar results to be found in this area if one considers other bases of V_d . This point of view will be taken up again in chapter 3 where we will see some very interesting and promising bases. The second observation is that the PF -sets of the six fundamental bases are those that have the greatest number of inclusions among them. In other words, the PF property is the one that is “most likely” to be preserved under a linear transformation. The second best behaved notion is the PF_2 property which, as we have shown in section 2.5, has much nicer properties than simple log-concavity. One other remark that follows from our study is that there are *no inclusions* among the Unimodal-sets of the six fundamental bases. This is particularly striking because the transition matrices between the six fundamental bases are, as we have seen, rather simple. The fact that there are no inclusions between the Unimodal-sets of the six fundamental bases may in part “explain” why it is often difficult to prove that a sequence is unimodal when it has no stronger properties. The

notion of unimodality (and also the one of log-concavity) seems to be very “delicate” and easily destroyed. It would be an interesting problem to try to classify all the linear transformations that preserve the unimodal property. We will briefly say something about this problem in section 3.5.

Taken together the above remarks seem to imply that it may often be “easier” to prove that a polynomial has the PF property even if one is actually interested only in the PF_2 or unimodal property (and this has often actually been the case in recent years in Combinatorics) and that, if possible, it is much easier to work with the PF_2 property than with the log-concave property.

The “philosophical” motivation that led me to develop the theory that I have presented in this chapter (besides the motivations that led to it and that have been considered in chapter 1) is that, in Combinatorics, it often happens that, in order to study a sequence $\{a_0, \dots, a_d\}$, the polynomial $\sum_{i=0}^d a_i x^i$ may not always be the best choice. It is sometimes the case that a different generating function, like $\sum a_i \binom{x}{i}$ or $\sum a_i \binom{x+d-i}{d}$ etc. , has a nice closed expression when the ordinary generating function doesn’t, and it seemed to me that the recent works in unimodality and log-concavity questions in Combinatorics lacked “tools” to tackle with this situation. My main purpose in developing this theory has been to look for, and provide, some basic such tools. We will see in chapters 5 and 6 many combinatorial situations in which we will be able to apply successfully the machinery developed in this chapter to obtain new interesting combinatorial results.

Finally, I should remark that the inclusions that hold among the 24 subsets are more if d is ≤ 10 . I have not checked if each of the counterexamples listed in the Appendix is of smallest possible dimension but this is probably true for most of them.

Chapter 3

Ramifications of the general theory

3.1 Introduction

In this chapter we examine several directions of further research that are suggested by the results presented in the last chapter. We only give some examples of the results that can be obtained along these lines and we present several conjectures. None of the results presented in this chapter, except those in section 3.3, will be used anywhere in the remainder of this work so the reader can, if he wants, skip all those sections without compromising in any way his understanding of the following chapters.

In the next section we study the inclusions that hold among unions and intersections of the 24 subsets. This is of course a very vast field of research and we only give a few representative results. In section 3.3 we specialize the general theory developed in chapter 2 to the special case of polynomials with no constant terms which is, of course, a case of particular interest for applications to Combinatorics. The general theory becomes nicer in this case and, in fact, we obtain an important result (Theorem 3.3.2) that will be used often in later chapters. In the fourth section we look beyond the six fundamental bases “universe” and we examine three other bases of V_d . These are the bases consisting of the Eulerian polynomials, the Lagrange interpolation polynomials and the Krawtchouk polynomials. Some of the inclusions between their associated sets

and those of some of the six fundamental bases are studied and several results and conjectures are presented. Finally, in section 3.5, we state three general problems that include as special cases *all* the results that we have obtained so far as well as many others and we collect what seems to be known about these three problems.

3.2 Further properties of the six fundamental bases

In the last chapter we almost completed the classification of the inclusions that hold between any two of the 24 subsets associated with the six fundamental bases. A closer look, however, shows that much more can be said about the 24 subsets (and about closely related others) than just which inclusions hold among them. In this section I will give some results of this kind. It seems to me that the results presented are interesting and that they represent only a fraction of the results that can be obtained in this area. Our first result is an easy (though important) observation about Theorem 2.3.3, (i).

Theorem 3.2.1 *The following equality holds in V_d :*

$$PF \left[\binom{x+d-i}{d} \right] = PF \left[\binom{x}{i} \right] \cap PF_1 \left[\binom{x+d-i}{d} \right] . \square$$

The preceding theorem tells us much more than just that $PF \left[\binom{x+d-i}{d} \right] \subset PF \left[\binom{x}{i} \right]$ and will in fact prove to be very useful. The above result naturally suggests the problem of studying which inclusions hold between unions and intersections of the 24 subsets and, more generally, to characterize in some useful way such unions and intersections. This is of course a vast field of possible research that is, so far, largely unexplored.

Unfortunately, it is easy to see that Theorem 3.2.1 is the only result of the

form $PF[p_i(x)] \cap PF_1[q_i(x)] = PF[q_i(x)]$ where $\{p_i(x)\}_{i=0,\dots,d}$ and $\{q_i(x)\}_{i=0,\dots,d}$ are two of the six fundamental bases. However, even though results of the sharpness and elegance of Theorem 3.2.1 are probably a rare occurrence, there are interesting results that can be found. The following is one example of the kind of results that we have in mind.

Proposition 3.2.2 *Let*

$$P(x) = \sum_{i=0}^d w_i \binom{x+d-i}{d} = \sum_{i=0}^d c_i \binom{x}{i}, \quad (1)$$

be a polynomial of degree $d \geq 1$. Suppose that $\{w_0, \dots, w_d\}$ is PF_1 and that $\{c_0, \dots, c_d\}$ is unimodal. Then $w_0 = c_0 = c_1 = \dots = c_{d-1} \leq c_d$ and $w_2 = w_3 = \dots = w_d = 0$.

Proof: We proceed by induction on d . It is easy to see that the result holds for $d = 1$. So assume that $d \geq 2$. From (4) of Theorem 2.3.3 there follows in particular that

$$w_1 = \sum_{i=1}^d c_i - d c_0, \quad (2)$$

$$w_{d-1} = (-1)^{d-1} (d c_0 - (d-1) c_1 - c_2), \quad (3)$$

$$w_d = (-1)^d (c_0 - c_1). \quad (4)$$

We now want to prove that $w_d = 0$. Suppose on the contrary that $w_d > 0$. We then have to distinguish two cases.

(i) d is odd. Then from (3) and (4) there follows that

$$c_0 < c_1,$$

$$(d-1)(c_0 - c_1) \geq c_2 - c_0. \quad (5)$$

Hence $c_0 > c_2$. Since $\{c_0, \dots, c_d\}$ is unimodal this implies that $c_1 > c_0 > c_2 \geq c_3 \geq \dots \geq c_d$. Therefore we have, from (2), that

$$\begin{aligned} w_1 &= \sum_{i=1}^d (c_i - c_0) < \\ &< (d-1)(c_1 - c_0) + (c_2 - c_0) \leq 0, \end{aligned}$$

by (5), which is a contradiction since $w_1 \geq 0$ by hypothesis.

(ii) d is even. Then from (4) there follows that $c_0 > c_1$. Since $\{c_0, \dots, c_d\}$ is unimodal this implies that $c_0 > c_1 \geq c_2 \geq c_3 \geq \dots \geq c_d$. Therefore we have, from (3), that

$$w_{d-1} = (d-1)(c_1 - c_0) + c_2 - c_0 < 0,$$

which is again a contradiction by our hypotheses.

This proves that $w_d = 0$ and hence, by (4), also that $c_0 = c_1$. Now, from (1), there follows that

$$\begin{aligned} P(x) - P(x-1) &= \sum_{i=0}^{d-1} w_i \left[\binom{x+d-i}{d} - \binom{x-1+d-i}{d} \right] \\ &= \sum_{i=0}^{d-1} w_i \binom{x+d-1-i}{d-1}. \end{aligned}$$

Analogously, (1) also implies that

$$\begin{aligned} P(x) - P(x-1) &= \sum_{i=0}^d c_i \left[\binom{x}{i} - \binom{x-1}{i} \right] \\ &= \sum_{i=1}^d c_i \binom{x}{i-1}. \end{aligned}$$

From these two equalities we deduce that

$$\sum_{i=0}^{d-1} w_i \binom{x+d-1-i}{d-1} = \sum_{i=0}^{d-1} c_{i+1} \binom{x}{i}.$$

Now, by our hypotheses, $\{w_0, \dots, w_{d-1}\}$ is a PF_1 sequence and $\{c_1, \dots, c_d\}$ is unimodal. So by the induction hypothesis we can conclude that $w_0 = c_1 = c_2 = \dots = c_{d-1} \leq c_d$ and $w_2 = \dots = w_{d-1} = 0$. Since we know that $w_d = 0$ and $c_0 = c_1$ this concludes the induction step and hence the proof. $\square$

As a special case of the above theorem we get the following result.

Theorem 3.2.3 *The following equality holds in V_d :*

$$PF_1 \left[\binom{x+d-i}{d} \right] \cap UNI \left[\binom{x}{i} \right] = \left\{ w_0 \binom{x+d}{d} + w_1 \binom{x+d-1}{d} : w_0, w_1 \geq 0 \right\}.$$

In particular,

$$PF_1 \left[\binom{x+d-i}{d} \right] \cap UNI \left[\left(\binom{x}{i} \right) \right] \subset PF \left[\binom{x+d-i}{d} \right] . \square$$

I should say that I have not investigated all possible results of the form $PF_2[p_i(x)] \cap PF[q_i(x)] = PF[p_i(x)]$ or $PF_2[p_i(x)] \cap PF[q_i(x)] = PF_2[q_i(x)]$ where $\{p_i(x)\}_{i=0,\dots,d}$ and $\{q_i(x)\}_{i=0,\dots,d}$ are two of the six fundamental bases nor results that involve four or more of the 24 subsets. One of the reasons why I have chosen not to do so is that as one goes deeper in the study of the 24 subsets the connection with combinatorial problems and applications seem to get more tenuous. However, I am convinced that more results await to be discovered in this area and I would not be surprised if some of them turned out to have combinatorial applications.

3.3 The special case of polynomials with no constant term

In this section we specialize the general theory developed in chapter 2 to polynomials that have no constant term. The basic reason for doing this is that polynomials with no constant term occur frequently in Combinatorics, for example the order polynomial of a labeled poset, or the chromatic polynomial of a graph, both have this property. The second reason is that multiplying a polynomial by x only translates the sequence of coefficients. Therefore, whenever we are considering a polynomial $P(x)$ because of the combinatorial significance of its coefficients, we may just as well consider $xP(x)$. Of course, specializing the general theory of chapter 2 to polynomials that have no constant term is equivalent to studying the inclusions that hold among the 24 subsets when they are “restricted” (i. e. intersected) with the set $\{A(x) \in V_d : A(0) = 0\}$. For this

reason it will be convenient to introduce a short notation for these restricted sets. For a subset A of V_d we will let $\overline{A} \stackrel{\text{def}}{=} \{A(x) \in A : A(x) = 0\}$, so that $\overline{A} = A \cap \overline{V}_d$ for all $A \subseteq V_d$. We will also write $\overline{PF}[p_i(x)], \overline{PF}_2[p_i(x)], \dots$, instead of $\overline{PF}[p_i(x)], \overline{PF}_2[p_i(x)], \dots$, if $\{p_i(x)\}_{i=0,\dots,d}$ is a basis of V_d . It is obvious that if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$ so that all the results obtained in chapter 2 about the inclusions that hold between the 24 subsets (specifically Theorems 2.6.1, 2.6.2, and 2.6.3) continue to hold now. However, it is a priori possible that some new inclusions may be created when restricting to $\overline{V}_d$. This is indeed the case and it is these additional inclusions that we will examine in this section. We start with a computational result that follows easily from Theorem 2.3.3.

Proposition 3.3.1 *Let*

$$P(x) = \sum_{i=1}^d w_i \binom{x+d-i}{d} = \sum_{i=1}^d c_i \binom{x}{i},$$

be a polynomial in $\overline{V}_d$. Then we have that

$$w_i = (-1)^{i-1} \sum_{j=i}^d \binom{j-1}{i-1} c_{d-j+1}, \quad (6)$$

$$c_{d-i+1} = (-1)^{i-1} \sum_{j=i}^d \binom{j-1}{i-1} w_j, \quad (7)$$

for $i = 1, \dots, d$. In particular,

$$\overline{PF}_1 \left[\binom{x+d-i}{d} \right] \subseteq \overline{PF}_1 \left[(-1)^{d-i} \binom{x}{i} \right], \quad (8)$$

$$\overline{PF}_1 \left[\binom{x}{i} \right] \subseteq \overline{PF}_1 \left[(-1)^{i-1} \binom{x+d-i}{d} \right]. \square$$

The reader should note that, as it is easy to see, both inclusions in (8) are false if “ $\overline{PF}_1$ ” is replaced by “ PF_1 ”. Also, the two equations in (6) and (7) show that the bases $\left\{ \binom{x}{i} \right\}_{i=1,\dots,d}$ and $\left\{ \binom{x+d-i}{d} \right\}_{i=1,\dots,d}$ of $\overline{V}_d$ are in a sense “dual” to each other and closely related by a very simple relation, a situation that sharply contrast that in V_d (cfr. section 3.2). The nice relations in (6) and

(7) indeed yield some inclusions between the PF sets of $\left(\begin{smallmatrix} x \\ i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x+d-i \\ d \end{smallmatrix}\right)$ when restricted to $\overline{V}_d$. These inclusions are collected in the next theorem which is the main result of this section.

Theorem 3.3.2 *The following equalities hold in V_d :*

$$\begin{aligned} \overline{PF} \left[\left(\begin{smallmatrix} x+d-i \\ d \end{smallmatrix} \right) \right] &= \overline{PF} \left[\left(\begin{smallmatrix} x \\ i \end{smallmatrix} \right) \right] \cap \overline{PF}_1 \left[\left(\begin{smallmatrix} x+d-i \\ d \end{smallmatrix} \right) \right] \\ &= \overline{PF} \left[(-1)^{d-i} \left(\begin{smallmatrix} x \\ i \end{smallmatrix} \right) \right] \cap \overline{PF}_1 \left[\left(\begin{smallmatrix} x+d-i \\ d \end{smallmatrix} \right) \right], \\ \overline{PF} \left[\left(\left(\begin{smallmatrix} x \\ i \end{smallmatrix} \right) \right) \right] &= \overline{PF} \left[(-1)^{i-1} \left(\begin{smallmatrix} x+d-i \\ d \end{smallmatrix} \right) \right] \cap \overline{PF}_1 \left[\left(\left(\begin{smallmatrix} x \\ i \end{smallmatrix} \right) \right) \right] \\ &= \overline{PF} \left[\left(\begin{smallmatrix} x \\ i \end{smallmatrix} \right) \right] \cap \overline{PF}_1 \left[\left(\left(\begin{smallmatrix} x \\ i \end{smallmatrix} \right) \right) \right]. \end{aligned}$$

Proof: All equalities follow easily from (4) and (5) of Proposition 2.3.3, from (6) and (7) of Proposition 3.3.1 and from Theorem 3.2.1. $\square$

The “duality” between the bases $\left(\begin{smallmatrix} x \\ i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x+d-i \\ d \end{smallmatrix}\right)$ in $\overline{V}_d$ and the consequent inclusions of Theorem 3.3.2 are the main advantages that one obtains by working in $\overline{V}_d$ instead than in V_d . One can check, in fact, that these are the only inclusions that hold in $\overline{V}_d$ among the 24 subsets and which don’t follow from corresponding inclusions in V_d .

3.4 Eulerian, Lagrange, and Krawtchouk polynomials

Up to now we have concentrated our attention on the six fundamental bases and on closely related others (see section 3.3). In this section we will consider other bases that, although not so commonly used as those considered so far, are also used in practice and especially in some specific areas of Mathematics. We will consider three bases that seem to have very interesting properties but whose study we have not fully pursued because it leads too far from the combinatorial

motivations that we are really interested in. Nonetheless, they may very well have interest and applications in other branches of Mathematics. My aim is principally to give to the reader some results that I have obtained in these directions and to present to him some open questions and conjectures that seem to me quite interesting.

The first “non fundamental” basis of V_d that comes to mind to a combinatorialist is without doubt the basis consisting of the Eulerian polynomials $\{A_i(x)\}_{i=0,\dots,d}$, (see, e. g. , [63], §1.3 for the definition and first values of the Eulerian polynomials). The combinatorial interpretation of these polynomials is very well known and it will not concern us here. However, the first basic fact is that the Eulerian polynomials have nonnegative coefficients so that we have the inclusion $PF_1[A_i(x)] \subset PF_1[x^i]$. This immediately makes us look for other inclusions between the corresponding PF_2 or PF sets. However, the identity

$$16 + 8 A_1(x) + 4 A_2(x) + 2 A_3(x) + A_4(x) = 16 + 15 x + 23 x^2 + 13 x^3 + x^4$$

shows that, in fact, $PF_2[A_i(x)] \not\subseteq UNI[x^i]$. We therefore remain with just three possible inclusions and, although I haven’t been able to prove them, I have not found any counterexample to them and I venture the following conjecture.

Conjecture: $PF[A_i(x)] \subset PF[x^i]$.

Equivalently, the preceding conjecture states that if the polynomial $\sum_{i=0}^d k_i x^i$ has only real zeros and nonnegative coefficients then the polynomial $\sum_{i=0}^d k_i A_i(x)$ also does.

I should also mention that I have no proofs either for the two weaker statements “ $PF[A_i(x)] \subseteq PF_2[x^i]$ ” and “ $PF[A_i(x)] \subseteq UNI[x^i]$ ”. However, I feel that the three statements are either all true or all false.

The second basis of V_d that I will take into consideration is one that is very often used in numerical analysis, namely the basis consisting of the Lagrange interpolation polynomials, $\{L_i(x)\}_{i=0,\dots,d}$. Instead of defining them explicitly

in terms of the basis $\{x^i\}_{i=0,\dots,d}$ it is more convenient for us to define, for $i = 0, \dots, d$, $L_i(x)$ to be the unique polynomial of degree i such that

$$L_i(x) = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{if } x = 0, 1, 2, \dots, i-1. \end{cases}$$

It then follows immediately that the basis $\{L_i(x)\}_{i=0,\dots,d}$ has the following characteristic property.

Proposition 3.4.1 *Let $A(x) \in V_d$, then*

$$A(x) = \sum_{i=0}^d A(i) L_i(x). \square$$

In other words, the coefficient of $L_i(x)$ in the expansion of a polynomial $A(x)$ is given by the polynomial itself evaluated at i . This property makes it of course very easy to expand a polynomial with respect to this basis and also implies that $PF_1[p_i(x)] \subseteq PF_1[L_i(x)]$ if $\{p_i(x)\}_{i=0,\dots,d}$ is any one of the six fundamental bases. In fact, it is easy to see that if $A(x)$ is a polynomial of degree $\leq d$ and $A(x) \in PF_1[p_i(x)]$ where $\{p_i(x)\}_{i=0,\dots,d}$ is as above then the sequence $\{A(0), A(1), \dots, A(d)\}$ is nondecreasing so that, in particular, $PF_1[p_i(x)] \subseteq UNI[L_i(x)]$. This again suggests that one looks at inclusions between the corresponding PF_2 or PF -sets of the six fundamental bases and of $\{L_i(x)\}_{i=0,\dots,d}$. I have only investigated such possible inclusions in the case that $p_i(x) = \binom{x+d-i}{d}$, $i = 0, \dots, d$. The reason why I considered the $\left\{\binom{x+d-i}{d}\right\}_{i=0,\dots,d}$ basis will become clear in the next chapter, particularly in section 4.6. For the moment I will just present to the reader the results that I have found, and an open question.

Theorem 3.4.2 *The following inclusions hold:*

$$PF_2 \left[\binom{x+d-i}{d} \right] \subseteq PF_2[L_i(x)]; \quad (9)$$

$$PF[L_i(x)] \cap PF_1 \left[\binom{x+d-i}{d} \right] \subseteq PF_2 \left[\binom{x+d-i}{d} \right]. \quad (10)$$

Proof: By Theorem 2.6.2 we have that $PF_2 \left[\binom{x+d-i}{d} \right] \subset PF_2 \left[\binom{x}{i} \right]$. But by Theorem 2.5.7 any polynomial $P(x) \in PF_2 \left[\binom{x}{i} \right]$ is such that the sequence $\{P(n)\}_{n \in \mathbb{N}}$ is PF_2 . Hence, in particular, the sequence $\{P(n)\}_{n=0, \dots, d}$ is PF_2 and this, by Proposition 3.4.1, implies that $P(x) \in PF_2[L_i(x)]$ and (9) follows. We now prove (10). Let

$$A(x) \stackrel{\text{def}}{=} \sum_{i=0}^d w_i \binom{x+d-i}{d} \quad (11)$$

be a polynomial in $PF[L_i(x)] \cap PF_1 \left[\binom{x+d-i}{d} \right]$. This implies that $w_i \geq 0$ for $i = 0, \dots, d$ and, by Proposition 3.4.1, that the sequence $\{A(0), \dots, A(d)\}$ is PF . By (2) of Theorem 2.3.3 we know that (11) is equivalent to

$$\sum_{n \geq 0} A(n) z^n = \frac{W(z)}{(1-z)^{d+1}},$$

where $W(z) \stackrel{\text{def}}{=} \sum_{i=0}^d w_i z^i$. Hence we have that

$$(1-z)^{d+1} \left[Q(z) + \sum_{n \geq d+1} A(n) z^n \right] = W(z),$$

where $Q(z) \stackrel{\text{def}}{=} \sum_{i=0}^d A(i) z^i$. So we conclude that

$$W(z) \equiv Q(z)(1-z)^{d+1} \pmod{z^{d+1}}. \quad (12)$$

Now, since the sequence $\{A(0), \dots, A(d)\}$ is PF , the polynomial $Q(z)$ and hence also $Q(z)(1-z)^{d+1}$ have only real zeros. Therefore, by Newton's inequalities (see, e. g. , [12], p.270), the coefficients of $Q(z)(1-z)^{d+1}$ are a log-concave sequence. Hence, in particular, the sequence $\{w_0, \dots, w_d\}$ is log-concave. Furthermore, from (12) there follows that

$$\left(\frac{d}{dz} \right)^{d+1} \left[z^{2d+1} W \left(\frac{1}{z} \right) \right] = \left(\frac{d}{dz} \right)^{d+1} \left[z^d Q \left(\frac{1}{z} \right) (z-1)^{d+1} \right]. \quad (13)$$

Since $Q(z)(1-z)^{d+1}$ has only real zeros we have, from (13), that the polynomial on the LHS of (13) does. But, by hypothesis the coefficients of $W(z)$ are nonnegative. Hence $\left(\frac{d}{dz} \right)^{d+1} \left[z^{2d+1} W \left(\frac{1}{z} \right) \right] \in PF[z^i]$ and therefore the sequence $\{w_0(2d+1)_{(d+1)}, \dots, w_d(d+1)_{(d+1)}\}$ is PF . In particular the sequence

$\{w_0, \dots, w_d\}$ has no internal zeros and the thesis follows from Proposition 2.5.1. $\square$

It should be noted that the above proof also shows that if the sequence $\{A(0), \dots, A(d)\}$ is PF (where $A(x)$ is defined by (11)) then the sequence $\{w_0, \dots, w_d\}$ is log-concave even if it is not PF_1 . However, we have no use of this fact here. The second inclusion in Theorem 3.4.2 is in our opinion quite remarkable. What is even more remarkable, however, is that the following much stronger statement seems to be true.

Conjecture: $PF[L_i(x)] \cap PF_1 \left[\binom{x+d-i}{d} \right] \subseteq PF \left[\binom{x+d-i}{d} \right]$.

The third and last basis of V_d that I want to consider in this section is the basis consisting of the Krawtchouk polynomials. I will not define these polynomials in the greatest possible generality but only in the special case that is presently of interest to us. Fix $d \in \mathbf{P}$; then the i -th *Krawtchouk polynomial* of V_d is defined by

$$K_i^{(d)}(x) \stackrel{\text{def}}{=} \sum_{j=0}^i (-2)^j \binom{d-j}{i-j} \binom{x}{j}. \quad (14)$$

It is clear from the definition that $\{K_i^{(d)}(x)\}_{i=0, \dots, d}$ is a basis of V_d and we will usually omit the superscript “(d)” when no confusion is possible.

The definition (14) shows that the transition matrix between the two bases $\{K_i^{(d)}(x)\}_{i=0, \dots, d}$ and $\left\{ \binom{x}{i} \right\}_{i=0, \dots, d}$ is particularly simple. This naturally suggests that we look at possible inclusions between the eight subsets associated to these two bases of V_d . It is immediate from the definition (14), for example, that $PF_1[K_i(x)] \subseteq PF_1 \left[(-1)^i \binom{x}{i} \right]$. On the other hand, from (14) we can easily compute that

$$K_2^{(2)}(x) = 1 - 2 \binom{x}{1} + 4 \binom{x}{2},$$

which shows that $PF[K_i^{(d)}(x)] \not\subseteq PF \left[(-1)^i \binom{x}{i} \right]$. There is, however, at least one non-trivial inclusion that does hold.

Theorem 3.4.3 *Let $d \in \mathbf{P}$. Then we have that $PF_2[K_i^{(d)}(x)] \subseteq PF_2 \left[(-1)^i \binom{x}{i} \right]$*

in V_d .

Proof: Let $P(x) \stackrel{\text{def}}{=} \sum_{j=0}^d a_j \binom{x}{j} = \sum_{i=0}^d k_i K_i^{(d)}(x) \in PF_2[K_i^{(d)}(x)]$. Then, from the definition (14) of $K_i^{(d)}(x)$, we deduce that

$$\sum_{j=0}^d a_j \binom{x}{j} = \sum_{i=0}^d k_i \sum_{j=0}^i (-2)^j \binom{d-j}{i-j} \binom{x}{j},$$

and therefore that

$$a_j = (-2)^j \sum_{i=j}^d k_i \binom{d-j}{i-j}, \quad (15)$$

for $j = 0, \dots, d$, or, equivalently, that

$$(-1)^j a_j = 2^j Q(d-j), \quad (16)$$

for $j = 0, \dots, d$, where

$$Q(x) \stackrel{\text{def}}{=} \sum_{i=0}^d k_{d-i} \binom{x}{i}. \quad (17)$$

Now, since $\{k_0, \dots, k_d\}$ is a PF_2 sequence by hypothesis, there follows that $\{k_d, \dots, k_0\}$ is also PF_2 . Therefore, from (17) and Theorem 2.5.7, we conclude that the sequence $\{Q(n)\}_{n \in \mathbb{N}}$ is PF_2 . Hence, in particular, the sequence $\{Q(d), \dots, Q(1), Q(0)\}$ is PF_2 and this, by (16), implies that the sequence $\{a_0, -a_1, a_2, \dots, (-1)^d a_d\}$ is PF_2 , as desired. $\square$

One well known property of the Krawtchouk polynomials is that they always have only real nonnegative zeros (see, e. g. [67], or [11] which also give many further properties of these zeros), or equivalently, that $K_j^{(d)}(x) \in PF[(-1)^i x^i]$ for $j = 0, \dots, d$. Now, it is easy to see that (14) implies that $PF_1[K_i^{(d)}(x)] \subseteq PF_1[(-1)^i x^i]$. This, again, suggests that $\{(-1)^i x^i\}_{i=0, \dots, d}$ might also be a good basis to consider for looking at inclusions between its associated sets and those of $\{K_i^{(d)}(x)\}_{i=0, \dots, d}$. I have not looked very much at this problem except in the case of the two associated PF -sets, for which I have the following conjecture.

Conjecture: $PF[K_i^{(d)}(x)] \subseteq PF[(-1)^i x^i]$.

In other words, the conjecture states that if a polynomial $\sum_{i=0}^d a_i x^i$ has only real nonpositive zeros then the polynomial $\sum_{i=0}^d a_i K_i^{(d)}(x)$ has only real

nonnegative zeros.

3.5 Three general problems

In this section we present a general problem that includes as special cases all the results that we have obtained so far. This is the problem of determining which inclusions hold between the eight sets associated to two arbitrary bases $\{p_i(x)\}_{i=0,\dots,d}$ and $\{q_i(x)\}_{i=0,\dots,d}$ of V_d . In section 2.3 we had remarked that, since there are infinitely many bases of V_d , the above problem is probably hopeless. Nonetheless, in this section we will present a reduction of this problem that, even though still probably impossible to solve, looks more tractable. This reduction is based on a simple and elementary observation which on the one hand allows us to see all the results that we have obtained so far as special cases of a very precise problem, and on the other hand allows us to deduce from each one of the results that we have obtained so far infinitely many others that, though equivalent, may sometimes look very different.

The key observation is the following. Suppose that $\mathcal{L} : V_d \rightarrow V_d$ is an invertible linear transformation. Then $\mathcal{L}$ sends a basis $\{p_i(x)\}_{i=0,\dots,d}$ of V_d into another basis $\{\mathcal{L}(p_i(x))\}_{i=0,\dots,d}$. Furthermore, it is immediate from our definitions in section 2.3 that $\mathcal{L}(PF[p_i(x)]) = PF[\mathcal{L}(p_i(x))]$ and analogously for the PF_2 , UNI and PF_1 sets of $\{p_i(x)\}_{i=0,\dots,d}$. Therefore, given two bases $\{p_i(x)\}_{i=0,\dots,d}$ and $\{q_i(x)\}_{i=0,\dots,d}$ of V_d , the inclusion $PF[p_i] \subseteq PF[q_i]$ holds if and only if the inclusion $PF[\mathcal{L}(p_i)] \subseteq PF[x^i]$ holds where $\mathcal{L}$ is the invertible linear transformation defined by $\mathcal{L}(q_i) \stackrel{\text{def}}{=} x^i$ for $i = 0, \dots, d$. So we see that the general problem of determining all pairs of bases $\{p_i\}_{i=0,\dots,d}$ and $\{q_i\}_{i=0,\dots,d}$ such that $PF[p_i] \subseteq PF[q_i]$ is actually equivalent to the problem of determining all bases $\{p_i\}_{i=0,\dots,d}$ such that $PF[(p_i)] \subseteq PF[x^i]$. This second problem looks easier also because, by

Theorem 2.3.1, we know that the polynomials in $PF[x^i]$ are exactly those that have only real zeros and nonnegative coefficients. For example, by Theorem 2.3.3 the linear map $\mathcal{L} : V_d \rightarrow V_d$ defined by $\mathcal{L}\left(\binom{x}{i}\right) \stackrel{\text{def}}{=} x^i, i = 0, \dots, d$, has the property that $\mathcal{L}\left(\binom{x+d-i}{d}\right) = (x+1)^i, i = 0, \dots, d$. Thus the inclusion $PF\left[\binom{x+d-i}{d}\right] \subseteq PF\left[\binom{x}{i}\right]$ is equivalent to the inclusion $PF[(x+1)^i] \subseteq PF[x^i]$ which we immediately recognize to be true. On the other hand, if we define $\mathcal{L}$ by $\mathcal{L}((x)_i) \stackrel{\text{def}}{=} x^i$ then the basis $\{\mathcal{L}(x^i)\}$ seems to have no particularly easy description. So the method has both advantages and disadvantages to it, depending on the particular situation at hand. However, from the abstract point of view, we now see that all the problems that we have considered so far concerning PF -sets are part of the following general and very fundamental problem: for all $d \geq 1$, classify all (real) $(d \times d)$ -matrices that send PF sequences into PF sequences.

Let's call a matrix having the above property a *PF-preserving* matrix, and define PF_2 , UNI or PF_1 -preserving matrices in the analogous way. We then have the following three fundamental problems.

Problem 1: *Classify all PF-preserving matrices.*

Problem 2: *Classify all PF_2 -preserving matrices.*

Problem 3: *Classify all UNI -preserving matrices.*

Let's now denote with $PF[M_d]$ the set of all PF -preserving matrices and define $PF_2[M_d]$ and $UNI[M_d]$ analogously. Then we have the following elementary but important result.

Proposition 3.5.1 *The sets $PF[M_d]$, $PF_2[M_d]$ and $UNI[M_d]$ are monoids (i. e. semigroups with identity) with respect to matrix multiplication. $\square$*

The preceding proposition immediately suggests some interesting questions. For example, are any of the above monoids free? Or finitely generated? Or simplicial? (in the sense of [63], p.226) Do they enjoy any special property? I have not investigated myself any of the above questions because they seem to

belong more to the theory of total positivity than to Combinatorics. Nonetheless, I will now briefly summarize what seems to be known about Problems 1, 2 and 3.

One immediate, elementary observation about them is that, since the unit coordinate vectors $e_i \stackrel{\text{def}}{=} (0, \dots, 0, \underbrace{1}_i, 0, \dots, 0)$, $i = 0, \dots, d$, are PF sequences a necessary condition for a matrix to be in $PF[M_d]$ (respectively, $PF_2[M_d]$, $UNI[M_d]$) is that each of its columns be a PF (respectively, PF_2 , unimodal and PF_1) sequence when read from top to bottom. Easy examples show, however, that these conditions are not in general sufficient. A second remark is that, from what we have seen in this and the preceding chapter we expect $PF[M_d]$ to be the largest set and $UNI[M_d]$ to be the smallest. So it is probably true that the problem to classify or determine in some implicit or explicit way the matrices in $UNI[M_d]$ is the easiest and the one concerning $PF[M_d]$ the most difficult.

Concerning now Problem 3, specifically, very little seems to be known about it. In fact, even in this work, we have not encountered so far even one matrix in $UNI[M_d]$ besides the identity. It is, however, clear that any matrix $A = (a_{ij})_{0 \leq i, j \leq d}$ with the property that $a_{i0} \leq a_{i1} \leq \dots \leq a_{ir} \geq a_{ir+1} \geq \dots \geq a_{id}$ for all $i = 0, \dots, d$ (for a fixed $0 \leq r \leq d$), sends any PF_1 sequence to a unimodal sequence and hence is in particular in $UNI[M_d]$. By the semigroup property of Proposition 3.5.1 these matrices already generate quite a large set of matrices that are in $UNI[M_d]$, however I have no idea on how one can prove that they generate all of $UNI[M_d]$ or if that is true.

Concerning Problem 2 we should say that, confirming the impression that $PF_2[M_d]$ is larger than $UNI[M_d]$, many matrices are known that belong to $PF_2[M_d]$. First, there are the matrices produced by Theorem 2.6.2 (to which all the matrices that we have seen so far to be in $PF_2[M_d]$, like F^{-1} , M , B and $B^\#$, belong). Then, since the componentwise product of two PF_2 sequences

is, clearly, still a PF_2 sequence, we have that $\text{diag}(c_0, c_1, \dots, c_d) \in PF_2[M_d]$ if $\{c_0, c_1, \dots, c_d\}$ is PF_2 . Apart from these examples we should also mention the ones given in [53] where a matrix in $PF_2[M_d]$ is obtained for each pair of PF_2 sequences with the property that their componentwise sum is also PF_2 .

For what concerns Problem 1 the situation seems extremely complicated. There are, of course, the matrices produced by Theorem 2.6.1 (i. e. , F^{-1} , C , S and $B^\#$) and the general class of matrices produced by Theorem 2.4.6. Because of the relationship between PF sequences and polynomial with only real zeros, however, there are countless other examples of matrices in $PF[M_d]$ that come from the many (often classical) results about polynomials with only real zeros, a subject that has been studied extensively especially at the beginning of this century and at the end of the last one.

Many of these results can be found in the famous and wonderful book [46], especially in part 5 of Volume II and the reader can easily translate them, if he wishes, into results about matrices that are in $PF[M_d]$. We give here only a few examples just to give to the reader a flavor of the general character and variety of these results. One example is the following generalization of Theorem 2.4.1, which is Problem 63 in Part V of Volume II of [46].

Theorem 3.5.2 *Let $\sum_{i=0}^d a_i x^i$ and $P(x)$ be two polynomials in $PF[x^i]$. Then the polynomial $\sum_{i=0}^n a_i P^{(i)}(x)$ is in $PF[x^i]$.*

The proof of Theorem 3.5.2 is a very easy generalization of that of Theorem 2.4.1 and we omit it. Theorem 3.5.2 clearly produces a matrix in $PF[M_d]$ for every PF sequence $\{a_0, a_1, \dots, a_d\}$. One other example is the following classical result that appears as Problem 156 in Part V of Volume II of [46] and that is originally due to Schur (see [55]). A proof of it can also be found in [37].

Theorem 3.5.3 *Let $f(x) \stackrel{\text{def}}{=} \sum_{i=0}^n a_i x^i$ and $g(x) \stackrel{\text{def}}{=} \sum_{i=0}^n b_i x^i$ be two polynomials of degrees m and n , respectively. Suppose that $f(x)$ and $g(x)$ have only real*

zeros and that the zeros of $g(x)$ are all of the same sign. Then the polynomial

$$\sum_{i=0}^k a_i b_i (i!) x^i,$$

where $k \stackrel{\text{def}}{=} \min(m, n)$, has only real zeros.

So Theorem 3.5.3 implies that $F \text{diag}(a_0, \dots, a_d) \in PF[M_d]$ (and hence, in particular, $\text{diag}(a_0, \dots, a_d) \in PF[M_d]$) whenever $\{a_0, \dots, a_d\}$ is a PF sequence. In all the above cases matrices in $PF[M_d]$ are produced from results in Analysis. It is interesting to note, however, that sometimes non-trivial analytic results can be proved using matrices that are easily seen to be in $PF[M_d]$. For example, by Proposition 2.2.3 we know that if a sequence $\{a_i\}_{i \in \mathbf{N}}$ is PF then also the sequence $\{a_{ki}\}_{i \in \mathbf{N}}$ is PF , for all $k \in \mathbf{P}$. This implies, in particular, that the matrices A_k defined, for $0 \leq i, j \leq d$, by

$$(A_k)_{i,j} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j = ki, \\ 0 & \text{otherwise.} \end{cases}$$

are in $PF[M_d]$ (note, by the way, that they are also easily seen to be in $PF_2[M_d]$ and $UNI[M_d]$). Using now Theorem 2.3.1 we see that what we have in fact proved is the following analytic result.

Theorem 3.5.4 *Let $\sum_{i=0}^d a_i x^i$ be a polynomial having only real nonpositive zeros. Then, for each $k \in \mathbf{P}$, the polynomial $\sum_{i=0}^{\lfloor \frac{d}{k} \rfloor} a_{ki} x^i$ has only real nonpositive zeros. $\square$*

Theorem 3.5.4 can be found in Problem 171.4 of Part V of Volume II of [46] and the fact that the proof given there is longer and more difficult than the one that we have given illustrates the power of Theorem 2.3.1. The examples that we have shown so far (that are only an infinitesimal sample of the immense classical literature existing on the subject) will certainly have convinced the reader of the tremendous complexity of Problem 1. On the other hand, it would certainly be a gigantic achievement in Mathematics if one could find a unified proof of this incredible variety of results by finding a (possible infinite) set of generators

for the monoid $PF[M_d]$ and, consequently, a “decision procedure” for deciding if a given matrix is in the monoid or not.

Chapter 4

Polynomials in $PF\left[\binom{\mathbf{x} + \mathbf{d} - \mathbf{i}}{\mathbf{d}}\right]$

4.1 Introduction

In the last chapter we have seen different ways in which we can deepen our understanding of the 24 subsets and of their interrelationships. However, we have just touched upon these ways, because of the more tenuous link that they seem to have to combinatorial problems, and in particular to the poset conjecture. In this chapter, instead, we will examine a direction of investigation that is very much connected to combinatorial problems, and in particular to the poset conjecture. For this reason we will pursue this direction much farther than we have pursued the others. As we will see later, the results that we will obtain in this way will prove to be the most powerful and widely applicable of all the results that we have obtained so far.

In this chapter we will concentrate on one of the 24 subsets and we will try to describe it as explicitly as possible. The definitions of the 24 subsets are quite explicit if one knows already the coefficients of the polynomial with respect to the given basis. However, to find these coefficients may not always be easy and can sometimes be very difficult. So in many cases one would like to have some alternative description of the given set in terms of other properties of the polynomial other than its coefficients. One example of such a characterization is Theorem 2.3.1 which, by Theorem 2.6.1, also tells us something about the sets $PF\left[\left(\binom{x}{i}\right)\right]$, $PF[\langle x \rangle_i]$ and $PF[(x)_i]$. However we still don't know much

about $PF\left[\binom{x+d-i}{d}\right]$ and $PF\left[\binom{x}{i}\right]$ because they neither contain nor are contained in $PF[x^i]$. On the other hand, Theorem 3.2.1 tells us that $PF\left[\binom{x+d-i}{d}\right]$ and $PF\left[\binom{x}{i}\right]$ are very closely related and gives us a very efficient method of deciding whether a given polynomial is in one of them if we know whether it is or not in the other. Since the poset conjecture asserts exactly that a vast class of polynomials (all order polynomials of labeled posets) is contained in $PF\left[\binom{x+d-i}{d}\right]$ and since (as we remarked above) we still don't know much about it, it seems natural now to concentrate our efforts on the set $PF\left[\binom{x+d-i}{d}\right]$. In the next section we derive some elementary properties of $PF\left[\binom{x+d-i}{d}\right]$. In section 4.3 we will prove a fundamental result about it (Theorem 4.3.4) which is one of the two main results of this chapter. In the fourth section we will derive many consequences of it that will "elucidate" more its meaning and that will describe quite explicitly some classes of polynomials that are contained in $PF\left[\binom{x+d-i}{d}\right]$. In later chapters we will apply these results to many combinatorial situations and we will see that many combinatorially defined polynomials, including the order polynomial of many infinite families of posets, are in $PF\left[\binom{x+d-i}{d}\right]$, for some d . In the fifth section we introduce and discuss in detail the Fundamental Representation Theorem for PF sequences from the Theory of Total Positivity. This remarkable theorem will be used in an essential way in section 4.6 to derive the second main result of the chapter. This result gives a characterization of $PF\left[\binom{x+d-i}{d}\right]$ that does not depend on the knowledge of the coefficients of a polynomial with respect to the basis $\left\{\binom{x+d-i}{d}\right\}_{i=0,\dots,d}$ and is, for this reason, extremely useful. Finally, in the last section, we examine products of polynomials that are in $PF\left[\binom{x+d-i}{d}\right]$.

4.2 Elementary properties of $PF\left[\binom{x+d-i}{d}\right]$

In this section we derive some elementary but fundamental properties of $PF\left[\binom{x+d-i}{d}\right]$ that will be used often in the rest of the chapter and throughout the rest of this work. We begin with an “inversion formula” that follows easily from Theorem 2.3.3.

Proposition 4.2.1 *Let $P(x) = \sum_{i=0}^d w_i \binom{x+d-i}{d}$ be a polynomial of degree d . Then*

$$w_i = \sum_{j=0}^i (-1)^j \binom{d+1}{j} P(i-j), \quad (1)$$

for $i = 0, \dots, d$.

Proof: By Theorem 2.3.3 our hypothesis implies that

$$(1-z)^{d+1} \sum_{n \geq 0} P(n) z^n = \sum_{i=0}^d w_i z^i. \quad (2)$$

Equating now the coefficients of z^i in (2) for $i = 0, \dots, d$ yields (1). $\square$

The following result can be considered a sort of “reciprocity theorem” for $PF\left[\binom{x+d-i}{d}\right]$.

Proposition 4.2.2 *Let $P(x)$ be a polynomial of degree d . Then the following are equivalent:*

- (i) $P(x) \in PF\left[\binom{x+d-i}{d}\right]$;
- (ii) $(-1)^d P(-x-1) \in PF\left[\binom{x+d-i}{d}\right]$.

Proof: Suppose $P(x) = \sum_{i=0}^d w_i \binom{x+d-i}{d}$, then

$$\begin{aligned} (-1)^d P(-x-1) &= (-1)^d \sum_{i=0}^d w_i \binom{-x-1+d-i}{d} \\ &= \sum_{i=0}^d w_i \binom{x+i}{d}, \end{aligned}$$

the thesis now follows from the fact that a sequence $\{w_0, w_1, \dots, w_d\}$ is PF if and only if $\{w_d, \dots, w_1, w_0\}$ is PF . $\square$

The next proposition gives the behaviour of $PF \left[\binom{x+d-i}{d} \right]$ under a unit translation and under the difference operator Δ . Recall that $\Delta : V_d \rightarrow V_{d-1}$ is defined by $(\Delta P)(x) \stackrel{\text{def}}{=} P(x+1) - P(x)$ for $P \in V_d$.

Proposition 4.2.3 *Let $P(x)$ be a real polynomial of degree d such that $P(0) = 0$. Then the following are equivalent:*

- (i) $P(x) \in PF \left[\binom{x+d-i}{d} \right];$
- (ii) $P(x+1) \in PF \left[\binom{x+d-i}{d} \right];$
- (iii) $(\Delta P)(x) \in PF \left[\binom{x+d-1-i}{d-1} \right].$

Proof: Suppose $P(x) = \sum_{i=0}^d w_i \binom{x+d-i}{d}$. Then we can easily compute that

$$P(x+1) = \sum_{i=0}^{d-1} w_{i+1} \binom{x+d-i}{d},$$

and

$$(\Delta P)(x) = \sum_{i=0}^{d-1} w_{i+1} \binom{x+d-1-i}{d-1},$$

and the thesis follows. $\square$

The reader should note that the hypothesis that $P(0) = 0$ in Proposition 4.2.3 is essential, as easy examples show.

Our last result in this section is just a simple observation, but one that should be kept in mind in the reading of this chapter.

Proposition 4.2.4 *Let $P(x) \in PF_1 \left[\binom{x+d-i}{d} \right]$ be a polynomial of degree $\leq d-1$. Then $P(x) \equiv 0$.*

Proof: Suppose $P(x) = \sum_{i=0}^d w_i \binom{x+d-i}{d}$. Then the leading coefficient of $P(x)$ is $\frac{1}{d!}(w_1 + \dots + w_d)$. Therefore, since $\deg(P) \leq d-1$, we have that $w_1 + \dots + w_d = 0$. Since the w_i 's are nonnegative this implies that $w_1 = \dots = w_d = 0$, as desired. $\square$

Proposition 4.2.4 is the reason why, in some of the results in this chapter we assume that the degree of $P(x)$ is *exactly* d . The reader should note that the preceding proposition implies, in particular, a property that none of the

others five PF sets have, namely, that $PF \left[\binom{x+d-i}{d} \right] \not\subseteq PF \left[\binom{x+d+1-i}{d+1} \right]$. In fact, Proposition 4.2.4 shows that a non-zero polynomial $P(x) \in PF \left[\binom{x+d-i}{d} \right]$ is never in $PF \left[\binom{x+d+1-i}{d+1} \right]$.

4.3 A Fundamental Theorem

In this section we derive the first of the two fundamental theorems of the chapter about $PF \left[\binom{x+d-i}{d} \right]$. Before we can state this result, however, we need some new pieces of notation and several intermediate results that are also interesting in their own right. We begin by introducing a notation for some particular subspaces of V_d that will be encountered often in what follows.

If l , t and d are integers such that $0 \leq l \leq t \leq d$ then we will denote by $W_l^t \left[\binom{x+d-i}{d} \right]$ the linear subspace of V_d spanned by $\left\{ \binom{x+d-i}{d} : l \leq i \leq t \right\}$. Our first result is the key to the fundamental theorem and uses Theorem 2.4.6 which was, in fact, discovered, in my attempts to prove the result that follows.

Theorem 4.3.1 *Let $P(x) \in PF \left[\binom{x+d-i}{d} \right] \cap W_l^t \left[\binom{x+d-i}{d} \right]$, where $0 \leq l \leq t \leq d$. Then*

$$(x-a)P(x) \in PF \left[\binom{x+d+1-i}{d+1} \right],$$

for all $a \in [t-d-1, l]$.

Proof: Let $P(x) \stackrel{\text{def}}{=} \sum_{i=l}^t w_i \binom{x+d-i}{d}$ and let $(x-a)P(x) \stackrel{\text{def}}{=} \sum_{i=0}^{d+1} u_i \binom{x+d+1-i}{d+1}$. Then, using the fact that

$$x \binom{x+d-i}{d} = i \binom{x+d+1-i}{d+1} + (d+1-i) \binom{x+d-i}{d+1}$$

we easily see that

$$(x-a) \sum_{i=l}^t w_i \binom{x+d-i}{d} = \sum_{i=0}^{d+1} u_i \binom{x+d+1-i}{d+1}$$

is equivalent to

$$u_i = \begin{cases} (i-a)w_i + (d-i+a+2)w_{i-1}, & \text{if } l \leq i \leq t+1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $t-d-1 \leq a \leq l$ this implies, in particular, that $u_i \geq 0$ for all $0 \leq i \leq d+1$. Now, letting $\alpha \stackrel{\text{def}}{=} 1$, $\beta \stackrel{\text{def}}{=} a$ and $\gamma \stackrel{\text{def}}{=} d+1+a$, we see that all the hypotheses of Theorem 2.4.6 are satisfied and this implies that the sequence $\{u_l, \dots, u_{t+1}\}$ is PF , as desired. $\square$

The proof of the preceding theorem also yields the following elementary but important result.

Lemma 4.3.2 *Let $A(x) \in W_l^t \left[\binom{x+d-i}{d} \right]$, where $0 \leq l \leq t \leq d$, and let $a \in \mathbf{R}$. Then*

$$A(x)(x-a) \in \begin{cases} W_l^t \left[\binom{x+d+1-i}{d+1} \right], & \text{if } a = t-d-1, \\ W_{l+1}^{t+1} \left[\binom{x+d+1-i}{d+1} \right], & \text{if } a = l, \\ W_l^{t+1} \left[\binom{x+d+1-i}{d+1} \right], & \text{otherwise.} \end{cases} \square$$

Using this lemma we can easily give an alternative description of the subspaces $W_l^t \left[\binom{x+d-i}{d} \right]$ which is sometimes more convenient to use than the original definition and that will be used in chapter 6.

Proposition 4.3.3 *Let l, t, d be integers such that $0 \leq l \leq t \leq d$ and let $A(x) \in V_d$. Then $A(x) \in W_l^t \left[\binom{x+d-i}{d} \right]$ if and only if $A(x) = 0$ for all $x \in [t-d, l-1] \cap \mathbf{Z}$.*

Proof: Suppose that $A(x) \in V_d$ and that $A(x) = 0$ for all $x \in [t-d, l-1] \cap \mathbf{Z}$. Then there exists $B(x) \in W_0^{t-l} \left[\binom{x+t-l-i}{t-l} \right]$ such that

$$A(x) = B(x) \prod_{i=t-d}^{l-1} (x-i).$$

Using now Lemma 4.3.2 repeatedly we conclude that $A(x) \in W_l^t \left[\binom{x+d-i}{d} \right]$, as desired. The converse statement is immediate from the definition of $W_l^t \left[\binom{x+d-i}{d} \right]$. $\square$

Using Theorem 4.3.1 and Lemma 4.3.2 we can now prove the fundamental result of this section.

Theorem 4.3.4 *Let $A(x)$ and $B(x)$ be two polynomials of degrees a and b , respectively. Suppose that $A(x) \in PF \left[\binom{x+a-i}{a} \right] \cap W_l^t \left[\binom{x+a-i}{a} \right]$, where $0 \leq l \leq t \leq a$, that $B(x)$ has only real zeros and that $B(x) = 0$ for all $x \in ([\lambda(B), t-a-1] \cup [l, \Lambda(B)]) \cap \mathbf{Z}$. Then $A(x)B(x) \in PF \left[\binom{x+a+b-i}{a+b} \right]$.*

Proof: We may clearly assume that $B(x)$ is monic. Let $h \stackrel{\text{def}}{=} |[l, \Lambda(B)] \cap \mathbf{Z}|$, $k \stackrel{\text{def}}{=} |[\lambda(B), t-a-1] \cap \mathbf{Z}|$. Let $\alpha_1, \dots, \alpha_b$ be the zeros of $B(x)$ and assume that they are indexed in such a way such that

$$\alpha_r = \begin{cases} l+r-1 & \text{if } 1 \leq r \leq h, \\ t-a+h-r & \text{if } h+1 \leq r \leq h+k. \end{cases}$$

Then, since $A(x) \in W_l^t \left[\binom{x+a-i}{a} \right]$, it follows from Lemma 4.3.2 that

$$A(x) \prod_{i=1}^r (x - \alpha_i) \in \begin{cases} W_{l+r}^{t+r} \left[\binom{x+a+r-i}{a+r} \right] & \text{if } 0 \leq r \leq h \\ W_{l+h}^{t+h} \left[\binom{x+a+r-i}{a+r} \right] & \text{if } h \leq r \leq h+k \\ W_{l+h}^{t+r-k} \left[\binom{x+a+r-i}{a+r} \right] & \text{if } h+k \leq r \leq b. \end{cases}$$

But

$$\alpha_r \in \begin{cases} [t-a-1, l+r-1] & \text{if } 1 \leq r \leq h, \\ [t-a+h-r, l+h] & \text{if } h \leq r \leq h+k, \\ [t-k-a-1, l+h] & \text{if } h+k \leq r \leq b. \end{cases}$$

Hence, by Theorem 4.3.1, we have that, for $r = 1, \dots, b$, $A(x) \prod_{i=1}^r (x - \alpha_i) \in PF \left[\binom{x+a+r-i}{a+r} \right]$ if $A(x) \prod_{i=1}^{r-1} (x - \alpha_i) \in PF \left[\binom{x+a+r-1-i}{a+r-1} \right]$. Since $A(x) \in PF \left[\binom{x+a-i}{a} \right]$ by hypothesis we can conclude that $A(x) \prod_{i=1}^b (x - \alpha_i) \in PF \left[\binom{x+a+b-i}{a+b} \right]$, as desired. $\square$

It may at first sight seem that Theorem 4.3.4 should be of limited usefulness because it assumes that we already have a polynomial $A(x)$ which is in $PF \left[\binom{x+a-i}{a} \right]$. However, the real strength of the theorem lies in the fact that it

produces an *infinite number* of polynomials in $PF \left[\binom{x+b-i}{b} \right]$ starting with just *one* polynomial in $PF \left[\binom{x+a-i}{a} \right]$. In the next section we will specialize Theorem 4.3.4 in many different ways and we will see, for example, how the special case $A(x) \equiv 1$ already produces interesting and non trivial results.

However, even though the great majority of combinatorial applications of Theorem 4.3.4 will come from its specializations that will be presented in the following section (especially Theorem 4.4.1), we will still need, in some circumstances, the full generality of Theorem 4.3.4. Furthermore, since some of these specializations have such a wide range of applicability, we think that there may be other “intelligent” specializations of Theorem 4.3.4, also having wide applicability. It is for these reasons that I have chosen to state Theorem 4.3.4 in the most general form.

I conclude this section with some remarks about possible directions of further research along the lines of the preceding results. First it should be noted that, by Theorem 3.2.1, Theorem 4.3.4 also applies, with minor obvious modifications, to $PF \left[\binom{x}{i} \right]$. The remaining three PF -sets are all contained in or contain $PF[x^i]$ and we therefore have good information about them. However, it would be interesting to try to analyze differences such as $PF[(x)_i] \setminus PF[x^i]$ or $PF[x^i] \setminus PF[(x)_i]$. One other aspect which I believe would be worthwhile to investigate is to try to obtain analogs of the results in this section for the PF_2 -sets of the six fundamental bases, in particular for $PF_2 \left[\binom{x+d-i}{d} \right]$, which, again because of the poset conjecture, would be particularly interesting. I personally feel that this last one is the most promising line of investigation. Of course, one can also try to obtain analogs for the UNI -sets of the six fundamental bases but, since the unimodal property is so badly behaved even in the simple cases considered in chapter 2, it seems to me that this would probably be a very difficult task. Finally, it should be noted that Theorem 4.3.4 is based on Theorem 4.3.1 which in turn depends on Theorem 2.4.6 which, as one can easily see, fails

for PF_2 or unimodal sequences.

4.4 Consequences of the Fundamental Theorem

In this section we derive some general consequences of Theorem 4.3.4. These specializations will not only make clear the strength of Theorem 4.3.4 and exhibit explicit classes of polynomials that are in $PF \left[\binom{x+d-i}{d} \right]$ but will also yield some remarkable properties of the set $PF \left[\binom{x+d-i}{d} \right]$. Furthermore, many of these specializations will be used in the combinatorial applications of Theorem 4.3.4 that will be given in chapter 5.

Our first result is obtained by taking $A(x) \equiv 1$ in Theorem 4.3.4.

Theorem 4.4.1 *Let $B(x)$ be a polynomial of degree b . Suppose that $B(x)$ has only real zeros and that $B(x) = 0$ for all $x \in ([\lambda(B), -1] \cup [0, \Lambda(B)]) \cap \mathbf{Z}$. Then $B(x) \in PF \left[\binom{x+b-i}{b} \right]$. $\square$*

The reader should be careful not to confuse the condition on the zeros of $B(x)$ stated in the theorem with the (seemingly equivalent) condition that “ $B(x) = 0$ for all $x \in [\lambda(B), \Lambda(B)] \cap \mathbf{Z}$ ”. The two are, in fact, equivalent, if and only if $\lambda(B) \leq 0$ and $\Lambda(B) \geq -1$. Theorem 4.4.1 not only describes explicitly a large class of polynomials that are in $PF \left[\binom{x+d-i}{d} \right]$ but also has a wide applicability in Combinatorics as we will see in chapter 5.

Our second result gives in precise form a curious property of $PF \left[\binom{x+d-i}{d} \right]$ that follows easily from the preceding theorem; namely, that any polynomial with only real zeros is a divisor of some polynomial in $PF \left[\binom{x+d-i}{d} \right]$.

Proposition 4.4.2 *Let $A(x)$ be a polynomial of degree a with only real zeros and let $M \stackrel{\text{def}}{=} \max \{ \lfloor \Lambda(A) \rfloor + 1, 0 \}$, $m \stackrel{\text{def}}{=} \min \{ \lceil \lambda(A) \rceil, 0 \}$. Then $A(x)(x-m)_{M-m} \in PF \left[\binom{x+b-i}{b} \right]$ where $b \stackrel{\text{def}}{=} a + M - m$.*

Proof: Let $B(x) \stackrel{\text{def}}{=} A(x)(x-m)_{M-m}$. Then from the definitions of m and M

we have that

$$\Lambda(B) = \max \{ \Lambda(A), -1 \},$$

$$\lambda(B) = \min \{ \lambda(A), 0 \}.$$

Hence $m = \lceil \lambda(B) \rceil$ and $M = \lfloor \Lambda(B) \rfloor + 1$. Therefore

$$[\lambda(B), -1] \cap \mathbf{Z} = [m, -1] \cap \mathbf{Z},$$

$$[0, \Lambda(B)] \cap \mathbf{Z} = [0, M - 1] \cap \mathbf{Z},$$

and the thesis follows from Theorem 4.4.1. $\square$

Our next result is a consequence of Theorem 4.3.4 and of the characterization given in Proposition 4.3.3.

Theorem 4.4.3 *Let $A(x) \in PF \left[\binom{x+a-i}{a} \right]$ and let $s \stackrel{\text{def}}{=} \max \{ x \in \mathbf{Z} : A(x) \neq 0, x \leq -1 \}$, $r \stackrel{\text{def}}{=} \min \{ x \in \mathbf{Z} : A(x) \neq 0, x \geq 0 \}$. Let $\alpha_1, \dots, \alpha_b \in [s, r]$, then $A(x) \prod_{i=1}^b (x - \alpha_i) \in PF \left[\binom{x+a+b-i}{a+b} \right]$.*

Proof: Let $B(x) \stackrel{\text{def}}{=} \prod_{i=1}^b (x - \alpha_i)$. Then from our hypotheses there follows that $\lambda(B) \geq s$ (and $\lambda(B) = s$ if and only if $\alpha_j = s$ for some $1 \leq j \leq b$) and $\Lambda(B) \leq r$ (and $\Lambda(B) = r$ if and only if $\alpha_i = r$ for some $1 \leq i \leq b$). Now, by Proposition 4.3.3 and the definitions of s and r there follows that $A(x) \in W_r^{a+s+1} \left[\binom{x+a-i}{a} \right]$, but

$$[\lambda(B), s] = \begin{cases} \{s\} & \text{if } \alpha_j = s \text{ for some } 1 \leq j \leq b, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$[r, \Lambda(B)] = \begin{cases} \{r\} & \text{if } \alpha_i = r \text{ for some } 1 \leq i \leq b, \\ \emptyset & \text{otherwise.} \end{cases}$$

So in any case $([\lambda(B), s] \cup [r, \Lambda(B)]) \cap \mathbf{Z} \subseteq \{\alpha_1, \dots, \alpha_b\}$ and the thesis follows from Theorem 4.3.4. $\square$

Our last result in this section is obtained by taking $A(x) \equiv 1$ in the last theorem and also follows immediately from Theorem 4.4.1. It is nonetheless worthy of mention for its simplicity, and wide range of applicability.

Theorem 4.4.4 *Let $B(x)$ be a polynomial of degree b . Suppose that all the zeros of $B(x)$ lie in the interval $[-1, 0]$. Then $B(x) \in PF \left[\binom{x+b-i}{b} \right]$. $\square$*

4.5 The characterization of PF-sequences

We have seen in section 2.2, the basic connection (Theorem 2.2.4) between the theory of total positivity and polynomials with only real zeros. Theorem 2.2.4 can be seen as a characterization result of a finite PF sequence in terms of its generating function. In this section we will present what is probably the most important result in the theory of PF sequences. This result gives a characterization of infinite PF sequences in terms of their generating functions and reduces to Theorem 2.2.4 in the case of finite PF sequences.

Before we can state this result we need the following preliminary observation, which also appears in [33], p. 394.

Proposition 4.5.1 *Let $\{a_n\}_{n \in \mathbb{N}}$ be a PF_2 sequence. Then the power series $\sum_{n \geq 0} a_n z^n$ has a positive radius of convergence.*

Proof: By Proposition 2.5.1 the sequence $\{a_n\}_{n \in \mathbb{N}}$ has no internal zeros so that there is some $N \in \mathbb{N}$ such that either $a_n = 0$ for all $n \geq N$ (and the proposition is trivial in this case) or $a_n > 0$ for all $n \geq N$. In the latter case the sequence $\left\{ \frac{a_{n+1}}{a_n} \right\}_{n \geq N}$ is (again by Proposition 2.5.1) a non-increasing sequence of positive real numbers and, therefore, has a limit $\beta \geq 0$. As it is well known from elementary calculus (see, e. g. [4]) this implies that the power series $\sum_{n \geq 0} a_n z^n$ has radius of convergence $\frac{1}{\beta}$ (where, of course, $\frac{1}{\beta} \stackrel{\text{def}}{=} \infty$ if $\beta = 0$). $\square$

So the preceding proposition tells us that, given a PF_2 sequence $\{a_n\}_{n \in \mathbb{N}}$, the power series $\sum_{n \geq 0} a_n z^n$ is a well defined analytic function of the complex variable z in an open disk centered at the origin. We can now state the Fundamental Representation Theorem for PF sequences.

Theorem 4.5.2 *Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers with $a_0 = 1$ and let $f(z) \stackrel{\text{def}}{=} \sum_{n \geq 0} a_n z^n$. Then $\{a_n\}_{n \in \mathbb{N}}$ is a *PF* sequence if and only if*

$$f(z) = e^{\gamma z} \frac{\prod_{i \geq 1} (1 + \alpha_i z)}{\prod_{i \geq 1} (1 - \beta_i z)}, \quad (3)$$

in some open disk centered at the origin, where $\gamma \geq 0$, $\alpha_i, \beta_i \geq 0$ for all $i \geq 1$ and $\sum_{i \geq 1} (\alpha_i + \beta_i) < \infty$.

Before proceeding further we should make some observations about the statement of Theorem 4.5.2 in order to make it absolutely clear. First, it should be noted that, under the conditions that $\alpha_i, \beta_i \geq 0$ for all $i \geq 1$ and $\sum_{i \geq 1} (\alpha_i + \beta_i) < \infty$, the two infinite products $\prod_{i \geq 1} (1 + \alpha_i z)$ and $\prod_{i \geq 1} (1 - \beta_i z)$ are convergent in the complex plane (see, e. g. ,[13], chap.7, Thm.5.9) and the function on the RHS of (3) is a well defined analytic function in some open disk centered at the origin. Theorem 4.5.2 states that, under the additional condition that $\gamma \geq 0$, the coefficients of the Taylor expansion around $z = 0$ of the analytic function on the RHS of (3) form a *PF* sequence and that any *PF* sequence (normalized so that $a_0 = 1$) can be obtained in this way. Let us here also remark that equation (3) is *not* an equality between two formal power series but is an equality between two analytic functions of one complex variable.

Theorem 4.5.2 was conjectured by Schoenberg in 1951 and first proved by Edrei in 1952 (see, [54], [17], and [18]). The proof of Theorem 4.5.2 is long and relies on deep results from the Nevańlinna theory of meromorphic functions (see, e. g. ,[43] or [44]). No proof of it seems to be known that avoids these deep analytic results. Theorem 4.5.2 appears in [33], p. 412, Theorem 5.3, and additional references concerning it can be found there on p. 453. There is no reason for us to give here a proof of Theorem 4.5.2 since a very lucid exposition appears in [33] pp. 406-412. However we should remark that one implication of

Theorem 4.5.2 is almost immediate. Suppose that $\gamma \geq 0$, $\alpha_i, \beta_i \geq 0$ for all $i \geq 1$ and $\sum_{i \geq 1} (\alpha_i + \beta_i) < \infty$, it then follows easily from the definitions that the two formal power series $(1 + \alpha_i z)$ and $\frac{1}{1 - \beta_i z} = \sum_{n \geq 0} (\beta_i)^n z^n$ are *PF*. By Theorem 2.2.2 any finite product of them $\prod_{i \geq 1}^N \left(\frac{1 + \alpha_i z}{1 - \beta_i z} \right)$ is also a *PF* formal power series.

Now, if a family of sequences $\{a_n(\epsilon)\}_{n \in \mathbb{N}}$, dependent on a parameter $\epsilon \geq 0$, are *PF* for all $\epsilon > 0$ and $a_n(\epsilon) \rightarrow a_n$ for all $n \in \mathbb{N}$ as $\epsilon \rightarrow 0$, then clearly the sequence $\{a_n\}_{n \in \mathbb{N}}$ is also *PF*. This implies that the Taylor expansions around $z = 0$ of the functions $e^{\gamma z} = \lim_{n \rightarrow +\infty} \left(1 + \frac{\gamma z}{n}\right)^n$, $\prod_{i=1}^{\infty} (1 + \alpha_i z)$ and $\prod_{i=1}^{\infty} (1 - \beta_i z)^{-1}$, are also *PF* formal power series. Therefore, by Theorem 2.2.2, the Taylor expansion of $e^{\gamma z} \frac{\prod_{i \geq 1} (1 + \alpha_i z)}{\prod_{i \geq 1} (1 - \beta_i z)}$ is a *PF* formal power series, as desired.

From the point of view of Combinatorics, the most serious drawback of Theorem 4.5.2 is that it concerns analytic functions and not formal power series. Our first step, therefore, will be to deduce a “formal version” of Theorem 4.5.2 which will be the one that we will actually use in all the applications that follow. We will say that a formal power series $\sum_{n \geq 0} a_n z^n$ is *rational* if there exist two polynomials $W(z), V(z)$ with $V(0) \neq 0$ (i. e. $V(z)$ is invertible as a formal power series) such that $\sum_{n \geq 0} a_n z^n = \frac{W(z)}{V(z)}$ as formal power series. We refer the reader to [63], chapter IV, for further information about rational formal power series. The next theorem characterizes in a purely algebraic way those rational formal power series that are *PF*.

Theorem 4.5.3 *Let $\sum_{n \geq 0} a_n z^n \in \mathbf{R}[[z]]$ be a rational formal power series and let $W(z), V(z)$ be two relatively prime polynomials such that $\sum_{n \geq 0} a_n z^n = \frac{W(z)}{V(z)}$. Then $\{a_n\}_{n \in \mathbb{N}}$ is a *PF* sequence if and only if $W(z)$ has only real nonpositive zeros, $V(z)$ has only real positive zeros and $\{a_n\}_{n \in \mathbb{N}}$ is a PF_1 sequence.*

Proof: Let $t \stackrel{\text{def}}{=} \min \{n \geq 0 : a_n \neq 0\} = \max \{n \geq 0 : z^n | W(z)\}$ and let $W_1(z) z^t = W(z)$. Suppose that $\{a_n\}_{n \geq 0}$ is a *PF* sequence. Then $\left\{ \frac{a_{n+t}}{a_t} \right\}_{n \geq 0}$ is also a *PF* sequence and by Theorem 4.5.2 there exist $\gamma \geq 0$ and $\alpha_i, \beta_i \geq 0$ for

$i \geq 1$, with $\sum_{i \geq 1} (\alpha_i + \beta_i) < \infty$, such that

$$\sum_{n \geq 0} a_{n+t} z^n = a_t e^{\gamma z} \frac{\prod_{i \geq 1} (1 + \alpha_i z)}{\prod_{i \geq 1} (1 - \beta_i z)},$$

for all z in an open disk centered at the origin. On the other hand, by hypothesis, we also have that

$$\sum_{n \geq 0} a_{n+t} z^n = \frac{W_1(z)}{V(z)},$$

for all z in an open disk centered at the origin. Therefore, for all z in an open neighborhood of the origin we have

$$W_1(z) \prod_{i \geq 1} (1 - \beta_i z) = a_t V(z) e^{\gamma z} \prod_{i \geq 1} (1 + \alpha_i z). \quad (4)$$

Since both functions, on the left and right hand side of (4), are entire, it follows that (4) holds for all $z \in \mathbf{C}$. Let now $z_0 \in \mathbf{C}$ be a zero of $W_1(z)$. Then $V(z_0) \neq 0$ and from (4) we have that

$$\prod_{i \geq 1} (1 + \alpha_i z_0) = 0.$$

It then follows from standard results in the theory of one complex variable (see, e. g. [13], p. 167, Theorem 5.9) that $z_0 = -\frac{1}{\alpha_i}$ for some i . Hence $z_0 \in \mathbf{R}$ and $z_0 < 0$. Analogously, if $z_1 \in \mathbf{C}$ is a zero of $V(z)$ then $W_1(z_1) \neq 0$ and from (4) we have that $\prod_{i \geq 1} (1 - \beta_i z_1) = 0$ which implies $z_1 = \frac{1}{\beta_i}$ for some i . Therefore $z_1 \in \mathbf{R}$ and $z_1 > 0$, as desired.

Conversely, suppose $W(z)$ has only nonpositive real zeros and $V(z)$ has only positive real zeros. Let $r = \deg(V(z))$, $s = \deg(W(z))$. Then $W_1(z) = w \prod_{i=1}^{s-t} (z + \delta_i)$ and $V(z) = v \prod_{i=1}^r (z - \gamma_i)$ where $w, v \in \mathbf{R} \setminus \{0\}$ and $\gamma_i, \delta_i > 0$ for all i . Note that $W_1(0) = w \prod_{i=1}^{s-t} \delta_i$, $V(0) = v \prod_{i=1}^r (-\gamma_i)$ and hence $\frac{w \prod_{i=1}^{s-t} \delta_i}{v \prod_{i=1}^r (-\gamma_i)} = \frac{W_1(0)}{V(0)} = a_t$. Therefore we have

$$\sum_{n \geq 0} a_{n+t} z^n = \frac{W_1(z)}{V(z)} = a_t \frac{\prod_{i=1}^{s-t} \left(1 + \frac{z}{\delta_i}\right)}{\prod_{i=1}^r \left(1 - \frac{z}{\gamma_i}\right)}.$$

By Theorem 4.5.2 this implies that $\left\{\frac{a_{n+i}}{a_i}\right\}_{n \geq 0}$ is a PF sequence and hence that $\{a_n\}_{n \geq 0}$ is a PF sequence. This concludes the proof. $\square$

In the special case that the sequence $\{a_n\}_{n \in \mathbf{N}}$ is eventually vanishing Theorem 4.5.3 reduces to the following result which implies Theorem 2.2.4 as a special case.

Theorem 4.5.4 *Let $W(z)$ be a polynomial. Then $W(z) \in PF[z^i]$ if and only if $W(z)$ has only real nonpositive zeros and $W(z) \in PF_1[z^i]$. $\square$*

We conclude this section by giving some consequences of Theorem 4.5.3 and Proposition 2.2.3 that, even though they won't be needed in this chapter, will be used in chapter 6 and are also of interest by themselves.

We first need an elementary observation about rational formal power series.

Proposition 4.5.5 *Let $\{a_n\}_{n \geq 0}$ be a sequence of complex numbers such that $\sum_{n \geq 0} a_n z^n$ is a rational formal power series. Then, for each $k \in \mathbf{P}$, $\sum_{n \geq 0} a_{n+k} z^n$ is also rational.*

Proof: It is well known, (see, e. g. [63] Theorem 4.1.1 and Proposition 4.2.2) that $\sum_{n \geq 0} a_n z^n$ is a rational formal power series if and only if there exist polynomials $P_1, \dots, P_r \in \mathbf{C}[z]$ and constants $\gamma_1, \dots, \gamma_r \in \mathbf{C}$ such that

$$a_n = \sum_{i=1}^r P_i(n) \gamma_i^n, \quad (5)$$

for all n sufficiently large. The thesis now follows by substituting $n+k$ for n in (5). $\square$

We can now prove the following result.

Theorem 4.5.6 *Let $\sum_{n \geq 0} a_n z^n \in \mathbf{R}[[z]]$ be a formal power series such that $\sum_{n \geq 0} a_n z^n = \frac{W(z)}{V(z)}$ for some polynomials $W(z), V(z)$. Suppose that $W(z)$ has only real nonpositive zeros and $V(z)$ has only real positive zeros. Then, for each $k \in \mathbf{P}$, there exist polynomials $W_k(z), V_k(z)$ such that*

$$\sum_{n \geq 0} a_{n+k} z^n = \frac{W_k(z)}{V_k(z)}, \quad (6)$$

$W_k(z)$ has only real nonpositive zeros and $V_k(z)$ has only real positive zeros.

Proof: It follows from our hypotheses that either $\{a_n\}_{n \in \mathbb{N}}$ or $\{-a_n\}_{n \in \mathbb{N}}$ is a PF_1 sequence. Hence we may assume, without loss of generality, that $\{a_n\}_{n \in \mathbb{N}}$ is a PF_1 sequence. By our hypotheses we also have that $W(z)$ and $V(z)$ are relatively prime. Therefore, by Theorem 4.5.3, we conclude that $\{a_n\}_{n \in \mathbb{N}}$ is a PF sequence. By Proposition 4.5.5, for each $k \in \mathbf{P}$ there exist two relatively prime polynomials $W_k(z)$ and $V_k(z)$ such that (6) holds. But, by Proposition 2.2.3, we know that $\{a_{nk}\}_{n \in \mathbb{N}}$ is also a PF sequence so another application of Theorem 4.5.3 establishes the desired result. $\square$

4.6 The characterization of $PF\left[\binom{x+d-i}{d}\right]$

In this section, using Theorem 4.5.3, we give a characterization of $PF\left[\binom{x+d-i}{d}\right]$ which does not depend on the explicit knowledge of the coefficients of a polynomial with respect to the basis $\left\{\binom{x+d-i}{d}\right\}_{i=0,\dots,d}$ but only on the values of the polynomial at nonnegative integers.

Let $A(x)$ be a polynomial in V_d . In Theorem 2.3.3 we have seen that then there exists a polynomial $W(z) \in V_d$ such that

$$\sum_{n \geq 0} A(n)z^n = \frac{W(z)}{(1-z)^{d+1}}, \quad (7)$$

as formal power series in z . In fact, we have seen that the coefficients of $W(z)$ are exactly the coefficients of $A(x)$ with respect to the basis $\left\{\binom{x+d-i}{d}\right\}_{i=0,\dots,d}$. So, in particular, the formal power series $\sum_{n \geq 0} A(n)z^n$ is rational for every polynomial $A(x)$. If we apply the characterization of rational PF formal power series that we have obtained in Theorem 4.5.3 to this special class of rational formal power series we obtain the following result.

Theorem 4.6.1 *Let $A(x)$ be a polynomial in V_d . Then $\{A(n)\}_{n \geq 0}$ is a PF sequence if and only if the polynomial $W(z)$ defined in (7) has only real non-*

positive zeros and $\{A(n)\}_{n \geq 0}$ is a PF_1 sequence. $\square$

Using the preceding result, Theorem 3.2.1, and Theorem 2.3.3, we can now easily prove the second main result of this chapter.

Theorem 4.6.2 *Let $A(x)$ be a polynomial in V_d . Then the following are equivalent:*

- (i) $A(x) \in PF \left[\binom{x+d-i}{d} \right];$
- (ii) $A(x) \in PF \left[\binom{x}{i} \right] \cap PF_1 \left[\binom{x+d-i}{d} \right];$
- (iii) *the sequence $\{A(n)\}_{n \in \mathbb{N}}$ is a PF sequence.*

Proof: We already know that (i) and (ii) are equivalent from Theorem 3.2.1. Suppose now that $A(x) \in PF \left[\binom{x+d-i}{d} \right]$. Then the polynomial $W(z)$ defined by (7) has only real nonpositive zeros and nonnegative coefficients. Therefore, by (7), the sequence $\{A(n)\}_{n \in \mathbb{N}}$ is nonnegative and (iii) follows from Theorem 4.6.1. Conversely, if $\{A(n)\}_{n \in \mathbb{N}}$ is a PF sequence then, by Theorem 4.6.1 the polynomial $W(z)$ defined by (7) has only real nonpositive zeros. This implies that the coefficients of $W(z)$ are either all ≥ 0 or all ≤ 0 . If they are all ≤ 0 then letting $\sum_{i=0}^d w_i z^i \stackrel{\text{def}}{=} W(z)$ and letting $z = 0$ in (7) we conclude that $w_0 = A(0) \geq 0$ and hence that $A(0) = w_0 = 0$. Then dividing both sides of (7) by z and setting $z = 0$ gives that $w_1 = A(1) \geq 0$ and hence that $A(1) = w_1 = 0$. Continuing in this way we conclude that $w_i = 0$ for $i = 0, \dots, d$. Hence the coefficients of $W(z)$ are all ≥ 0 and therefore $W(z) \in PF[z^i]$. By Theorem 2.3.3 this implies that $A(x) \in PF \left[\binom{x+d-i}{d} \right]$, as desired. $\square$

Theorem 4.6.2 is, together with Theorem 4.3.4, the most powerful result that we have about the set $PF \left[\binom{x+d-i}{d} \right]$. Both results will prove to be extremely useful in attacking combinatorial problems. The reason for their usefulness is, in the case of Theorem 4.3.4, that it describes quite explicitly large classes of polynomials that are in $PF \left[\binom{x+d-i}{d} \right]$, and in the case of Theorem 4.6.2, that it

gives a necessary and sufficient condition for a polynomial to be in $PF \left[\binom{x+d-i}{d} \right]$ that does not require the knowledge of the coefficients of the polynomial with respect to the basis $\left\{ \binom{x+d-i}{d} \right\}_{i=0,\dots,d}$ but only the knowledge of the values of the polynomial when evaluated at nonnegative integers. In many cases of interest in Combinatorics (including the poset conjecture) these values have a much simpler combinatorial interpretation than the coefficients with respect to the basis $\left\{ \binom{x+d-i}{d} \right\}_{i=0,\dots,d}$ and this makes Theorem 4.6.2 much easier to use.

We now derive some immediate consequences of Theorems 4.5.6 and 4.6.2 that will be used in chapter 6.

Theorem 4.6.3 *Let $A(x)$ be a polynomial of degree d . For each $k \in \mathbf{P}$ define a polynomial $W_k(z)$ by*

$$\sum_{n \geq 0} A(nk)z^n = \frac{W_k(z)}{(1-z)^{d+1}}. \quad (8)$$

Then, if $W_1(z)$ has only real nonpositive zeros, so does $W_k(z)$ for $k \geq 2$. $\square$

Comparing Theorem 4.6.2 and Proposition 2.2.3 yields the following remarkable property of the set $PF \left[\binom{x+d-i}{d} \right]$.

Theorem 4.6.4 *Let $A(x) \in PF \left[\binom{x+d-i}{d} \right]$. Then, for each $k \in \mathbf{P}$, $A(kx) \in PF \left[\binom{x+d-i}{d} \right]$. $\square$*

We close this section by deriving the analog of Theorem 4.6.2 for polynomials with no constant term.

Theorem 4.6.5 *Let $A(x)$ be a polynomial in $\overline{V}_d$. Then the following are equivalent:*

- (i) $A(x) \in PF \left[\binom{x+d-i}{d} \right]$;
- (ii) $A(x) \in PF \left[\binom{x}{i} \right] \cap PF_1 \left[\binom{x+d-i}{d} \right]$;
- (iii) $A(x) \in PF \left[(-1)^{d-i} \binom{x}{i} \right] \cap PF_1 \left[\binom{x+d-i}{d} \right]$;
- (iv) the sequence $\{A(n)\}_{n \in \mathbf{N}}$ is PF ;

(v) the sequence $\{(-1)^d A(-n)\}_{n \in \mathbb{N}}$ is PF.

Proof: The equivalence of (i), (ii) and (iv) is just the statement of Theorem 4.6.2. The equivalence of (i) and (iii) follows from Theorem 3.3.2 (since $A(x) \in \overline{V}_d$). Finally, applying Theorem 4.6.2 to $(-1)^d A(-x)$ we obtain that (v) is equivalent to $(-1)^d A(-x) \in PF\left[\binom{x+d-i}{d}\right]$ which by Propositions 4.2.2 and 4.2.3 is equivalent to (i). $\square$

4.7 Products of polynomials in $PF\left[\binom{x+d-i}{d}\right]$

In this section we take up again a question that was raised in section 1.3 and that is very relevant to the poset conjecture. In section 1.3 we have seen the formula

$$\Omega(P + Q, \omega_1 + \omega_2; x) = \Omega(P, \omega_1; x) \Omega(Q, \omega_2; x) \quad (9)$$

valid for any two labeled posets (P, ω_1) and (Q, ω_2) where $+$ denotes the operation of disjoint union of two posets. In view of the poset conjecture (in the form of Conjecture 5 on page 36) it is natural to investigate the following more general problem.

Problem 1: Let $A(x) \in PF\left[\binom{x+r-i}{r}\right]$ and $B(x) \in PF\left[\binom{x+s-i}{s}\right]$. Does it follow that $A(x)B(x) \in PF\left[\binom{x+r+s-i}{r+s}\right]$?

If this is true it would then follow, by (9), that it is enough to prove the poset conjecture for *connected* posets. Unfortunately, Problem 1 is still open. In this section we will prove some partial results that add evidence in favor of an affirmative answer to the above question and that also give some new properties of the set $PF\left[\binom{x+d-i}{d}\right]$.

Our first objective will be to give some reasonable formulas that enable us to compute the coefficients of the product AB of two polynomials A and B with respect to the basis $\left\{\binom{x+a+b-i}{a+b}\right\}_{i=0, \dots, a+b}$ given the coefficients of A and B

with respect to $\left\{\binom{x+a-i}{a}\right\}_{i=0,\dots,a}$ and $\left\{\binom{x+b-i}{b}\right\}_{i=0,\dots,b}$, respectively, where $a = \deg(A)$ and $b = \deg(B)$. Such a formula appears essentially already in [57] (see Theorem 4.7 (part (ii)), p. 134). Though it is there stated only in a special case (namely when A and B are the order polynomials of two labeled posets) the proof given generalizes verbatim to give the general case that we are presently interested in. That such a formula has a reasonably nice form is due to the following result which simplifies the general computation in a crucial point. The following binomial coefficient identity is the special case $q = 1$ of a q -identity appearing, e. g. , in [25] and also in [58] (see Lemma 12.5, part(ii), p. 40) and we refer the reader to these last two references for its proof.

Lemma 4.7.1 *For all $\alpha, \beta \in \mathbb{N}$ we have that*

$$\sum_{j \in \mathbb{N}} (-1)^j \binom{a+b+1}{j} \binom{a+\alpha-j}{\alpha-j} \binom{b+\beta-j}{\beta-j} = \binom{a+\alpha-\beta}{\alpha} \binom{b+\beta-\alpha}{\beta}.$$

We can now prove the first main result of this section.

Theorem 4.7.2 *Let $A(x) \stackrel{\text{def}}{=} \sum_{i=0}^a u_i \binom{x+a-i}{a}$ and $B(x) \stackrel{\text{def}}{=} \sum_{i=0}^b v_i \binom{x+b-i}{b}$ be two polynomials of degrees a and b , respectively. Then*

$$A(x)B(x) = \sum_{i=0}^{a+b} w_i \binom{x+a+b-i}{a+b}, \quad (10)$$

where

$$w_i = \sum_{k=0}^a \sum_{l=0}^b \binom{b+k-l}{i-l} \binom{a+l-k}{i-k} u_k v_l \quad (11)$$

for $i = 0, \dots, a+b$.

Proof: By Proposition 4.2.1 we know that (10) implies that

$$w_i = \sum_{j=0}^i \binom{a+b+1}{j} (-1)^j A(i-j) B(i-j),$$

for $i = 0, \dots, a+b$. By the definitions of $A(x)$ and $B(x)$ this becomes

$$\begin{aligned} w_i &= \sum_{k=0}^a \sum_{l=0}^b u_k v_l \left(\sum_{j=0}^i \binom{a+b+1}{j} (-1)^j \binom{i-j+a-k}{i-j-k} \binom{i-j+b-l}{i-j-l} \right) \\ &= \sum_{k=0}^a \sum_{l=0}^b u_k v_l \binom{b+k-l}{i-l} \binom{a+l-k}{i-k}, \end{aligned}$$

by Lemma 4.7.1 (with $\alpha = i - k$, $\beta = i - l$), as desired. $\square$

One immediate (and already non-trivial) consequence of this theorem is the following result.

Theorem 4.7.3 *Let $A(x) \in PF_1 \left[\binom{x+a-i}{a} \right]$ and $B(x) \in PF_1 \left[\binom{x+b-i}{b} \right]$ then $A(x)B(x) \in PF_1 \left[\binom{x+a+b-i}{a+b} \right]$. $\square$*

There is another equivalent way to express the relation given by Theorem 4.7.2 that is perhaps easier to grasp especially for the analytical minded. It's in the form of a differential operator.

Theorem 4.7.4 *Let $A(x) \stackrel{\text{def}}{=} \sum_{i=0}^a u_i \binom{x+a-i}{a}$ and $B(x) \stackrel{\text{def}}{=} \sum_{i=0}^b v_i \binom{x+b-i}{b}$ be two polynomials of degrees a and b , respectively. Then $A(x)B(x) = \sum_{i=0}^{a+b} w_i \binom{x+a+b-i}{a+b}$, where*

$$\sum_{i=0}^{a+b} w_i z^i \stackrel{\text{def}}{=} \frac{(1-z)^{a+b+1}}{b!} \left[\left(\frac{d}{dx} \right)^b \left[\frac{\left(\sum_{i=0}^a u_i x^i \right) \left(\sum_{i=0}^b v_i z^i x^{b-i} \right)}{(1-x)^{a+1}} \right] \right]_{|x=z} \quad (12)$$

Proof: It is easy to see (e. g. by induction on b) that the rational function on the RHS of (12) is actually a polynomial of degree $\leq a+b$ so that the definition in (12) makes sense. Now, from the definition of $A(x)$ and from Theorem 2.3.3 we know that

$$\sum_{n \geq 0} A(n) x^n = \frac{\sum_{i=0}^a u_i x^i}{(1-x)^{a+1}}. \quad (13)$$

Multiplying now both sides of (13) by $\sum_{i=0}^b v_i z^i x^{b-i}$ we get

$$\sum_{n \geq 0} \sum_{i=0}^b A(n) v_i z^i x^{n+b-i} = \frac{\left(\sum_{i=0}^a u_i x^i \right) \left(\sum_{i=0}^b v_i z^i x^{b-i} \right)}{(1-x)^{a+1}},$$

deriving now both sides b times with respect to x and then setting $x = z$ we obtain, by the definition (12) of the w_i 's,

$$\sum_{n \geq 0} A(n) \left(\sum_{i=0}^b (n+b-i)_b v_i \right) z^n = \frac{b! \sum_{i=0}^{a+b} w_i z^i}{(1-z)^{a+b+1}}.$$

This, by the definition of $B(x)$, reduces to

$$\sum_{n \geq 0} A(n) B(n) z^n = \frac{\sum_{i=0}^{a+b} w_i z^i}{(1-z)^{a+b+1}},$$

which, by Theorem 2.3.3, is equivalent to

$$A(x) B(x) = \sum_{i=0}^{a+b} w_i \binom{x+a+b-i}{a+b},$$

as desired. $\square$

The reader should note that, even though it is more compact, the formula given in (12) is not “simpler” than that given in (11); in the sense that if the computations in (12) are carried out and the coefficients of z^i on both sides are then equated one still has to use Lemma 4.7.1 to simplify the resulting expression for w_i in order to reduce it to (11). So, by the preceding theorem, we can restate Problem 1 equivalently as follows:

Problem 2: Let $V(z), U(z) \in PF[z^i]$ be two polynomials of degrees $\leq a$ and $\leq b$, respectively. Define a polynomial $W(z) \in V_{a+b}$ by

$$W(z) \stackrel{\text{def}}{=} (1-z)^{a+b+1} \left[\left(\frac{d}{dx} \right)^b \left[\frac{V(x) U\left(\frac{z}{x}\right) x^b}{(1-x)^{a+1}} \right] \right]_{|x=z}. \quad (14)$$

Does it follow that $W(z) \in PF[z^i]$?

Note that Theorem 4.7.3 assures us that $W(z) \in PF_1[z^i]$ if $V(z), U(z) \in PF_1[z^i]$; a non-trivial fact also from the point of view of (14). The differential expression given in (14) does seem to be easier to deal with than the explicit formula (11). In fact, our second main result of this section about Problems 1 and 2 is obtained using the formulation (14).

Theorem 4.7.5 Let $V(z) \stackrel{\text{def}}{=} \sum_{i=l}^t v_i z^i$ and $U(z) \stackrel{\text{def}}{=} \sum_{i=0}^s u_i z^i$ be two polynomials of degrees t and s , respectively, where $0 \leq l \leq t \leq a$ and $r \geq s \geq 0$. Suppose that $V(z) \in PF[z^i]$, that $U(1) \neq 0$ and that all the zeros of $\sum_{i=0}^s u_{s-i}(z+i)_r$ lie

in the real interval $[t - a - 1, \max\{l, r - s\}]$. Define a polynomial $W(q)$ by

$$W(q) \stackrel{\text{def}}{=} (1 - q)^{a+r+1} \left[\left(\frac{d}{dz} \right)^r \left[\frac{V(z) U\left(\frac{q}{z}\right) z^s}{(1 - z)^{a+1}} \right] \right]_{|z=q}$$

Then $W(q) \in PF[z^i]$.

Proof: Let $A(x) \stackrel{\text{def}}{=} \sum_{i=l}^t v_i \binom{x+a-i}{a}$ and $B(x) \stackrel{\text{def}}{=} \sum_{i=0}^s u_{s-i} (x+i)_r$. Since $U(1) \neq 0$ we have that $\deg(B(x)) = r$. Then, by our hypotheses, $A(x) \in PF\left[\binom{x+a-i}{a}\right] \cup W_l^t\left[\binom{x+a-i}{a}\right]$. Also, by our hypotheses we know that the zeros of $B(x)$ are all real and $\Lambda(B) \leq \max\{l, r - s\}$, $\lambda(B) \geq t - a - 1$. Furthermore, $\Lambda(B) \geq r - s - 1$ since $(x)_{r-s}$ divides $B(x)$ by the definition of $B(x)$. Therefore

$$[\lambda(B), t - a - 1] \cap \mathbf{Z} = \begin{cases} \emptyset & \text{if } \lambda(B) > t - a - 1 \\ \{t - a - 1\} & \text{if } \lambda(B) = t - a - 1 \end{cases}$$

and

$$[l, \Lambda(B)] \cap \mathbf{Z} = \begin{cases} \emptyset & \text{if } r - s \leq l \text{ and } \Lambda(B) < l \\ \{l\} & \text{if } r - s \leq l \text{ and } \Lambda(B) = l \\ [l, r - s - 1] \cap \mathbf{Z} & \text{if } r - s \geq l \text{ and } \Lambda(B) < r - s \\ [l, r - s] \cap \mathbf{Z} & \text{if } r - s \geq l \text{ and } \Lambda(B) = r - s \end{cases}$$

Since $(x)_{r-s}$ divides $B(x)$ we see that in any case all the conditions of Theorem 4.3.4 are satisfied. Therefore we conclude that

$$A(x) B(x) \in PF\left[\binom{x + a + r - i}{a + r}\right]. \quad (15)$$

Now, by the definitions of $A(x)$ and $V(z)$ and by Theorem 2.3.3 we know that

$$\sum_{n \geq 0} A(n) z^n = \frac{V(z)}{(1 - z)^{a+1}}.$$

Multiplying both sides of this equality by $U\left(\frac{q}{z}\right) z^s$ we then get

$$\sum_{n \geq 0} A(n) \sum_{i=0}^s u_i q^i z^{n+s-i} = \frac{V(z) U\left(\frac{q}{z}\right) z^s}{(1 - z)^{a+1}}. \quad (16)$$

Differentiating now both sides of (16) r times with respect to z and then letting $z = q$ we obtain, by the definitions of $W(q)$ and $B(x)$,

$$\sum_{n \geq 0} A(n) B(n) q^n = \frac{q^{r-s} W(q)}{(1-q)^{a+r+1}}. \quad (17)$$

On the other hand, since $A(x)B(x)$ is a polynomial of degree $a+r$, by (15) and (17), using Theorem 2.3.3, we conclude that $q^{r-s}W(q) \in PF[q^i]$ and hence that $W(q) \in PF[q^i]$, as desired. $\square$

Theorem 4.7.5 is the result “closest” to an affirmative solution of Problem 2 that I have been able to obtain. The reader should notice that if the hypothesis that $U(1) \neq 0$ is waived in Theorem 4.7.5 then a slight variation of the above proof yields the weaker, but still interesting result, that $V(q)$ has only real zeros.

As an immediate corollary of Theorem 4.7.5 we obtain an affirmative answer to Problem 2 (and hence Problem 1) in the case that one of the two polynomials $U(z)$, $V(z)$ has degree less than or equal to one.

Theorem 4.7.6 *Let $r, a \in \mathbf{N}$ $u, v \in \mathbf{R}$, $u, v \geq 0$ and let $V(z) \in V_a \cap PF[z^i]$. Define a polynomial $W(q)$ by*

$$W(q) \stackrel{\text{def}}{=} (1-q)^{a+r+1} \left[\left(\frac{d}{dz} \right)^r \left[\frac{V(z)(uq + vz)}{(1-z)^{a+1}} \right] \right]_{|z=q}.$$

Then $W(q) \in PF[z^i]$.

Proof: We may clearly assume $u + v > 0$ to avoid trivialities. Now, the polynomial $u(z)_r + v(z+1)_r = (z)_{r-1} [u(z-r+1) + v(z+1)]$ has zeros $0, 1, 2, \dots, r-2, \frac{u(r-1)-v}{u+v}$. Since $u, v \geq 0$ we have that $-1 \leq \frac{u(r-1)-v}{u+v} \leq r-1$, so that all the zeros are contained in $[-1, r-1]$. This shows that all the hypotheses of Theorem 4.7.5 are satisfied (with $U(z) = v + uz$, $l = 0$, $t = a$, $s = 1$) and the thesis follows. $\square$

The preceding result has the following curious consequence.

Theorem 4.7.7 *Let $s, t \in \mathbf{N}$ and let $V(z) \in V_t$, $U(z) \in V_s$. Suppose that $V(z) \in PF[z^i]$ and that $sU(1) \geq U'(1) \geq (s-1)U(1)$. Define a polynomial*

$W(q)$ by

$$W(q) \stackrel{\text{def}}{=} (1-q)^{t+2} \left[\left(\frac{d}{dz} \right) \left[\frac{V(z)U\left(\frac{q}{z}\right)z^s}{(1-z)^{t+1}} \right] \right]_{|z=q}, \quad (18)$$

then $W(q) \in PF[q^i]$.

Proof: Let $u \stackrel{\text{def}}{=} U'(1) - (s-1)U(1)$, $v \stackrel{\text{def}}{=} sU(1) - U'(1)$ and $\tilde{U}(z) \stackrel{\text{def}}{=} uz^s + vz^{s-1}$.

Then $\tilde{U}(1) = U(1)$ and $\tilde{U}'(1) = U'(1)$. So by (18) we have that

$$\begin{aligned} W(q) &= (1-q)^{t+2} \left[\left(\frac{d}{dz} \right) \left[\frac{V(z)U\left(\frac{q}{z}\right)z^s}{(1-z)^{t+1}} \right] \right]_{|z=q} \\ &= (1-q)^{t+2} \left[\left(\frac{d}{dz} \right) \left[\frac{V(z)\tilde{U}\left(\frac{q}{z}\right)z^s}{(1-z)^{t+1}} \right] \right]_{|z=q} \\ &= q^{s-1}(1-q)^{t+2} \left[\left(\frac{d}{dz} \right) \left[\frac{V(z)(uq + vz)}{(1-z)^{t+1}} \right] \right]_{|z=q}. \end{aligned} \quad (19)$$

Since, by hypothesis, $u, v \geq 0$, the proof now follows from Theorem 4.7.6. $\square$

In general, it is easy to see that the polynomial $W(q)$ defined by (18) only depends on $(V(z), r, t, s)$ and $U(1), U'(1), \dots, U^{(r)}(1)$. However, we have no use of this fact here.

There is a third equivalent way to formulate Problems 1 and 2 that can be interesting to consider. This is obtained from Problem 1 using the characterization of $PF \left[\binom{x+d-i}{d} \right]$ given by Theorem 4.6.2. Using this characterization we can easily see that Problem 1 can be reformulated as follows.

Problem 3 Let $A(x)$ and $B(x)$ be two polynomials such that the sequences $\{A(n)\}_{n=0,1,2,\dots}$ and $\{B(n)\}_{n=0,1,2,\dots}$ are both PF . Does it follow that the sequence $\{A(n)B(n)\}_{n=0,1,2,\dots}$ is also PF ?

There are a couple of observations to make about Problem 3. The first one is that the corresponding statement for arbitrary PF sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ is false. This was already known to Karlin (see Remark 1.1 on p. 157 of [33]) and a counterexample can be found in the last paragraph of section 6.5

of the present work. This counterexample shows that the above statement is false even if one of the two sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ is finite. However, if *both* sequences are finite then the *PF* property is preserved by the operation of Hadamard (i. e. componentwise) product. This is, in fact, equivalent to a classical (though half forgotten) result in complex analysis due originally to Maló (see [41]) and we have already presented to the reader a slightly stronger result in section 3.5. Theorem 3.5.3 is, in fact, easily seen to imply the following result.

Theorem 4.7.8 *Let $\{a_0, \dots, a_d\}$ and $\{b_0, \dots, b_d\}$ be two *PF* sequences. Then the sequence $\{a_0b_0, (1!)a_1b_1, \dots, (d!)a_db_d\}$ is also *PF*. In particular, the sequence $\{a_0b_0, \dots, a_db_d\}$ is *PF*. $\square$*

Now, if $P(x)$ is a polynomial of degree d then the sequence $\{P(n)\}_{n \in \mathbb{N}}$ is completely determined by the finite sequence $\{P(0), \dots, P(d)\}$. So it is conceivable that Theorem 4.7.8 might play a role in Problem 3.

Concluding this section I would like to remark that Problem 1 (and hence 2 and 3) is still open even in the case $\deg A = \deg B = 2$ and that my efforts to obtain some information on Problem 3 (see, e. g. [9]) have yielded, to this date, no replies.

Chapter 5

Applications to the Poset Conjecture

5.1 Introduction

In this chapter we apply the general results obtained in the last three chapters to the problem that motivated them in the first place, namely, the poset conjecture. We prove that the conjecture holds for four infinite families of labeled posets. In almost all cases the proofs are straightforward applications of the results obtained in the last chapter (especially Theorems 4.4.1 and 4.6.2) to formulas that were previously known.

In the next section we introduce an infinite family of posets arising from the theory of plane partitions which we call Ferrers posets, and we state some fundamental formulas for their order polynomials with respect to some particular classes of labelings. These formulas will then be used in the third section to prove the poset conjecture for these labeled posets; we then show how a conjecture of R.Stanley follows as a special case of these results.

In section 5.4 we use an unpublished result of R.Proctor to prove the poset conjecture for a class of naturally labeled Ferrers posets. In the fifth section we prove that the poset conjecture holds for a disjoint union of chains thus generalizing a previous result of R.Simion. In section 5.6 we prove that the poset conjecture holds for all naturally labeled Gaussian posets. Finally, in the last section, we give some new necessary and sufficient conditions for the poset

conjecture in the general case. In all cases we will prove the poset conjecture in the form given by Conjecture 5 in section 2.3.

5.2 Ferrers Posets

In the next two sections we will study an infinite family of posets arising from the theory of plane partitions which we call Ferrers posets. In this section we give all the necessary definitions and notations and we state some important results about the posets that will be used in the next two sections. Because of the connections between Ferrers posets and plane partitions we will use, in this and the next section, the theory of symmetric functions. We follow the notations and terminology of [38], Chapter I, for this theory; in particular we denote by s_λ the Schur symmetric function associated to the partition λ .

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l)$ we can associate to it a partially ordered set in a very natural way. We let $P_\lambda \stackrel{\text{def}}{=} \{(i, j) \in \mathbf{P} \times \mathbf{P} : 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}$ and we endow it with the partial ordering induced by the product ordering on $\mathbf{P} \times \mathbf{P}$. So, for example, the poset $P_{(4,3,2,2)}$ is shown in Figure 1. We will call the posets P_λ *Ferrers posets*.

Figure 1: the poset $P_{(4,3,2,2)}$

Our main goal here is to give some formulas for the order polynomials of these posets with respect to some particular labelings of them. Among all possible labelings of a Ferrers poset P_λ two general classes are particularly important: natural labelings and column strict labelings.

A labeling ω of P_λ is called *column strict* if $\omega((i, j)) > \omega((i + 1, j))$ and $\omega((i, j)) < \omega((i, j + 1))$ for all $(i, j) \in P_\lambda$ for which both sides of the above inequalities are defined. So, for example, the labeling in Figure 2 is natural, the one in Figure 3 is column strict and the one in Figure 4 is neither one. One can, of course, define row strict labelings in the analogous way but their theory is clearly equivalent to that of column strict labelings and therefore we will only consider the latter. The reason why these two classes of labelings are important and interesting to consider is that in those two cases there are formulas (coming from the theory of plane partitions) for the order polynomials of the corresponding labeled posets.

We will consider natural labelings first. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0)$ be a partition and ω a natural labeling of P_λ . It is then easy to recognize that a (P_λ, ω) -partition $\sigma : P_\lambda \rightarrow [n]$ with largest part $\leq n$ is just a plane partition of shape λ and largest part $\leq n$ or, equivalently, a plane partition of shape contained in λ and largest part $\leq n - 1$. A formula for the number of such plane partitions was given by MacMahon in 1915 (see [40], p.243).

Theorem 5.2.1 *Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_l > 0)$ be a partition. Then the number $f_\lambda(n)$ of plane partitions of shape contained in λ and largest part $\leq n$ is given by*

$$f_\lambda(n) = \det \left[\binom{\lambda_i + n}{\lambda_i - i + j} \right]_{1 \leq i, j \leq l}.$$

A proof of a generalization of Theorem 5.2.1 was given by Krewerās in 1965 (see [35]) and also appears in [63] (see Theorem 2.7.1 and ex. 63, chapter 3). From the remarks preceding it and from the definition of the order polynomial

Figure 2

Figure 3

Figure 4

we obtain the following equivalent formulation of Theorem 5.2.1.

Theorem 5.2.2 *Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_l > 0)$ be a partition. Then the order polynomial of P_λ is given by*

$$\Omega(P_\lambda; x) = \det \left[\binom{\lambda_i + x}{\lambda_i - i + j} \right]_{1 \leq i, j \leq l}. \square$$

This formula for the order polynomial of a Ferrers poset is not very explicit but it is nonetheless of some use as we will see in section 5.4.

Fix now a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ (where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$) and a column strict labeling ω of P_λ . Then it is easy to recognize that a (P_λ, ω) -partition $\sigma : P_\lambda \rightarrow [n]$ with largest part $\leq n$ is just a column strict plane partition of shape λ and largest part $\leq n$. By the combinatorial definition of the Schur functions this number is also given by $s_\lambda(n)$ where $s_\lambda(n) \stackrel{\text{def}}{=} s_\lambda(\underbrace{1, 1, \dots, 1}_n, 0, 0, \dots)$. We therefore have the following fundamental relation between the order polynomial of (P_λ, ω) and the Schur function associated to λ .

Theorem 5.2.3 *Let λ be a partition and ω a column strict labeling of P_λ . Then $\Omega(P_\lambda, \omega; n) = s_\lambda(n)$ for all $n \geq 1$. $\square$*

This shows, in particular, that $s_\lambda(n)$ is a polynomial function of n (a fact that can also be proved directly, of course). Using the theory of symmetric functions it is, in fact, possible to give a closed explicit expression for this polynomial $s_\lambda(n)$. Before we can state this expression, however, we need to define two important invariants attached to any element of a Ferrers posets. If λ is a partition and $(i, j) \in P_\lambda$ then we let $c(i, j) \stackrel{\text{def}}{=} j - i$ and $h(i, j) \stackrel{\text{def}}{=} |\{(i, n) \in P_\lambda : n \geq j\} \cup \{(m, j) \in P_\lambda : m \geq i\}|$. We call $c(i, j)$ the *content* and $h(i, j)$ the *hook-length* of (i, j) . The concept of the hook length of an element in a poset can actually be defined for a much larger class of posets than just Ferrers posets (see, e. g., [57], [51], or [52]), however, we have no use for this fact here.

We can now state the closed explicit expression for $s_\lambda(n)$ referred to above.

Theorem 5.2.4 *Let λ be a partition, then*

$$s_\lambda(n) = \prod_{(i,j) \in P_\lambda} \left(\frac{n + c(i,j)}{h(i,j)} \right)$$

for all $n \in \mathbb{N}$.

The proof of the preceding theorem relies on the equivalence of the combinatorial and algebraic definitions of the Schur functions, one of the central results in the theory of symmetric functions. For this reason we don't prove it here but instead refer the reader to [38] Ex. 4, chapter I, section 3. From the last two theorems we can now derive the following result.

Theorem 5.2.5 *Let λ be a partition and ω a column strict labeling of P_λ . Then*

$$\Omega(P_\lambda, \omega; x) = \prod_{(i,j) \in P_\lambda} \left(\frac{x + c(i,j)}{h(i,j)} \right) \square \quad (1)$$

5.3 Column strict labeled Ferrers posets and the solution of a conjecture by R.Stanley

In this section we prove that the poset conjecture holds for all column strict labeled Ferrers posets. We then show how a conjecture by R.Stanley is a special case of this result.

From the explicit closed formula obtained in Theorem 5.2.5 we can see almost “by inspection” that the poset conjecture holds for column strict labeled Ferrers posets.

Theorem 5.3.1 *Let λ be a partition and ω a column strict labeling of P_λ . Then*

$$\Omega(P_\lambda, \omega; x) \in PF \left[\binom{x+|\lambda|-i}{|\lambda|} \right].$$

Proof: Let $\lambda = (\lambda_1, \dots, \lambda_l)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ Then from (1) it is immediate that

$$\Lambda(\Omega(P_\lambda, \omega; x)) = l - 1,$$

$$\lambda(\Omega(P_\lambda, \omega; x)) = 1 - \lambda_1.$$

Therefore

$$[0, \Lambda(\Omega(P_\lambda, \omega; x))] \cap \mathbf{Z} = \{-c(i, 1)\}_{i=1, \dots, l}$$

and

$$[\lambda(\Omega(P_\lambda, \omega; x)), -1] \cap \mathbf{Z} = \{-c(1, j)\}_{j=2, \dots, \lambda_1}.$$

So all the conditions of Theorem 4.4.1 are satisfied and the thesis follows. $\square$.

We will now show how the preceding theorem implies the conjecture by R.Stanley mentioned in the beginning of the section. The original conjecture (Conjecture 7.9 in [65]) was stated as follows.

Conjecture *For each partition λ define a polynomial $T_\lambda(q)$ by*

$$\sum_{\lambda} T_\lambda(q) s_\lambda = \frac{\sum_{n \geq 0} (1-q)^n s_n}{1 - q \sum_{n \geq 1} (1-q)^{n-1} s_n} \quad (2)$$

Then the polynomial $T_\lambda(q)$ is unimodal.

In other words, the conjecture states that if we expand the symmetric function on the RHS of (2) with respect to the basis consisting of the Schur functions then all the coefficients in this expansion are unimodal polynomials. We will show that they actually have only real zeros. At first sight it might seem that the above conjecture has nothing to do with the poset conjecture, but this is not the case. In the next result we prove the above conjecture by showing that it follows from Theorem 5.3.1.

Theorem 5.3.2 *Let λ be a partition. Then the polynomial $T_\lambda(q)$ defined by (2) has nonnegative coefficients and has only real zeros. In particular, $T_\lambda(q)$ is unimodal.*

Proof: Let ω be a column strict labeling of P_λ . By equation (2) of Theorem 2.3.3 we know that Theorem 5.3.1 is equivalent to the statement that the

polynomial $W(P_\lambda, \omega; q)$ defined by

$$\sum_{n \geq 0} \Omega(P_\lambda, \omega; n) q^n = \frac{W(P_\lambda, \omega; q)}{(1 - q)^{|\lambda|+1}}, \quad (3)$$

has nonnegative coefficients and only real zeros. We will prove the theorem by showing that $qT_\lambda(q) = W(P_\lambda, \omega; q)$. By Theorem 5.2.3 equation (3) is equivalent to

$$\sum_{n \geq 1} s_\lambda(n) q^n = \frac{W(P_\lambda, \omega; q)}{(1 - q)^{|\lambda|+1}}. \quad (4)$$

If we now multiply both sides of (4) by s_λ and we sum over all partitions λ we obtain

$$\sum_{n \geq 1} \left(\sum_{\lambda} s_{\lambda} s_{\lambda}(n) \right) q^n = \sum_{\lambda} \frac{s_{\lambda} W(P_{\lambda}, \omega; q)}{(1 - q)^{|\lambda|+1}}. \quad (5)$$

Now, it is a well known fact in the theory of symmetric functions (see, e. g. , [38], p.33) that the following identity holds

$$\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1}, \quad (6)$$

where the sum is over all partitions λ . Therefore, letting $y_j = 1$ if $j = 1, \dots, n$ and $y_j = 0$ if $j > n$ in (6) we obtain the identity

$$\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(n) = \left[\prod_{i \geq 1} (1 - x_i)^{-1} \right]^n. \quad (7)$$

Substituting this in (5) we obtain

$$\sum_{n \geq 1} \left[\prod_{i \geq 1} (1 - x_i)^{-1} q \right]^n = \sum_{\lambda} \frac{s_{\lambda} W(P_{\lambda}, \omega; q)}{(1 - q)^{|\lambda|+1}},$$

which is equivalent to

$$\sum_{\lambda} \frac{s_{\lambda} W(P_{\lambda}, \omega; q)}{(1 - q)^{|\lambda|+1}} = \frac{\prod_{i \geq 1} (1 - x_i)^{-1} q}{1 - \prod_{i \geq 1} (1 - x_i)^{-1} q}. \quad (8)$$

Letting $n = 1$ in (7) we get, using Theorem 5.2.4, that

$$\sum_{n \geq 0} s_n(x) = \prod_{i \geq 1} (1 - x_i)^{-1}.$$

Substituting this into the RHS of (8) we then get

$$\sum_{\lambda} \frac{s_{\lambda} W(P_{\lambda}, \omega; q)}{q(1-q)^{|\lambda|+1}} = \frac{\sum_{n \geq 0} s_n}{1 - q \sum_{n \geq 0} s_n}$$

and making the substitution $x_i \mapsto (1-q)x_i$ for $i \geq 1$ we get

$$\begin{aligned} \sum_{\lambda} s_{\lambda} W(P_{\lambda}, \omega; q) &= \frac{q \sum_{n \geq 0} (1-q)^{n+1} s_n}{1 - q \sum_{n \geq 0} (1-q)^n s_n} \\ &= \frac{q \sum_{n \geq 0} (1-q)^n s_n}{1 - q \sum_{n \geq 1} (1-q)^{n-1} s_n}. \end{aligned} \tag{9}$$

Comparing now (9) with (2) and using the fact that the Schur functions are a basis for the ring of symmetric functions we finally obtain that

$$qT_{\lambda}(q) = W(P_{\lambda}, \omega; q),$$

for all partitions λ , as desired. $\square$

5.4 A class of naturally labeled Ferrers Posets

In this section we verify the poset conjecture for a large class of naturally labeled Ferrers posets. In Theorem 5.2.2 we have given a determinantal formula for the order polynomial of a Ferrers poset. In general it seems unlikely that a simpler formula than that exists. However, there is at least one special class of Ferrers posets for which it is possible to evaluate explicitly the determinant in Theorem 5.2.1. This class was discovered by R.Proctor and is the following.

Let $r, s \in \mathbf{P}$ and $b \in \mathbf{N}$. We define a partition $\lambda(r, s, b)$ by $\lambda(r, s, b) \stackrel{\text{def}}{=} (s + (r-1)b, s + (r-2)b, \dots, s + b, s)$. Note that $P_{\lambda(r, s, 0)} \cong \mathbf{r} \times \mathbf{s}$ and that $|\lambda(r, s, b)| = rs + b \binom{r}{2}$. We will often write $P_{r, s, b}$ instead of $P_{\lambda(r, s, b)}$.

For the partitions $\lambda(r, s, b)$ the determinant in Theorem 5.2.2 (and hence the order polynomial of the posets $P_{r,s,b}$) can be explicitly evaluated and has a nice closed expression. The following result was first conjectured and later proved by R.Proctor (see [48]).

Theorem 5.4.1 *For $r, s \in \mathbf{P}$ and $b \in \mathbf{N}$ we have that*

$$\Omega(P_{r,s,b}; x) = \prod_{(i,j) \in \tilde{P}} \left(\frac{x + r + j - i - 1}{r + j - i} \right) \times \prod_{(i,j) \in P_{r,s,b} \setminus \tilde{P}} \left(\frac{(b+1)(x-1) + r + j - i}{r + j - i} \right), \quad (10)$$

where

$$\tilde{P} = \begin{cases} P_{r,s,b} & \text{if } b = 0, \\ P_{r,s,b-1} & \text{if } b \geq 1. \end{cases}$$

The proof of the above formula is purely computational and we omit it. With the above formula and Theorem 4.4.1 we can again see almost “by inspection” that the poset conjecture holds for these naturally labeled posets.

Theorem 5.4.2 *For $r, s \in \mathbf{P}$ and $b \in \mathbf{N}$ we have that $\Omega(P_{r,s,b}; x) \in PF \left[\binom{x+p-i}{p} \right]$ where $p = |P_{r,s,b}|$.*

Proof: It is easy to see from (10) that

$$\Lambda(\Omega(P_{r,s,b}; x)) = \beta_{r,1} = 0$$

$$\lambda(\Omega(P_{r,s,b}; x)) = \beta_{1,s} = -(r + s - 2), \quad \text{if } b = 0,$$

$$\lambda(\Omega(P_{r,s,b}; x)) = \beta_{1,s+(r-1)(b-1)} = -(s + (r-1)b - 1), \quad \text{if } b \geq 1,$$

where b_{ij} denotes the zero of $\Omega(P_{r,s,b}; x)$ corresponding to $(i, j) \in P_{r,s,b}$. This is easily seen to imply that $\Omega(P_{r,s,b}; x) = 0$ for all $x \in [\lambda(\Omega(P_{r,s,b}; x)), -1] \cap \mathbf{Z}$ and the thesis follows from Theorem 4.4.1. $\square$

It should be remarked that small examples seem to suggest that the partitions $\lambda(r, s, b)$ are the “most general” class of partitions for which the determinant in Theorem 5.2.2 is expressible as a product of linear factors.

5.5 Disjoint unions of chains

In this section we prove that the poset conjecture is true for posets that are the disjoint union of chains.

Theorem 5.5.1 *Let P be a poset that is the disjoint union of chains and let ω be a labeling of P . Then $\Omega(P, \omega; x) \in PF \left[\binom{x+p-i}{p} \right]$, where $p = |P|$.*

Proof: Let $P = \mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_r$ where $\mathbf{a}_i$ denotes the chain $x_{i,1} < x_{i,2} < \dots < x_{i,a_i}$. It is an easy exercise (which we leave to the reader) in the theory of (P, ω) -partitions to show that

$$\Omega(P, \omega; x) = \prod_{i=1}^r \binom{x + b_i}{a_i}, \quad (11)$$

where $b_i \stackrel{\text{def}}{=} a_i - r_i - 1$ and where r_i is the number of descents of the sequence $\omega(x_{i,1}) \dots \omega(x_{i,a_i})$, for $i = 1, \dots, r$. In particular, this implies that $0 \leq b_i < a_i$ for $i = 1, \dots, r$. We then have, for some $1 \leq k, l \leq r$, that

$$\Lambda(\Omega(P, \omega; x)) = r_k,$$

$$\lambda(\Omega(P, \omega; x)) = -b_l,$$

where $r_k = \max_{1 \leq i \leq r} \{r_i\}$ and $b_l = \max_{1 \leq i \leq r} \{b_i\}$. Therefore every element of

$$[\lambda(\Omega(P, \omega; x)), -1] \cap \mathbf{Z} = [-b_l, -1] \cap \mathbf{Z},$$

is a zero of $\binom{x+b_l}{a_l}$, and every element of

$$[0, \Lambda(\Omega(P, \omega; x))] \cap \mathbf{Z} = [0, r_k] \cap \mathbf{Z}$$

is a zero of $\binom{x+b_k}{a_k}$. So, by (11), all the hypotheses of Theorem 4.4.1 are satisfied, and the thesis follows. $\square$

Theorem 5.5.1 reduces to a result of R. Simion (see [56]) if ω is a natural labeling.

5.6 Gaussian Posets

In this section we prove, using Theorem 4.4.1 that the poset conjecture holds for any naturally labeled Gaussian poset.

Given a (finite) poset P and $m \in \mathbf{N}$ we may define a polynomial $U_m(P; q)$ by

$$q^{|P|} U_m(P; q) \stackrel{\text{def}}{=} \sum_{\sigma: P \rightarrow [m+1]} q^{|\sigma|}, \quad (12)$$

where the sum is over all order reversing maps $\sigma : P \rightarrow [m+1]$ and where $|\sigma| \stackrel{\text{def}}{=} \sum_{x \in P} \sigma(x)$. The polynomial $U_m(P; q)$ was first introduced by Stanley (who actually defined it for any labeled poset) in [57] and is a “q-analogue” of the order polynomial. In fact, letting $q = 1$ in (12) yields the following result.

Theorem 5.6.1 *Let P be a poset. Then $U_m(P; 1) = \Omega(P; m+1)$ for all $m \in \mathbf{N}$. $\square$*

A poset P is said to be *Gaussian* if there exist $h_1, \dots, h_p \in \mathbf{P}$ (where $p \stackrel{\text{def}}{=} |P|$) such that

$$U_m(P; q) = \prod_{i=1}^p \left(\frac{1 - q^{m+h_i}}{1 - q^{h_i}} \right)$$

for all $m \in \mathbf{N}$. The integers $h_1, \dots, h_p$ are called the *hook lengths* of the Gaussian poset P . Gaussian posets were first introduced and studied by Stanley in [57] who was led to their definition by his studies in the theory of plane partitions. Since then, they have been widely studied (see, e. g. , [49]) and many non trivial properties of them have been discovered. In what follows we will only need two of them. The first one is the following and its proof can be found in [63], ex. 4.25(a), p. 270.

Theorem 5.6.2 *Let P be a poset. Then P is Gaussian if and only if every connected component of P is Gaussian.*

The second result that we need is the one that will enable us to use Theorem 4.4.1. Its proof can be found in [63], ex. 4.25 (a), (b).

*

Theorem 5.6.3 *Let P be a connected, Gaussian poset. Then P is graded and*

$$U_m(P; q) = \prod_{x \in P} \left(\frac{1 - q^{m+1+\rho(x)}}{1 - q^{\rho(x)+1}} \right) \quad (13)$$

for all $m \in \mathbf{N}$, where $\rho : P \rightarrow \mathbf{N}$ is the rank function of P .

So the preceding theorem says that the multiset $\{h_1, \dots, h_p\}$ of hook lengths of a Gaussian poset P is actually equal to the multiset $\{\rho(x) + 1 : x \in P\}$, (where ρ is the rank function of P) if P is connected. Using Theorems 5.6.1 and 5.6.3 we can give a nice closed expression of the order polynomial of a connected Gaussian poset.

Theorem 5.6.4 *Let P be a connected, Gaussian poset. Then*

$$\Omega(P; x) = \prod_{y \in P} \left(\frac{x + \rho(y)}{1 + \rho(y)} \right) \quad (14)$$

(as polynomials in $\mathbf{R}[x]$).

Proof: By (13) we have that

$$U_m(P; q) = \prod_{y \in P} \left(\frac{1 + q + q^2 + \dots + q^{m+\rho(y)}}{1 + q + q^2 + \dots + q^{\rho(y)}} \right).$$

Letting $q = 1$ and using Theorem 5.6.1 yields (14). $\square$

It would be interesting to classify all connected graded posets for which equation (14) holds.

Using Theorems 5.6.2 and 5.6.4 we can again see almost “by inspection” that the poset conjecture holds for naturally labeled Gaussian posets.

Theorem 5.6.5 *Let P be a Gaussian poset. Then $\Omega(P; x) \in PF \left[\binom{x+p-i}{p} \right]$, where $p \stackrel{\text{def}}{=} |P|$.*

Proof: By Theorem 5.6.2 there exist connected Gaussian posets $P_1, \dots, P_t$ such that $P = P_1 + \dots + P_t$. Therefore we have, using Theorem 5.6.4 that

$$\begin{aligned} \Omega(P; x) &= \prod_{i=1}^t \Omega(P_i; x) \\ &= \prod_{i=1}^t \prod_{y \in P_i} \left(\frac{x + \rho_i(y)}{1 + \rho_i(y)} \right) \end{aligned} \quad (15)$$

where $\rho_i : P_i \rightarrow \mathbf{N}$ is the rank function of P_i , for $i = 1, \dots, t$. From (15) we then deduce that $\Omega(P; x) = 0$ if and only if $x \in [-r(P_s), 0] \cap \mathbf{Z}$ where $1 \leq s \leq t$ is such that $r(P_s) = \max_{1 \leq i \leq t} \{r(P_i)\}$, (where $r(P_i)$ denotes the rank of P_i). Therefore all the conditions of Theorem 4.4.1 are satisfied and the thesis follows. $\square$

It would be interesting to know whether Theorem 5.6.5 can be generalized to *labeled* Gaussian posets. Using Theorem 5.6.5 and Proposition 1.3.1 we obtain the second main result of this section.

Theorem 5.6.6 *Let P be a naturally labeled poset such that $P = G_1 \oplus \dots \oplus G_t$ where $G_1, \dots, G_t$ are Gaussian posets. Then $\Omega(P; x) \in PF \left[\binom{x+p-i}{p} \right]$, where $p \stackrel{\text{def}}{=} |P|$. $\square$*

Theorems 5.6.5 and 5.6.6 are very elegant and interesting results. However, they become even more interesting when one takes into account some more facts that are known about Gaussian posets. The following (highly non-trivial) result appears in [63], ex. 25 (f), p. 270 and an elegant proof is given in [49] by Proctor using the representation theory of semisimple Lie algebras.

Theorem 5.6.7 *The following posets are Gaussian:*

- (i) $\mathbf{r} \times \mathbf{s}$, for all $r, s \in \mathbf{P}$;
- (ii) $J(\mathbf{2} \times \mathbf{r})$, for all $r \in \mathbf{P}$;
- (iii) $\mathbf{r} \oplus (\mathbf{1} + \mathbf{1}) \oplus \mathbf{r}$, for all $r \in \mathbf{P}$;
- (iv) $J(J(\mathbf{2} \times \mathbf{3}))$;
- (v) $J(J(J(\mathbf{2} \times \mathbf{3})))$.

From Theorems 5.6.5 and 5.6.7 we obtain the last main result of this section.

Theorem 5.6.8 *The poset conjecture holds for all the naturally labeled posets listed in parts (i) through (v) of Theorem 5.6.7. $\square$*

The reader should note that the fact that the poset conjecture holds for the naturally labeled posets $\mathbf{r} \oplus (\mathbf{1} + \mathbf{1}) \oplus \mathbf{r}$ for all $r \in \mathbf{P}$ can also be easily

proved directly using Theorem 5.5.1 and Proposition 1.3.1, and that we have already seen in section 5.4, Theorem 5.4.2, that the poset conjecture holds for the naturally labeled posets $\mathbf{r} \times \mathbf{s}$ for all $r, s \in \mathbf{P}$. For the posets listed in parts (ii), (iv) and (v), however, I don't know of any proof (even using the machinery developed in chapters 2 through 4) that avoids Theorem 5.6.7. Combining Theorems 5.6.6 and 5.6.7 we can produce explicitly even more naturally labeled posets for which the poset conjecture holds.

Finally, it should be mentioned that the posets listed in Theorem 5.6.7 are the *only* known connected Gaussian posets to this date. It has been conjectured in 1984 by Proctor (see [49], sect.9) that there are no other connected Gaussian posets and this conjecture is still open at this time.

5.7 Necessary and sufficient conditions for the poset conjecture

In this section we give, using several results obtained in the preceding chapters, some necessary and sufficient conditions for the poset conjecture. These conditions will include the equivalent statements of the poset conjecture given in chapter 1 and section 2.3 as well as some new ones.

Before stating these equivalent conditions we will recall the combinatorial interpretation of the coefficients of the order polynomial with respect to three of the six fundamental bases.

Theorem 5.7.1 *Let (P, ω) be a labeled poset of size p and let*

$$\begin{aligned} \Omega(P, \omega; x) &= \sum_{i=1}^p w_i(P, \omega) \binom{x+p-i}{p} = \sum_{i=1}^p e_i(P, \omega) \binom{x}{i} \\ &= \sum_{i=1}^p c_i(P, \omega) \left(\binom{x}{i} \right), \end{aligned} \tag{16}$$

be its order polynomial. Then:

- (i) $w_i(P, \omega)$ equals the number of linear extensions τ of P such that $d(\tau, \omega) = i - 1$;
- (ii) $e_i(P, \omega)$ equals the number of surjective (P, ω) -partitions with largest part $\leq i$;
- (iii) $\Omega(P, \omega; i)$ equals the number of (P, ω) -partitions with largest part $\leq i$;
- (iv) $c_i(P, \omega) = (-1)^{p-i} e_i(P, \bar{\omega})$;
- (v) $\Omega(P, \omega; -i) = (-1)^p \Omega(P, \bar{\omega}, i)$.

Proof: (i), (ii) and (iii) have already been stated in sections 1.2 and 1.4. The only statements that remain to be proved are (iv) and (v). Now, for any linear extension τ of P we have that $d(\tau, \omega) = p - 1 - d(\tau, \bar{\omega})$. Hence from (i) there follows that $w_i(P, \omega) = w_{p-i+1}(P, \bar{\omega})$ for $i = 1, \dots, p$. Hence we have that

$$\begin{aligned}
 \Omega(P, \omega; -n) &= \sum_{i=1}^p w_i(P, \omega) \binom{-n + p - i}{p} \\
 &= \sum_{i=1}^p w_i(P, \bar{\omega}) \binom{i - n - 1}{p} \\
 &= (-1)^p \Omega(P, \bar{\omega}; n),
 \end{aligned}$$

as desired; (iv) now follows easily from (v). $\square$

We should remark that statement (v) in Theorem 5.7.1 is sometimes known as the reciprocity theorem for order polynomials. We can now state the main result of this section.

Theorem 5.7.2 *Let (P, ω) be a labeled poset of size p and let*

$$\Omega(P, \omega; x) = \sum_{i=1}^p w_i(P, \omega) \binom{x + p - i}{p} = \sum_{i=1}^p e_i(P, \omega) \binom{x}{i} = \sum_{i=1}^p c_i(P, \omega) \left(\binom{x}{i} \right),$$

be its order polynomial. Then the following are equivalent:

- (i) $\{w_1(P, \omega), \dots, w_p(P, \omega)\}$ is a PF sequence;
- (ii) $\{e_1(P, \omega), \dots, e_p(P, \omega)\}$ is a PF sequence;

- (iii)) $\{(-1)^{p-1}c_1(P, \omega), (-1)^{p-2}c_2(P, \omega), \dots, c_p(P, \omega)\}$ is a *PF sequence*;
- (iv) $\{\Omega(P, \omega; i)\}_{i \in \mathbb{N}}$ is a *PF sequence*;
- (v) $\{(-1)^p \Omega(P, \omega; -i)\}_{i \in \mathbb{N}}$ is a *PF sequence*.

Proof: By the combinatorial interpretation of $w_1(P, \omega), \dots, w_p(P, \omega)$ there follows that $\Omega(P, \omega; x) \in PF_1 \left[\binom{x+p-i}{p} \right]$, so the result follows immediately from Theorem 4.6.5 $\square$.

Note that, by (v) of Theorem 5.7.1, the preceding theorem implies in particular that the poset conjecture holds for (P, ω) if and only if it holds for $(P, \overline{\omega})$.

It should be noted that the most non-trivial implication in Theorem 5.7.2 is that (iv) $\Rightarrow$ (i). Since a great deal of labor and deep theorems in analysis are involved in the proof of this implication it is reasonable to expect that (given a specific labeled poset (P, ω)) proving (iv) should be easier than proving (i). This is also confirmed by the fact that the combinatorial interpretation of $\Omega(P, \omega; i)$ is much simpler than that of $w_i(P, \omega)$ or even of $e_i(P, \omega)$; this should make it easier to try to interpret combinatorially the determinants involved in proving (iv) than those relative to (i) or (ii). It should also be noted that more consequences of (and hence necessary conditions for) the poset conjecture are obtained from Theorem 5.7.2 than just Conjectures 1.1, 1.2, 2.1 and 2.2. In particular there follows from (iv) that the sequence $\{\Omega(P, \omega; i)\}_{i \in \mathbb{N}}$ is PF_2 if the poset conjecture holds for (P, ω) . As we will see in chapter 7 the preceding sequence actually is PF_2 for all labeled posets. Virtually any minor involved in the definition of a *PF* sequence can be used to obtain from (iv) new necessary conditions for the poset conjecture. Many of these minors will be shown to have indeed nonnegative determinant for all labeled posets (P, ω) in chapter 7. Many more, however, remain in general undecided. The fact that many minors are known to have nonnegative determinant for the sequence in (iv) while not one (non-trivial) minor with the same property is known for the sequence in (i) (for

example, we don't even know whether $\{w_i(P, \omega)\}_{i=1, \dots, p}$ is PF_2 or not) is further evidence that a solution to the poset conjecture (whether positive or negative) will probably be found by analyzing further the sequence $\{\Omega(P, \omega; i)\}_{i \in \mathbb{N}}$. Finally, Theorem 5.7.2 prompts again a question that we had already posed in general at the end of section 2.2; we have seen in this chapter four general classes of labeled posets for which the equivalent conditions of Theorem 5.7.2 hold. Is there a combinatorial interpretation of the nonnegative determinants involved in any of these four cases? Maybe from one such interpretation one would be led to discover a general one. We will say something about this problem in chapter 7.

Chapter 6

Applications to Enumerative Combinatorics

6.1 Introduction

In this chapter we present several applications to enumerative combinatorics of the general theory developed in chapters 2 through 4. In all cases the applications consist in showing that certain combinatorially defined sequences are PF and hence in particular log-concave and unimodal. Some of the applications that we present improve on already existing results either by generalizing them or by providing a simpler and shorter proof.

The sections in this chapter are pretty much independent with one another and they can be read in any order, except for section 6.7 that uses some results from section 6.6.

In the next section we present some polynomials having only real zeros that arise from symmetric functions and especially from Jack polynomials. The main result of the section is a generalization of Theorem 5.3.1. In the third section we consider some PF sequences arising from the zeta polynomials of two infinite families of posets. In section 6.4 we present an analogue of the well known fact that the signless Stirling numbers of the first kind are a PF sequence. In the fifth section we give a very simple proof of a generalization of a concavity result for associated Lah numbers originally proved some years ago. In section 6.6 some sequences obtained from the enumeration of Stirling

permutations are shown to be PF , thus providing another vast generalization of the fact that Eulerian numbers form PF sequences. In section 6.7, we give very simple and direct proofs of the PF property for some sequences of associated Stirling numbers. Finally, in section 6.8 we prove some log-concavity results for sequences arising from the enumeration of colorings of a graph.

6.2 PF sequences arising from symmetric functions and Jack polynomials

In this section we use the theory of symmetric functions. We will use the terminology and notations of [38], Chap.I. In particular, for any partition λ , we denote by h_λ (respectively e_λ , p_λ , m_λ , and s_λ) the *complete homogeneous* (respectively, *elementary*, *power sum*, *monomial*, and *Schur*) symmetric functions.

In section 5.2 we have seen that $s_\lambda(n) \stackrel{\text{def}}{=} s_\lambda(\underbrace{1, \dots, 1}_n, 0, 0, \dots)$ is a polynomial function of n of degree $|\lambda|$. More generally, we have the following fact.

Proposition 6.2.1 *For every symmetric function $a \in \Lambda$ there exists a polynomial $a[x] \in \mathbb{Z}[x]$ such that*

$$a[n] = a(\underbrace{1, \dots, 1}_n, 0, 0, \dots),$$

for all $n \in \mathbb{P}$.

Proof: Clearly, for all $n \in \mathbb{P}$ we have that $p_\lambda(\underbrace{1, \dots, 1}_n, 0, 0, \dots) = n^{l(\lambda)}$. Hence the proposition holds for power sum symmetric functions. But every $a \in \Lambda$ can be expressed as a linear combination of a finite number of p_λ 's and the thesis follows. $\square$

In section 5.3 we have seen that $s_\lambda[x] \in PF \left[\binom{x+|\lambda|-i}{|\lambda|} \right]$ for all partitions λ . It would be interesting to know which symmetric functions $a \in \Lambda$ have the property that $a[x] \in PF \left[\binom{x+d-i}{d} \right]$ where $d = \deg(a[x])$. For example, by what

we have proved in section 5.3 we can see that the following result holds.

Theorem 6.2.2 *For any partition λ we have that $s_\lambda[x], e_\lambda[x], h_\lambda[x] \in PF \left[\binom{x+|\lambda|-i}{|\lambda|} \right]$.*

Proof By Theorem 5.3.1 $s_\lambda[x]$ satisfies the hypotheses of Theorem 4.4.1. Hence any finite product $s_{\lambda_1}[x], \dots, s_{\lambda_n}[x]$ also satisfies those hypotheses and is therefore in $PF \left[\binom{x+d-i}{d} \right]$, where $d = |\lambda_1| + \dots + |\lambda_n|$, and the assertion follows. $\square$

It should be noted that, given two partitions μ, λ with $\mu \subseteq \lambda$, the arguments and definitions in section 5.2 can be generalized in a natural way to yield the result that $s_{\lambda/\mu}[x]$ is the order polynomial of a certain labeled poset $(P_{\lambda/\mu}, \omega)$ with $|P_{\lambda/\mu}| = |\lambda| - |\mu|$. Therefore it is a consequence of the poset conjecture that $s_{\lambda/\mu}[x] \in PF \left[\binom{x+|\lambda|-|\mu|-i}{|\lambda|-|\mu|} \right]$ and this is also an open problem.

In this section we generalize the result that $s_\lambda[x] \in PF \left[\binom{x+|\lambda|-i}{|\lambda|} \right]$ in another direction, by showing that, for each partition λ , there is a countable family of distinct polynomials in $PF \left[\binom{x+|\lambda|-i}{|\lambda|} \right]$ of which $s_\lambda[x]$ is only one. To state and prove this more general result we need some terminology from the theory of Jack polynomials. We will say only a few words here about this rich and fascinating subject and we refer the reader to [39], [31], [27] and [66] for further information. The fastest way to define Jack polynomials is probably the following. Let α be an indeterminate and define a scalar product on $\Lambda \otimes_{\mathbf{Q}} \mathbf{Q}(\alpha)$ by

$$\langle p_\lambda, p_\mu \rangle \stackrel{\text{def}}{=} \delta_{\lambda\mu} z_\lambda \alpha^{l(\lambda)}$$

where $z_\lambda \stackrel{\text{def}}{=} (1^{m_1} 2^{m_2} \dots) m_1! m_2! \dots$ if λ has exactly m_i parts equal to i , ($i = 1, 2, \dots$). Then it is not difficult to show (see, e. g. [66], Proposition 1.1) that the following result holds.

Proposition 6.2.3 *There is a unique family of symmetric functions $J_\lambda(x; \alpha) \in \Lambda \otimes \mathbf{Q}(\alpha)$ indexed by partitions λ such that:*

- (i) $\langle J_\lambda, J_\mu \rangle = 0$ if $\lambda \neq \mu$;

(ii) if $J_\lambda = \sum_\mu v_{\lambda\mu}(\alpha) m_\mu$ then $v_{\lambda\mu}(\alpha) = 0$ unless $|\lambda| = |\mu|$ and $\mu \leq \lambda$ (where $\leq$ denotes reverse lexicographic ordering);

(iii) if $|\lambda| = n$, then $v_{\lambda(1^n)}(\alpha) = n!$.

The functions $J_\lambda(x; \alpha)$ uniquely defined by the preceding proposition are called *Jack polynomials*. From the preceding result the following follows immediately.

Theorem 6.2.4 *For every partition λ we have that $J_\lambda(x; 1) = H_\lambda s_\lambda(x)$, where $H_\lambda \stackrel{\text{def}}{=} \prod_{(i,j) \in P_\lambda} h(i, j)$. $\square$*

So Jack polynomials essentially reduce to ordinary Schur functions when $\alpha = 1$. Since Jack polynomials are linear combinations of symmetric functions with coefficients in $\mathbf{Q}(\alpha)$ there follows from Proposition 6.2.1 that there exists a polynomial $J_\lambda[x; \alpha] \in \mathbf{Q}(\alpha)[x]$ such that

$$J_\lambda[n; \alpha] = J_\lambda(\underbrace{1, \dots, 1}_n, 0, 0, \dots; \alpha) \quad (1)$$

for all $n \in \mathbf{P}$. The exact form of this polynomial was first conjectured by Macdonald and later proved by Stanley. The following result appears in [66], Theorem 5.4.

Theorem 6.2.5 *For every partition λ and $n \in \mathbf{P}$ we have that*

$$J_\lambda(\underbrace{1, \dots, 1}_n, 0, 0, \dots; \alpha) = \prod_{(i,j) \in P_\lambda} (n + \alpha(j-1) - (i-1)). \quad (2)$$

We can now state and prove the main result of this section.

Theorem 6.2.6 *For every partition λ and $r \in \mathbf{Z} \setminus \{0\}$ we have that $J_\lambda \left[x; \frac{1}{r} \right] \in PF \left[\binom{x+|\lambda|-i}{|\lambda|} \right]$, where $J_\lambda[x; \alpha]$ is defined by (1).*

Proof: Fix a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots)$ and let $\alpha_{ij} = (i-1) - \frac{1}{r}(j-1)$ for $(i, j) \in \mu$. Then, from (1) and (2) we have that

$$\lambda \left(J_\mu \left[x; \frac{1}{r} \right] \right) = \begin{cases} -\frac{\mu_1 - 1}{r} & \text{if } r > 0, \\ 0 & \text{if } r < 0, \end{cases}$$

$$\Lambda \left(J_\mu \left[x; \frac{1}{r} \right] \right) = \begin{cases} l(\mu) - 1 & \text{if } r > 0, \\ \alpha_{i_0, j_0} & \text{if } r < 0, \end{cases}$$

for some $(i_0, j_0) \in \mu$. Therefore

$$\left[\lambda \left(J_\mu \left[x; \frac{1}{r} \right] \right), -1 \right] \cap \mathbf{Z} = \begin{cases} \{ \alpha_{1j} : 2 \leq j \leq \mu_1, j \equiv 1 \pmod{r} \} & \text{if } r > 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

$$\left[0, \Lambda \left(J_\mu \left[x; \frac{1}{r} \right] \right) \right] \cap \mathbf{Z} = \begin{cases} \{ \alpha_{i1} : i \in [l(\mu)] \} & \text{if } r > 0, \\ \{ \alpha_{ik} : i \in [i_0] \} \cup \{ \alpha_{1j} : j \in [k], j \equiv 1 \pmod{r} \} & \text{if } r < 0, \end{cases}$$

where $k \stackrel{\text{def}}{=} -r \left\lfloor \frac{j_0-1}{-r} \right\rfloor + 1$. So all the hypotheses of Theorem 4.4.1 are satisfied and the thesis follows. $\square$

Note that, by (1) and Theorem 6.2.5, $J_\lambda[x; \alpha]$ is a polynomial in x and α with coefficients in $\mathbf{Z}$. Therefore there exists (see, e. g. , [63], Cor.4.3.1) a polynomial $U_\lambda(z; \alpha) \in \mathbf{Z}[z, \alpha]$ such that

$$\sum_{n \geq 0} J_\lambda[n; \alpha] z^n = \frac{U_\lambda(z; \alpha)}{(1-z)^{|\lambda|+1}}, \quad (3)$$

as formal power series in $\mathbf{Z}[\alpha][[z]]$. By Theorems 2.3.3 and 4.6.2 we can therefore restate the last result in the following form.

Theorem 6.2.7 *For every partition λ and $r \in \mathbf{Z} \setminus \{0\}$ we have that:*

- (i) $U_\lambda \left(z; \frac{1}{r} \right)$ has nonnegative coefficients and only real zeros;
- (ii) $\left\{ J_\lambda \left[n; \frac{1}{r} \right] \right\}_{n \in \mathbf{N}}$ is a PF sequence. $\square$

We should remark that, as easy examples show, the polynomials $U_\lambda \left(z; \frac{1}{r} \right)$ have (nonnegative) integer coefficients, for all λ , only if $r = 1, -1$. If $r = 1$ then $J_\lambda[n; 1] = H_\lambda s_\lambda[n]$ and we have a nice combinatorial interpretation of $s_\lambda[n]$ for $n \in \mathbf{N}$ and of the coefficients of $U_\lambda(z; 1)$ (see Theorems 5.2.3 and 5.7.1). However, I don't know of any combinatorial interpretation of $J_\lambda[n; -1]$ and of the coefficients of $U_\lambda(z, -1)$. The first few polynomials $U_\lambda(z; -1)$ are given in

Table 3. For the reader interested in this last problem we state below one further property of the polynomials $U_\lambda(z; \alpha)$ which suggests that their coefficients might enumerate some kind of permutations.

Proposition 6.2.8 *For every partition λ we have that*

$$U_\lambda(1; \alpha) = n! ,$$

where $n = |\lambda|$. In particular, $U_\lambda(1; \alpha)$ does not depend on α .

Proof: It follows from (2) of Theorem 2.3.3 and the definition of $U_\lambda(z; \alpha)$ that $\frac{1}{n!}U_\lambda(1; \alpha)$ equals the leading coefficient of $J_\lambda[x; \alpha]$, where $n = |\lambda| = \deg(J_\lambda)$. The thesis now follows from (1) and (2). $\square$

For $r \neq 1, -1$ the polynomials $U_\lambda\left(z; \frac{1}{r}\right)$ have in general only rational (non-negative) coefficients. It would be interesting to know the minimum common denominator d_λ of these coefficients and then see if the coefficients of $d_\lambda U_\lambda\left(z; \frac{1}{r}\right)$ have a nice combinatorial interpretation.

We close this section by showing that the results of Theorem 6.2.7 are best possible.

Proposition 6.2.9 *Let $\alpha \in \mathbf{R}$ and suppose that $U_\lambda(q; \alpha)$ has nonnegative coefficients for all partitions λ . Then either $\alpha = 0$ or $\alpha = \frac{1}{r}$ for $r \in \mathbf{Z} \setminus \{0\}$.*

Proof: We will actually prove a stronger statement. Let $\lambda = (l)$, $l \in \mathbf{P}$. Then, by (2), $J_{(l)}[n; \alpha] = \prod_{j=1}^l (n + \alpha(j-1))$. Therefore, by (3), we have that

$$U_{(l)}(q; \alpha) = \sum_{i=0}^{l-1} (-\alpha)^i (1-q)^i s(l, l-i) A_{l-i}(q), \quad (4)$$

where $A_j(q)$ denotes the j -th Eulerian polynomial and $s(l, l-i)$ is a Stirling number of the first kind. Since $q|A_j(q)$ for all $j \geq 1$ there follows that $q|U_{(l)}(q; \alpha)$; so dividing both sides of (4) by q and then letting $q = 0$ we obtain that the coefficient of q in $U_{(l)}(q; \alpha)$ is given by

$$\sum_{i=0}^{l-1} (-\alpha)^i s(l, l-i) = \prod_{j=1}^{l-1} (1 + j\alpha). \quad (5)$$

The formula (4) also implies (by the symmetry of the Eulerian polynomials) (see, e. g. , [12], pp.241-242) that $U_{(l)}(q; -\alpha) = q^{l+1}U_{(l)}\left(\frac{1}{q}; \alpha\right)$. Hence the coefficient of q^l in $U_{(l)}(q; \alpha)$ is given by

$$\prod_{j=1}^{l-1} (1 - j\alpha). \quad (6)$$

(5) and (6) show that the coefficients of $U_{(l)}(q; \alpha)$ can only be nonnegative (for all $l \geq 1$) if $\alpha = \frac{1}{r}$ for some $r \in \mathbf{Z} \setminus \{0\}$ or if $\alpha = 0$. $\square$

Note that, by (1) and (2), we have that $J_\lambda[x; 0] = e_{\lambda'}[x]$ (where λ' denotes the conjugate partition of λ) and hence, by Theorem 6.2.2, $J_\lambda[x; 0] \in PF\left[\begin{pmatrix} x+|\lambda|-i \\ |\lambda| \end{pmatrix}\right]$.

6.3 Zeta polynomials of partially ordered sets

In this section we use the theory of the zeta polynomial of a partially ordered set. We use the notation and terminology of [63], sections 3.11 and 3.12.

In section 1.3 we stated the poset conjecture for natural labelings in the following form.

Conjecture 3 (*The distributive lattice conjecture*): *Let D be a (finite) distributive lattice and let, for $s \in \mathbf{N}$, $c_s(D)$ be the number of chains of length s from $\hat{0}$ to $\hat{1}$ in D . Then the polynomial $C(D, z) = \sum_{s \geq 0} c_s(D)z^s$ has only real zeros.*

An equivalent way of stating this conjecture is to state it in terms of the zeta polynomial of a partially ordered set. In fact, we have the following well known theorem about the zeta polynomial of a graded poset with $\hat{0}$ and $\hat{1}$ whose proof can be found e. g. in [63], Ex. 67, a,b, chapter 3.

Theorem 6.3.1 *Let P be a graded poset of rank m with $\hat{0}$ and $\hat{1}$. Then*

$$Z(P; x) = \sum_{i=1}^m c_i(P) \binom{x}{i} = \sum_{i=1}^m \omega_{i-1}(P) \binom{x+m-i}{m}, \quad (7)$$

where $c_i(P)$ equals the number of chains of length i from $\hat{0}$ to $\hat{1}$ in P and

$$\omega_i(P) \stackrel{\text{def}}{=} \sum_{S \subseteq [m-1] \mid |S|=i} \beta(P, S).$$

Using this result we see that we can state Conjecture 3 in the following equivalent way:

Conjecture 6 (*The distributive lattice conjecture*): *Let D be a (finite) distributive lattice, then $Z(D; x) \in PF \left[\binom{x}{i} \right]$.*

In fact, using Theorem 3.2.1 and the fact that, by (7), $Z(D; x) \in PF_1 \left[\binom{x+m-i}{m} \right]$ if D is a distributive lattice of rank m , (see, e. g. [63], Thm.3.12.1) we can state Conjecture 6 in yet another equivalent form:

Conjecture 7 (*The distributive lattice conjecture*): *Let D be a (finite) distributive lattice, then $Z(D; x) \in PF \left[\binom{x+m-i}{m} \right]$ where m is the rank of D .*

Since the zeta polynomial is a well defined object for any graded poset with $\hat{0}$ and $\hat{1}$ Conjectures 6 and 7 immediately suggest the following problems:

- (i) Characterize the class $\mathcal{A}$ of all graded, bounded posets P for which $Z(P; x) \in PF \left[\binom{x}{i} \right]$.
- (ii) Characterize the class $\mathcal{B}$ of all graded, bounded posets P for which $Z(P; x) \in PF \left[\binom{x+m-i}{m} \right]$, where $m = \text{rank}(P)$.

That problems (i) and (ii) have a combinatorial significance can be seen from Theorem 6.3.1 which, together with Theorem 4.6.2, immediately implies the following result, where $c_i(P)$ and $\omega_i(P)$ have the same meaning as in Theorem 6.3.1.

Theorem 6.3.2 *Let P be a graded poset of rank m , with $\hat{0}$ and $\hat{1}$, such that $Z(P; x) \in PF \left[\binom{x+m-i}{m} \right]$. Then we have that:*

- (i) $\{Z(P; n)\}_{n \in N}$ is a PF sequence;
- (ii) $\{c_1(P), \dots, c_m(P)\}$ is a PF sequence;
- (iii) $\{\omega_0(P), \dots, \omega_{n-1}(P)\}$ is a PF sequence.

In particular, the polynomials $\sum_{i=1}^m c_i(P)z^i$ and $\sum_{i=1}^m \omega_{i-1}(P)z^i$ have only real zeros. $\square$

The sequences in (i) and (ii) always have, by definition, an important combinatorial significance. However, also the sequence in (iii) has, even if P is not a distributive lattice, a combinatorial interpretation, for example if P is a CL-shellable poset (see, e. g. , [6], [7], and [8]). Note that, by Theorem 3.2.1, we have that $\mathcal{B} \subseteq \mathcal{A}$. It should also be noted, however, that while Conjectures 6 and 7 are equivalent it is not true that $\mathcal{A} = \mathcal{B}$. This is because $Z(P; x) \in PF_1 \left[\binom{x+m-i}{m} \right]$ if P is a distributive lattice of rank m but this is not true in general, as easy examples show. Nonetheless, as we have remarked above, very large classes $\mathcal{C}$ of posets are known for which the coefficients of the zeta polynomial are non-negative with respect to the basis $\left\{ \binom{x+m-i}{m} \right\}_{i=0, \dots, m}$ and for which, therefore, $\mathcal{A} \cap \mathcal{C} = \mathcal{B} \cap \mathcal{C}$. This fact suggests that, if Conjectures 6 and 7 are true, then they might actually be true for a larger class of posets than just distributive lattices. We will return on this problem in chapter 7.

In the last chapter we have seen several families of posets for which the poset conjecture for natural labelings hold. In other words, we have seen families of distributive lattices whose zeta polynomial is in $PF \left[\binom{x+m-i}{m} \right]$ for the appropriate m . In this section we look at two classes of posets that are *not* distributive lattices but are nonetheless in class $\mathcal{B}$. Besides yielding interesting combinatorial sequences that are PF these families might one day suggest the common feature that makes their zeta polynomials be in $PF \left[\binom{x+m-i}{m} \right]$.

Let $k, m \in \mathbf{P}$. A partition π of $[m]$ is called *non-crossing* if the following property holds: if $B, B' \in \pi$, $a, c \in B$, $b, d \in B'$ and $a < b < c < d$ then $B = B'$. A partition of $[km]$ is called *k-divisible* if every block of it has cardinality divisible by k . We will denote by $T_{k,m}$ the poset of all non-crossing k -divisible partitions of $[km]$, ordered by refinement. $T_{k,m}$ is a graded poset of rank $m - 1$ whose rank function is $m - |\pi|$ for $\pi \in T_{k,m}$. Note that $T_{k,m}$ always has a top

element and has a minimum element if and only if $k = 1$. The following result appears in [16], Corollary 4.4.

Theorem 6.3.3 *For $k, m \in \mathbf{P}$ we have that*

$$Z(T_{k,m}; x + 1) = \frac{1}{m} \binom{(kx + 1)m}{m - 1}.$$

Using the preceding formula and the general theory of chapter 4 it is now not hard to prove that the $\hat{T}_{k,m}$'s belong to $\mathcal{B}$. In what follows, for any poset P , we denote by $\hat{P}$ the poset obtained by adjoining a bottom element to P .

Theorem 6.3.4 *Let $T_{k,m}$ be the poset of k -divisible non-crossing partitions of $[mk]$. Then $Z(\hat{T}_{k,m}; x) \in PF \left[\binom{x+m-i}{m} \right]$.*

Proof: Theorem 6.3.3 implies that the polynomial $Z(T_{k,m}; x + 1)$ has all its zeros (real and) contained in $[-1, 0]$ and has nonnegative coefficients. By Theorem 4.4.4 this implies that $Z(T_{k,m}; x + 1) \in PF \left[\binom{x+m-1-i}{m-1} \right]$. Now, it is well known (see, e. g. , [15], Corollary 1.2.5) that, for any poset P , we have

$$\Delta Z(\hat{P}; x) = Z(P; x + 1). \quad (8)$$

Therefore, since $Z(\hat{T}_{k,m}; 0) = 0$, we conclude from Proposition 4.2.3 that $Z(\hat{T}_{k,m}; x) \in PF \left[\binom{x+m-i}{m} \right]$, as desired. $\square$

We now describe a second general class of posets that are in $\mathcal{B}$. Let $m \geq 1$, a *2-partition* of $[m]$ is a set $\pi = \{(A_1, B_1), \dots, (A_k, B_k)\}$ such that $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_k\}$ are partitions of $[m]$ and $|A_i| = |B_i|$ for $i = 1, \dots, k$. A 2-partition of $[m]$ as above is called *non-crossing* if $\{A_1, \dots, A_k\}$ is a non-crossing partition. We denote by T_m^2 the poset of all non-crossing 2-partitions of $[m]$ ordered by refinement (i. e. π as above is a refinement of $\sigma = \{(C_1, D_1), \dots, (C_r, D_r)\}$ if and only if $\{A_1, \dots, A_k\}$ is a refinement of $\{C_1, \dots, C_r\}$ and $\{B_1, \dots, B_k\}$ is a refinement of $\{D_1, \dots, D_r\}$). This makes T_m^2 a graded poset of rank $m - 1$ whose rank function is $m - |\pi|$ for $\pi \in T_m^2$. Note that T_m^2 always has a top

element but has never a bottom element, unless $m = 1$. The next result gives the zeta polynomial of the poset T_m^2 and a proof can be found in [16], Theorem 5.3.

Theorem 6.3.5 *For $m \in \mathbb{P}$ we have that*

$$Z(T_m^2; x + 1) = (mx + 1)^{m-1}.$$

Using this result and Theorem 4.4.4 we can see almost “by inspection” that the posets $\hat{T}_m^2$ belong to $\mathcal{B}$.

Theorem 6.3.6 *Let T_m^2 be the poset of non-crossing 2-partitions of $[m]$. Then $Z(\hat{T}_m^2; x) \in PF \left[\binom{x+m-i}{m} \right]$.*

Proof: Theorem 6.3.5 implies that the polynomial $Z(T_m^2; x + 1)$ has all its zeros contained in $[-1, 0]$ and has nonnegative coefficients and therefore, by Theorem 4.4.4, is in $PF \left[\binom{x+m-1-i}{m-1} \right]$. By (8) this implies that $\Delta Z(\hat{T}_m^2; x) \in PF \left[\binom{x+m-1-i}{m-1} \right]$ and, since $Z(\hat{T}_m^2; 0) = 0$, by Proposition 4.2.3 we conclude that $Z(\hat{T}_m^2; x) \in PF \left[\binom{x+m-i}{m} \right]$, as desired. $\square$

6.4 Functions of a finite set into itself

In this section we prove that certain combinatorially defined sequences arising from the enumeration of functions from a finite set into itself are PF sequences. Our results are analogue of the well known fact (see, e. g. [63], Prop.1.3.4) that the sequences $\{c(n, i)\}_{i=1, \dots, n}$ are PF , where $c(n, i)$ denotes a signless Stirling number of the first kind.

Let $f : [n] \rightarrow [n]$. The *functional digraph* of f is the digraph $Dig(f) \stackrel{\text{def}}{=} ([n], A)$ where $(i, j) \in A$ if and only if $f(i) = j$. In what follows we will freely identify f with $Dig(f)$. It is well known (see, e. g. , [26], Proposition 3.3.11, p. 175, or [12], p. 69) (and easy to see) that a digraph D on the vertex set $[n]$ is

the functional digraph of some function $f : [n] \rightarrow [n]$ if and only if the following two conditions are satisfied:

- (i) every connected component of D is a directed cycle with some tree incident to it;
- (ii) every tree as in (i) intersects the cycle in exactly one vertex and all its edges are directed towards this unique vertex.

Fix now $n \in \mathbf{P}$. For $k \in [n]$ we denote by $b(n, k)$ the number of functions $f : [n] \rightarrow [n]$ with exactly k cycles (or, equivalently such that $\text{Dig}(f)$ has exactly k connected components). As can be expected, these numbers are connected to the signless Stirling numbers of the first kind $c(n, k)$ by a relatively simple relation. The following result is known (see, e. g. [26], exercise 3.3.28, p. 192) and is also easy to prove combinatorially.

Proposition 6.4.1 *For $k \in [n]$ we have that*

$$b(n, k) = \sum_{i=1}^n \binom{n-1}{i-1} n^{n-i} c(i, k). \quad (9)$$

Proof: To choose an $f : [n] \rightarrow [n]$ with exactly k cycles we may first choose a tree on $[n+1]$ and then choose a permutation of the vertices that are adjacent to $n+1$ into exactly k cycles. Each f is obtained exactly once in this way. Formula (9) now follows from the fact that there are exactly $\binom{n-1}{i-1} n^{n-i}$ trees on $[n+1]$ in which $n+1$ has degree i (see, e. g. , [12], Theorem F, p. 65). $\square$

Using the preceding proposition and Theorem 2.6.1 we can now give an easy and elegant proof of the main result of this section.

Theorem 6.4.2 *For $n \in \mathbf{P}$ the sequence $\{b(n, k)\}_{k=1, \dots, n}$ is PF. In particular, the polynomial $\sum_{k=1}^n b(n, k) z^k$ has only real zeros.*

Proof: Let $B_n(z) \stackrel{\text{def}}{=} \sum_{k=1}^n b(n, k) z^k$. Then from (9) there follows that

$$B_n(z) = \sum_{i=1}^n \binom{n-1}{i-1} n^{n-i} \langle z \rangle_i. \quad (10)$$

Now, the sequence $\left\{ \binom{n-1}{i-1} n^{n-i} \right\}_{i=1, \dots, n}$ is a PF sequence because it is nonnegative and because $\sum_{i=1}^n \binom{n-1}{i-1} n^{n-i} z^i = z(n+z)^{n-1}$ has only real zeros. Therefore $B_n(z) \in PF[\langle z \rangle_i]$ and this, by Theorem 2.6.1, implies that $B_n(z) \in PF[z^i]$, as desired. $\square$

6.5 Associated Lah numbers

The associated Lah numbers have been first introduced and studied in 1979 in [1] and [2] where they arose from a problem in statistics. The main result of [2] is that certain sequences of associated Lah numbers are PF . In this section we prove a generalization of this result (Theorem 6.5.2) and we give a combinatorial interpretation of the numbers in these sequences. We then show how the associated Lah numbers are closely related by a “combinatorial reciprocity theorem” (equation (13) below) in the sense of [59], to some polynomials arising from the calculus of finite differences.

By a *board* we mean a set of the form $[i] \times [k]$ where $i, k \in \mathbf{P}$, we call the subset $\{j\} \times [k]$ ($j \in [i]$) the j -th row of the board $[i] \times [k]$. Fix now integers $k, n, i \in \mathbf{P}$. We let $P_k^*(n, i)$ (respectively $L_k^*(n, i)$) be the number of n -subsets of $[i] \times [k]$ (respectively n -multisets on $[i] \times [k]$) that have non empty intersection with each row of $[i] \times [k]$. So, for example, $P_k^*(n, i) = L_k^*(n, i) = 0$ unless $i \in [n]$, $P_k^*(n, n) = L_k^*(n, n) = k^n$ and $P_k^*(n, 1) = \binom{k}{n}$, $L_k^*(n, 1) = \left(\binom{k}{n} \right)$ for $k, n \in \mathbf{P}$. From the combinatorial definitions of $L_k^*(n, i)$ and $P_k^*(n, i)$ the following result follows almost immediately.

Proposition 6.5.1 *For $n \in \mathbf{P}$ and $i \in [n]$ we have that $P_k^*(n, i)$ and $L_k^*(n, i)$ are polynomial functions of k . Furthermore, we have that, for $n \in \mathbf{P}$,*

$$\binom{y \ x}{n} = \sum_{i=1}^n P_y^*(n, i) \binom{x}{i}, \quad (11)$$

$$\left(\binom{y \ x}{n}\right) = \sum_{i=1}^n L_y^*(n, i) \binom{x}{i}, \quad (12)$$

as polynomials in $\mathbf{R}[x, y]$; in particular,

$$P_y^*(n, i) = (-1)^n L_{-y}^*(n, i) \quad (13)$$

as polynomials in $\mathbf{R}[y]$.

Proof: We only prove (11) the proof of (12) being completely analogous. Fix $n \in \mathbf{N}$, $y \in \mathbf{P}$. If $x \in \mathbf{P}$ then we can prove (11) by noting that we can choose an n -subset of $[x] \times [y]$ by first choosing rows $1 \leq j_1 < \dots < j_i \leq x$ and then choosing an n -subset of $\{j_1, \dots, j_i\} \times [y]$ with non-empty intersection with every row. This proves (11) for $x \in \mathbf{P}$. Equivalently, we have that

$$\binom{y \ x}{n} = \sum_{i=0}^x P_y^*(n, i) \binom{x}{i} \quad (14)$$

for all $x \in \mathbf{N}$. By a well known inversion formula (see, e. g. , [63], pp. 66-67) (14) is equivalent to

$$P_y^*(n, x) = \sum_{i=0}^x (-1)^{x-i} \binom{x}{i} \binom{i \ y}{n}, \quad (15)$$

for all $x \in \mathbf{N}$. This shows that $P_y^*(n, i)$ is a polynomial function of y for every fixed $n, i \in \mathbf{N}$. Therefore both the left and right hand sides of equation (11) are polynomials in $\mathbf{R}[x, y]$ that agree on all $x, y \in \mathbf{N}$ and hence are equal as polynomials. The proof of (12) is analogous using the inversion formula

$$L_y^*(n, x) = \sum_{i=0}^x (-1)^{x-i} \binom{x}{i} \left(\binom{i \ y}{n}\right) \quad (16)$$

valid for $n, x, y \in \mathbf{N}$. Finally, (13) is obtained substituting “ $-y$ ” for “ y ” in (12) and comparing with (11). $\square$

The polynomials $P_k(n, i) \stackrel{\text{def}}{=} \frac{n!}{i!} P_k^*(n, i)$ are used often in the calculus of finite differences and have been studied extensively in [32], sections 76 and 131, although their combinatorial interpretation is not mentioned there (equation (15) is taken as the definition of the $P_k(n, i)$ in [32]). The numbers

$L_k(n, i) \stackrel{\text{def}}{=} \frac{n!}{i!} L_k^*(n, i)$ are called *associated Lah numbers* and have been first introduced and studied in [2] where they are defined by means of equation (16). They reduce to the ordinary Lah numbers (see, e. g. , [50]) when $k = 1$. The authors in [2], however, seem to be unaware of the connection with the polynomials $P_k(n, i)$ studied in [32] as well as of the combinatorial interpretation of the $L_k^*(n, i)$. The polynomials $L_k(n, i)$ for $1 \leq i \leq n \leq 5$ are given in Table 4.

Using the last proposition and the general theory of chapters 2-4 we can now prove, with almost no effort, the following theorem which is a generalization of the main result in [2] and is also the main result of this section.

Theorem 6.5.2 *For $k, n \in \mathbf{P}$ the sequences $\{L_k^*(n, i)\}_{i=1, \dots, n}$ and $\{P_k^*(n, i)\}_{i=1, \dots, n}$ are PF. In particular, the sequences $\{L_k(n, i)\}_{i=1, \dots, n}$ and $\{P_k(n, i)\}_{i=1, \dots, n}$ are PF.*

Proof: Clearly $\binom{x}{n}, \left(\binom{x}{n}\right) \in PF\left[\binom{x+n-i}{n}\right]$. By Theorem 4.6.4 this implies that $\binom{kx}{n}, \left(\binom{kx}{n}\right) \in PF\left[\binom{x+n-i}{n}\right]$. Hence, by Theorem 2.6.1, $\binom{kx}{n}, \left(\binom{kx}{n}\right) \in PF\left[\binom{x}{i}\right]$ and this, by (11) and (12), proves the first statement. The second statement follows from the first by the definitions of $L_k(n, i)$ and $P_k(n, i)$ and Theorem 2.6.1. $\square$

We close this section by answering a question that presents itself naturally after having looked at Table x, namely: “Do the polynomials $L_x(n, i) \in \mathbf{R}[x]$ have only real zeros?” Though this is the case for $1 \leq i \leq n \leq 5$ it is not so in general. There are many ways of computing effectively the polynomials $L_x(n, i)$ (a recursion for them is given in [2]) but the fastest is probably through the following proposition which also appears in [32].

Proposition 6.5.3 *For $n, i \in \mathbf{P}$ we have that*

$$P_x(n, i) = \sum_{j=i}^n s(n, j) S(j, i) x^j, \quad (17)$$

$$L_x(n, i) = \sum_{j=i}^n c(n, j) S(j, i) x^j, \quad (18)$$

as polynomials in $\mathbb{Q}[x]$ (where $c(n, j)$, $s(n, j)$ and $S(j, i)$ have the same meaning as on page 39)

Proof: Both equations are immediately obtained by expanding $\binom{yx}{n}$ and $\binom{x}{i}$ in (11) and $\left(\binom{yx}{n}\right)$ and $\binom{x}{i}$ in (12) in terms of the basis $\{x^j\}_{j=0,\dots,n}$, then equating the coefficients of x^j and then using a well known inversion formula (see, e. g. ,[63], Prop.1.4.1). $\square$

We can now easily compute that, for example,

$$L_x(9, 5) = 3x^5(x + 1)(2317x^3 + 10283x^2 + 15197x + 7483), \quad (19)$$

which is easily seen not to have only real zeros. The above counterexample also shows that the Hadamard (i. e. componentwise) product of two PF sequences is not in general PF even if one of the two sequences is finite. In fact, $\{c(n, j)\}_{j=1,\dots,n}$ and $\{S(j, i)\}_{j=i,i+1,\dots}$ are both PF sequences for all $n, i \in \mathbb{N}$ because $\sum_{j=1}^n c(n, j)x^j = \langle x \rangle_n$ has only real zeros and, as can be easily shown (see, e. g. [63], p. 34), $\sum_{j \geq i} S(j, i)x^j = \frac{x^i}{(1-x)(1-2x)\dots(1-ix)}$ and this, by Theorem 4.5.3, implies that the infinite sequence $\{S(j, i)\}_{j \geq i}$ is PF . However, (19) shows that the sequence $\{c(9, j)S(j, 5)\}_{j \in \mathbb{N}}$ is *not* PF .

6.6 Stirling permutations

Fix integers $k \geq 1$ and $m_i \geq 1$ for all $i \in \mathbb{P}$. A permutation $a_1, a_2, \dots, a_{m_1+\dots+m_k}$ of the multiset $M_k \stackrel{\text{def}}{=} \{1^{m_1}, 2^{m_2}, \dots, k^{m_k}\}$ is called a *Stirling permutation* if $u < v < w$ and $a_u = a_w$ imply $a_v \geq a_u$. We denote by Q_k the set of all Stirling permutations of M_k . So, for example, $Q_k = S_k$ (the symmetric group on k elements) if $m_1 = \dots = m_k = 1$. Stirling permutations have been first introduced and studied in [21] in the case $m_1 = \dots = m_k = 2$ and later in [20] in the case $m_1 = \dots = m_k$. The reason for their name will become apparent later on.

Given any permutation $\pi = a_1 a_2 \dots a_{m_1+\dots+m_k}$ of M_k a *descent* of π is an

index $j \in [|M_k| - 1]$ such that $a_j > a_{j+1}$. For $0 \leq i \leq |M_k| - 1$ we let $B_{k,i}$ be the number of Stirling permutations of M_k with exactly i descents. So, for example, if $m_1 = \dots = m_k = 1$, then $B_{k,i}$ is just the i -th Eulerian number $A(k, i)$. Now, it is well known that the Eulerian polynomials $\sum_{i=0}^{k-1} A(k, i)x^i$ have only real zeros (see, e. g. , [12], p.292, Ex.3); in what follows we will prove that the polynomials $\sum_{i=0}^{k-1} B_{k,i}x^i$ have only real zeros for any $m_1, \dots, m_k \geq 1$. Before we can do this, however, we need some elementary properties of the numbers $B_{k,i}$.

Proposition 6.6.1 *Fix integers $k \geq 1$ and $m_1, \dots, m_k \geq 1$. Then we have that $B_{k,i} = 0$ unless $0 \leq i \leq k - 1$ and*

$$B_{k,i} = (m_1 + \dots + m_{k-1} - i + 1)B_{k-1,i-1} + (i + 1)B_{k-1,i}, \quad (20)$$

for $0 \leq i \leq k - 1$ (where $B_{k,i} \stackrel{\text{def}}{=} 0$ if $i < 0$).

Proof: Let $a_1 a_2 \dots a_{|M_k|} \in Q_k$ and let $l \in [k]$. If $a_{j_1} = \dots = a_{j_{m_l}} = l$ then from the definition of a Stirling permutation there follows that $j_1, \dots, j_{m_l-1}$ cannot be descents of π . This proves that $B_{k,i} = 0$ if $k \leq i \leq |M_k| - 1$. To prove (20) note that every permutation $\pi \in Q_k$ is uniquely obtained from a permutation $\sigma = a_1 \dots a_{|M_{k-1}|} \in Q_{k-1}$ either by inserting m_k consecutive k 's between a_j and a_{j+1} for some $j \in [|M_{k-1}| - 1]$ or by adjoining them at the beginning or at the end of σ , (in particular, hence, $|Q_k| = \prod_{i=1}^{k-1} (|M_i| + 1)$). If the k 's are inserted between a_j and a_{j+1} and j is a descent of σ or if they are adjoined to the end of σ then π will have the same number of descents as σ , otherwise π will have one more descent than σ and (20) follows. $\square$

The numbers $B_{k,i}$ above defined are a typical case in which the most convenient generating function to consider for them is not $\sum_{i=0}^{k-1} B_{k,i}x^i$. Instead, using the general theory developed in chapters 2, 3 and 4, the generating function $f_k(x)$ defined by

$$f_k(x) \stackrel{\text{def}}{=} \sum_{i=1}^k B_{k,i-1} \binom{x + |M_k| - i}{|M_k|}. \quad (21)$$

turns out to be much easier to handle. In fact, we can now prove the main result of this section.

Theorem 6.6.2 *For $k \in \mathbf{P}$ and $m_1, \dots, m_k \in \mathbf{P}$ we have that $f_k(x) \in PF \left[\binom{x+|M_k|-i}{|M_k|} \right]$.*

Proof: We proceed by induction on $k \geq 1$. If $k = 1$ then $f_1(x) = \binom{x+m_1-1}{m_1}$ and the thesis is clearly true. So assume that $f_{k-1}(x) \in PF \left[\binom{x+|M_{k-1}|-i}{|M_{k-1}|} \right]$ for some $k \geq 2$. By Theorem 4.3.1 this implies that $x f_{k-1}(x) \in PF \left[\binom{x+|M_{k-1}|+1-i}{|M_{k-1}|+1} \right]$. But, from (20) and (21) we have that

$$\begin{aligned} x f_{k-1}(x) &= x \sum_{i=1}^{k-1} B_{k-1,i-1} \binom{x+|M_{k-1}|-i}{|M_{k-1}|} \\ &= \sum_{i=1}^{k-1} B_{k-1,i-1} \left[i \binom{x+|M_{k-1}|+1-i}{|M_{k-1}|+1} + \right. \\ &\quad \left. + (|M_{k-1}|-i+1) \binom{x+|M_{k-1}|+1-(i+1)}{|M_{k-1}|+1} \right] \\ &= \sum_{i=1}^k B_{k,i-1} \binom{x+|M_{k-1}|+1-i}{|M_{k-1}|+1}. \end{aligned}$$

Hence the sequence $\{B_{k,0}, \dots, B_{k,k-1}\}$ is PF and therefore $f_k(x) \in PF \left[\binom{x+|M_k|-i}{|M_k|} \right]$, as desired. $\square$

Using Theorem 4.6.5 we can now immediately deduce from the last theorem the following result.

Theorem 6.6.3 *For $k \in \mathbf{P}$ and $m_1, \dots, m_k \in \mathbf{P}$ we have that*

- (i) *the sequence $\{B_{k,i}\}_{i=0,\dots,k-1}$ is PF (in particular, the polynomial $\sum_{i=0}^{k-1} B_{k,i} x^i$ has only real zeros);*
- (ii) *the sequence $\{f_k(n)\}_{n \in \mathbf{N}}$ is PF ;*
- (iii) *the sequence $\{(-1)^{|M_k|} f_k(-n)\}_{n \in \mathbf{N}}$ is PF ;*
- (iv) *$f_k(x) \in PF \left[\binom{x}{i} \right]$;*

$$(v) \quad f_k(x) \in PF \left[(-1)^{|M_k|-i} \binom{x}{i} \right].$$

Proof: By the definition (21) we know that $f_k(x) \in PF_1 \left[\binom{x+|M_k|-i}{|M_k|} \right]$ and $f_k(0) = 0$. So the thesis follows from Theorem 4.6.5. $\square$

The most interesting part of Theorem 6.6.3 is, from the combinatorial point of view, part (i), since the integers $B_{k,i}$ have a well defined combinatorial meaning. However, it is conceivable (and for this reason we have stated also parts (ii) through (v)) that the integers $f_k(n)$ ($n \in \mathbf{Z}$) and the coefficients of $f_k(x)$ with respect to the bases $\left\{ \binom{x}{i} \right\}_{i=0, \dots, |M_k|}$ and $\left\{ (-1)^{|M_k|-i} \binom{x}{i} \right\}_{i=0, \dots, |M_k|}$ may also have an interesting combinatorial interpretation.

For the cases $m_1 = \dots = m_k = 1$ and $m_1 = \dots = m_k = 2$ it is actually possible to give a combinatorial interpretation of $f_k(n)$ for $n \in \mathbf{Z}$ as we will now show. In what follows $S(n, k)$ and $c(n, k)$ have the same meaning as on p. 39.

Proposition 6.6.4 *For $k \in \mathbf{P}$ we have that:*

- (i) $f_k(x) = x^k$ if $m_1 = \dots = m_k = 1$;
- (ii) $f_k(n) = S(n+k, n)$ if $m_1 = \dots = m_k = 2$ and $n \in \mathbf{N}$;
- (iii) $f_k(-n) = c(n, n-k)$ if $m_1 = \dots = m_k = 2$ and $n \in \mathbf{N}$.

Proof We prove (i) first, by induction on k . Note that in case (i) we have that $|M_k| = k$. Hence, by (21), $f_1(x) = x$. Furthermore, from (20) and (21) we have that

$$\begin{aligned} x f_{k-1}(x) &= \sum_{i=1}^k B_{k,i-1} \binom{x+k-i}{k} \\ &= f_k(x) \end{aligned}$$

for all $k \geq 2$, as desired. In case (ii) we have that $|M_k| = 2k$. We prove (ii) by induction on $n+k$. If $n+k = 1$ then $n = 0$, $k = 1$ and $f_1(0) = S(1, 0) = 0$. Assume now that (ii) holds for $n+k \leq l-1$. Then, from (20) and (21) we have

that

$$\begin{aligned} f_k(n) - f_k(n-1) &= \sum_{i=1}^k B_{k,i-1} \binom{n+2k-1-i}{2k-1} \\ &= n f_{k-1}(n), \end{aligned}$$

by the induction hypothesis $f_k(n-1) = S(n+k-1, n-1)$ and $f_{k-1}(n) = S(n+k-1, n)$, so we have that

$$\begin{aligned} f_{k-1}(n) &= S(n+k-1, n-1) + n S(n+k-1, n) \\ &= S(n+k, n) \end{aligned}$$

by a well known recurrence relation for the Stirling numbers of the second kind (see, e. g. , [63], p.33) which also follows easily from their combinatorial definition, and the proof follows. Finally, to prove (iii) we proceed by induction on $n \in \mathbf{N}$. The thesis is clear for $n = 0$ and $k \in \mathbf{P}$. Assume now that it is true for n (and all $k \in \mathbf{P}$). We have seen in the proof of part (ii) that

$$f_k(n) = f_k(n-1) + n f_{k-1}(n), \quad (22)$$

for all $n \in \mathbf{N}$. Hence (22) holds as a polynomial identity and therefore also for $n \in \mathbf{Z}$. So from (22) we can deduce that

$$f_k(-n) = f_k(-n-1) - n f_{k-1}(-n),$$

for all $n \in \mathbf{N}$. By the induction hypothesis $f_k(-n) = c(n, n-k)$ and $f_{k-1}(-n) = c(n, n-k+1)$; hence we obtain that

$$\begin{aligned} f_k(-n-1) &= c(n, n-k) + n c(n, n-k+1) \\ &= c(n+1, n+1-k) \end{aligned}$$

by a well known recurrence relation for the signless Stirling numbers of the first kind (see, e. g. , [63], Lemma 1.3.3) which also follows easily from their combinatorial definition, and the proof follows. $\square$

Part (i) of Proposition 6.6.4 is a well known classical result (see, e. g. , [12], Theorem D, p. 243); part (ii) which is the reason for the terminology “Stirling permutations” and part (iii) are also known (see, e. g. [21], Proposition 1.1 and 1.2, and [32]). The polynomials $S(x + k, x)$ are called *Stirling polynomials* in [21] and are closely related to, but not identical with, what are called Stirling polynomials in [32], section 77. Parts (i) and (ii) of the above proposition are the only cases in which a combinatorial interpretation of $f_k(n)$ seems to be known.

As an immediate consequence of the preceding proposition and of Theorem 6.6.3 we obtain the following result.

Theorem 6.6.5 *For $k \in \mathbf{P}$, the following sequences are PF:*

- (i) $\{S(n + k, n)\}_{n \in \mathbf{N}}$;
- (ii) $\{c(n + k, n)\}_{n \in \mathbf{N}}$. $\square$

6.7 Associated Stirling numbers

In this section, using the results obtained in the preceding one, we prove that certain sequences of associated Stirling numbers (defined below) are PF.

Given $n, k \in \mathbf{P}$ we denote by $S_2(n, k)$ (respectively, $c_2(n, k)$) the number of partitions (respectively, permutations) of $[n]$ with exactly k blocks (respectively, cycles) each of cardinality greater than or equal to 2. So, for example, $S_2(n, k) = c_2(n, k) = 0$ unless $n \geq 2k$, $S_2(n, 1) = 1$, $c_2(n, 1) = (n - 1)!$ if $n \geq 2$. The numbers $S_2(n, k)$ and $c_2(n, k)$ are known, respectively, as the *associated Stirling numbers of the second kind* (sometimes also called the *Ward numbers*) and as the *associated signless Stirling numbers of the first kind* (sometimes also called the *Jordan numbers*). The Jordan and Ward numbers have been studied, e. g.,

in [69], [32], Sects. 52,77, [10], [36]. Tables of them can be found in [12], p.256 and p. 222, respectively. As can be expected, these numbers are closely related to their ordinary counterparts by very simple relations.

Proposition 6.7.1 *For $n, k \in \mathbf{P}$ with $n \geq k$ we have that*

$$S(n, n - k) = \sum_{i=1}^k S_2(k + i, i) \binom{n}{k + i}, \quad (23)$$

$$c(n, n - k) = \sum_{i=1}^k c_2(k + i, i) \binom{n}{k + i}. \quad (24)$$

Proof: Both identities are proved by very simple counting arguments. To prove (23) just note that to choose a partition of $[n]$ into $n - k$ blocks we may first choose $n - k - i$ blocks of cardinality 1 and then choose a partition of $[k + i]$ into i blocks each of cardinality greater than or equal to 2. The proof of (24) is completely analogous. $\square$

We can now derive the main result of this section.

Theorem 6.7.2 *For $k \in \mathbf{P}$ the following sequences are PF:*

- (i) $\{S_2(k + i, i)\}_{i=1, \dots, k};$
- (ii) $\{c_2(k + i, i)\}_{i=1, \dots, k}.$

In particular, the polynomials $\sum_{i=1}^k S_2(k + i, i)x^{i-1}$ and $\sum_{i=1}^k c_2(k + i, i)x^{i-1}$ have only real zeros.

Proof: By part (ii) of Proposition 6.6.4 and equation (23) of Proposition 6.7.1 we immediately deduce that

$$f_k(n - k) = \sum_{i=1}^k S_2(k + i, i) \binom{n}{k + i}, \quad (25)$$

for all $n \geq k$, and hence as a polynomial identity. By Theorem 6.6.2 and the definition of $f_k(x)$ we have that $f_k(x) \in PF\left[\binom{x+2k-i}{2k}\right]$ and $f_k(x) = 0$ if $x \in [-k, 0] \cap \mathbf{Z}$. By using repeatedly Proposition 4.2.3 this implies that $f_k(x - k) \in PF\left[\binom{x+2k-i}{2k}\right]$. Therefore, by Theorem 2.6.1, $f_k(x - k) \in PF\left[\binom{x}{i}\right]$

and this, by (25), proves (i). To prove (ii) note that by part (iii) of Proposition 6.6.4 and equation (24) of Proposition 6.7.1 we have that

$$f_k(-n) = \sum_{i=1}^k c_2(k+i, i) \binom{n}{k+i}, \quad (26)$$

for all $n \geq k$, and hence as a polynomial identity. Substituting “ $-n$ ” for “ n ” in (26) we then get

$$f_k(n) = \sum_{i=1}^k c_2(k+i, i) \left(\binom{n}{k+i} \right) (-1)^{k+i}. \quad (27)$$

But, by Theorem 6.6.3 (part (v)), we know that $f_k(x) \in PF \left[\left(\binom{x}{i} \right) (-1)^i \right]$ and the thesis follows from (27). $\square$

It should be remarked that some other PF and log-concavity properties of associated Stirling numbers also appear in [12], Ex. 20(2), p. 295 and in [36].

Finally, the reader should note that equation (27) gives the combinatorial interpretation of the coefficients of $f_k(x)$ (when $m_1 = \dots, m_k = 2$) with respect to the basis $\left\{ \left(\binom{x}{i} \right) (-1)^i \right\}_{i=0, \dots, 2k}$. It would be interesting to find a simple combinatorial interpretation of the coefficients of $f_k(x)$ with respect to $\left\{ \binom{x}{i} \right\}_{i=0, \dots, 2k}$.

6.8 Colorings of graphs

In this section we prove that some combinatorially defined sequences arising from the enumeration of colorings of graphs are PF_2 . All graphs in this section have no loops or multiple edges. We will follow [28] for graph theory notation and terminology. In particular, we denote by $\chi(G; x)$ the chromatic polynomial of a graph G .

The two main results in this section both follow from the following, elementary, general fact.

Proposition 6.8.1 *Let $P(x) \in PF[(x)_i]$; then the sequence $\{P(n)\}_{n \in \mathbb{N}}$ is PF_2 .*

Proof: By the hypothesis and Theorem 2.6.3 we know that $P(x) \in PF_2 \left[\binom{x}{i} \right]$ and this, by Theorem 2.5.7 implies that $\{P(n)\}_{n \in \mathbb{N}}$ is a PF_2 sequence, as desired. $\square$

Given a graph $G = (V, E)$ we denote by $\chi_2(G, l)$ the number of l -colorings of G that use each color at most twice (i. e. at most two vertices of G can have the same color). The first main result of this section is the following.

Theorem 6.8.2 *Let $G = (V, E)$ be a graph. Then the sequence $\{\chi_2(G, l)\}_{l=0,1,2,\dots}$ is PF_2 .*

Proof: Let $n = |V|$ and let $\chi_{2,k}(G; l)$ be the number of l -colorings of G that use exactly k colors twice and all the others at most once. It is then easy to see that $\chi_{2,k}(G; l) = m_k(\overline{G})(l)_{n-k}$ where $\overline{G}$ is the complementary graph of G and where $m_k(\overline{G})$ is the number of matchings of $\overline{G}$ having k edges. Hence $\chi_2(G; l)$ is a polynomial function of l and we have that

$$\chi_2(G; l) = \sum_{k=0}^n m_k(\overline{G})(l)_{n-k}, \quad (28)$$

for $l = 0, 1, 2, \dots$. Consider now the modified matchings polynomial of $\overline{G}$,

$$\tilde{\mu}(\overline{G}; x) \stackrel{\text{def}}{=} \sum_{k=0}^n m_k(\overline{G}) x^{n-k}.$$

It is known (see, e. g., [30], or [68]) that $\tilde{\mu}(G; x)$ has only real nonpositive zeros for every graph G . Therefore $\tilde{\mu}(\overline{G}; x) \in PF[x^i]$ and hence, by (28), $\chi_2(G; x) \in PF[(x)_i]$ and the thesis follows from Proposition 6.8.1 $\square$

To prove the second main result of the section we need first to recall some definitions and to state two well known characterization results. Recall that, given a poset P , the set of all chains of P ordered by inclusion forms a simplicial complex denoted by $\Delta(P)$. Given a simplicial complex Δ then Δ is called an *order complex* if and only if there exists a poset P such that $\Delta = \Delta(P)$. Also recall that a graph $G = (V, E)$ is called a *comparability graph* if there exists a partial ordering $\leq$ on V whose Hasse diagram is isomorphic to G as a graph.

The first result that we need is a simple characterization of order complexes given in [60], p. 144.

Proposition 6.8.3 *Let Δ be a (finite) simplicial complex. Then Δ is an order complex if and only if Δ satisfies the following:*

- (i) *any minimal set of vertices which do not form a face of Δ has two elements;*
- (ii) *the 1-skeleton of Δ is a comparability graph.*

The second result is the well known characterization of comparability graphs given by Gilmore and Hoffman in [24].

Theorem 6.8.4 *Let G be a graph. Then G is a comparability graph if and only if each odd cycle has at least one triangular chord.*

Let now V be a finite set. A collection $\mathcal{F}$ of subsets of V is called a *set system* on V if $\emptyset \in \mathcal{F}$ and $\bigcup_{x \in \mathcal{F}} x = V$. Given a set system $\mathcal{F}$ on V we define, following [68], the *partition polynomial* of $\mathcal{F}$ by

$$\rho(\mathcal{F}; x) \stackrel{\text{def}}{=} \sum_{k \geq 0} p_k(\mathcal{F}) x^k ,$$

where $p_k(\mathcal{F})$ equals the number of partitions of V into k blocks such that each block is in $\mathcal{F}$.

We can now prove the second main result of this section.

Theorem 6.8.5 *Let $G = (V, E)$ be a graph such that its complementary graph $\overline{G}$ is a comparability graph. Then the sequence $\{\chi(G; l)\}_{l \in \mathbb{N}}$ is PF_2 .*

Proof: Let $n = |V|$ and let

$$\rho(I(G); x) = \sum_{k=0}^n p_k(I(G)) x^k ,$$

be the partition polynomial of the simplicial complex $I(G)$ of independent subsets of V . Then we clearly have that

$$\chi(G; x) = \sum_{k=0}^n p_k(I(G)) (x)_k .$$

Now, it is easy to see that a simplicial complex of the form $I(G)$ always satisfies (i) of Proposition 6.8.3 and satisfies (ii) if and only if $\overline{G}$ is a comparability graph. Hence, by Proposition 6.8.3, $I(G)$ is an order complex in our hypotheses. By Theorem 4 of [68] this implies that the polynomial $\rho(I(G); x)$ has only real nonpositive zeros. Hence, by Theorem 2.3.1, $\rho(I(G); x) \in PF[x^i]$. Therefore $\chi(G; x) \in PF[(x)_i]$ and the thesis follows from Proposition 6.8.1. $\square$

It would be interesting to know if the sequence $\{\chi(G; l)\}_{l \in \mathbf{N}}$ is PF_2 for any graph G , or at least to know for which graphs this is true. I have not checked very many examples but for all those that I have checked the sequence $\{\chi(G; l)\}_{l \in \mathbf{N}}$ has always turned out to be PF_2 .

Chapter 7

Pólya frequency digraphs

7.1 Introduction

In this chapter we introduce the concept of a Pólya frequency digraph and study the problem of classifying such digraphs. We then show that the distributive lattice conjecture is a special case of this more general problem. Even though the concept of a Pólya frequency digraph is new many of the results presented in this chapter are not new. My main contribution here is to show how all these different and already existing results (and problems) fit into this general scheme of Pólya frequency digraphs.

In the next section we give the definition of a Pólya frequency digraph of order r (or, a PF_r digraph, for short) and we prove, using Theorem 4.5.3, an algebraic characterization of PF digraphs. We then define two important classes of digraphs associated with posets and show their connection with the distributive lattice conjecture. In section 7.3 we present a general result that clarifies the meaning of PF digraphs and that is used in later sections. In the fourth section the connection existing between PF digraphs and the theory of symmetric functions is illustrated. This connection is then used to give a useful necessary and sufficient condition for a digraph to be PF . In section 7.5 we use again the theory of symmetric functions to prove that a digraph D is PF if and only if its complement $\overline{D}$ is PF . In the sixth section we briefly study PF_2 digraphs. We prove that a very large class of digraphs are PF_2 and

then we give two elegant characterizations of PF_2 digraphs. Finally, in section 7.7, we introduce a class of digraphs that are always PF . We then discuss the implications that this result has on the distributive lattice conjecture.

This chapter has been inspired largely by the work [23] of Gessel and Viennot and by many conversations had with Ira Gessel. Even though my point of view is slightly different from that of [23] all the main results and concepts presented here are already present there and it is a pleasure for me to acknowledge this.

7.2 PF digraphs and the distributive lattice conjecture

In this section we define the concept of a Pólya frequency digraph (or, PF digraph, for short) and derive an algebraic characterization of PF digraphs. We then introduce two general classes of digraphs derived from partially ordered sets and show their connection with the distributive lattice conjecture.

Let $D = (\mathcal{V}, \mathcal{R})$ be a finite directed graph (or, digraph, for short) on vertex set $\mathcal{V}$ and without multiple edges (so that we can identify $\mathcal{R}$ with a subset of $\mathcal{V} \times \mathcal{V}$). A *path* in D is a sequence $a_1, \dots, a_n$ of vertices such that $(a_{i-1}, a_i) \in \mathcal{R}$ for $i = 2, \dots, n$; the integer n is then called the cardinality of the path. For $n \in \mathbf{P}$ we let $A_D(n)$ be the number of paths of cardinality n in D ; it will also be convenient to set $A_D(0) \stackrel{\text{def}}{=} 1$ and $A_D(-n) \stackrel{\text{def}}{=} 0$ if $n \in \mathbf{P}$. We will write simply “ $A(n)$ ” for “ $A_D(n)$ ” when there is no danger of confusion. We are now already in a position to define the central objects that we will study in this chapter. For $r \in \mathbf{P}$, we call a digraph D as above a *Pólya frequency digraph of order r* (or, a PF_r digraph, for short) if the sequence $\{A_D(n)\}_{n \in \mathbf{N}}$ is a PF_r sequence. We call D a *Pólya frequency digraph* (or, a PF digraph, for short) if D is a PF_r digraph for all $r \in \mathbf{P}$. It should be noticed that, by our definitions, every digraph is

PF_1 . However, not every digraph is PF_2 , for example, for the digraph D given in Figure 1 we have that $\{A_D(n)\}_{n \in \mathbb{N}} = \{1, 5, 4, 4, \dots\}$ which is not PF_2 .

Figure 1: a digraph that is not PF_2

The sequence $\{A_D(n)\}_{n \in \mathbb{N}}$ is one of the most fundamental enumerative statistics of the digraph D and has been studied extensively. The most important result about it is probably the following one which is an easy consequence of the “Transfer Matrix Method” (see, e. g. , [63], §4.7).

Theorem 7.2.1 *For any digraph $D = (\mathcal{V}, \mathcal{R})$ we have that*

$$\sum_{n \geq 0} A_D(n) z^n = \frac{(-1)^{v+1} \det(J_{v+1} - z \tilde{A})}{z \det(I_v - z A)}, \quad (1)$$

where $v = |\mathcal{V}|$, A is the incidence matrix of D and, for $1 \leq i, j \leq v+1$,

$$\tilde{A}_{ij} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i = 1 \text{ or } j = v+1, \\ A_{ij} & \text{otherwise,} \end{cases}$$

$$(J_{v+1})_{i,j} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Define a new digraph $\hat{D} \stackrel{\text{def}}{=} (\mathcal{V} \cup \{\hat{0}, \hat{1}\}, \hat{\mathcal{R}})$ where $(u, v) \in \hat{\mathcal{R}}$ if and only if either $u, v \in \mathcal{V}$ and $(u, v) \in \mathcal{R}$, or $u = \hat{0}$ and $v \in \mathcal{V}$, or $u \in \mathcal{V}$ and $v = \hat{1}$ or $u = \hat{0}$ and $v = \hat{1}$. Then, for all $n \in \mathbb{N}$, $A_D(n)$ equals the number of paths from $\hat{0}$ to $\hat{1}$ in $\hat{D}$ of length $n+1$. By a well known result (see, e. g. [63], Theorem 4.7.2, p. 242) this is easily seen to imply that $\sum_{n \geq 0} A_D(n) z^n$ has the desired form. $\square$

One other (much more elementary) but useful property to note about the sequence $\{A_D(n)\}_{n \in \mathbf{N}}$ is the following.

Proposition 7.2.2 *Let D be a digraph. Then the sequence $\{A_D(n)\}_{n \in \mathbf{N}}$ has no internal zeros.*

Proof: By the combinatorial definition of $A_D(n)$ there follows at once that if $A_D(n) > 0$ and $n \in \mathbf{P}$ then $A_D(n-1) > 0$. $\square$

Now, Theorem 7.2.1 implies in particular that the formal power series $\sum_{n \geq 0} A_D(n) z^n$ is always rational. Applying then Theorem 4.5.3 to this rational formal power series we immediately get, from our definitions, the following characterization of PF digraphs.

Theorem 7.2.3 *Let $D = (\mathcal{V}, \mathcal{R})$ be a digraph and let $W(z), V(z) \in \mathbf{R}[z]$ be the (unique) relatively prime polynomials such that*

$$z W(z) \det(I_v - z A) = (-1)^{v+1} V(z) \det(J_{v+1} - z \tilde{A}),$$

where $A, \tilde{A}, v$ and J_{v+1} have the same meaning as in Theorem 7.2.1. Then D is a PF digraph if and only if $W(z)$ has only nonpositive real zeros and $V(z)$ has only positive real zeros. $\square$

Theorem 7.2.3 is one way in which the problem of classifying PF digraphs can be attacked. Unfortunately, easy examples show that, in general, the polynomials $z \det(I_v - z A)$ and $\det(J_{v+1} - z \tilde{A})$ may have more factors in common than just z so that the computation of $V(z)$ and $W(z)$ may not always be easy. It would be interesting to classify all digraphs for which the numerator and denominator on the RHS of (1) are relatively prime (except for z). There are, however, two important general classes for which this can be accomplished. These are digraphs that are derived naturally from a poset P . There are, of course, many digraphs that can be associated to a poset but here we will only consider what are maybe the two “most natural” ones. Namely, given a poset $(P, \leq)$ we associate to it the two digraphs $O(P) \stackrel{\text{def}}{=} (P, \mathcal{R})$ and $S(P) \stackrel{\text{def}}{=} (P, \overline{\mathcal{R}})$

where $(x, y) \in \mathcal{R}$ (respectively, $(x, y) \in \overline{\mathcal{R}}$) if and only if $x \leq y$ (respectively, $x < y$). We call $O(P)$ (respectively, $S(P)$) the *order digraph* (respectively, the *strict order digraph*) of P . The next result gives the relation between the functions $A_{O(P)}$ and $A_{S(P)}$ and some fundamental enumerative statistics of $(P, \leq)$. In what follows $c_n(P)$ has the same meaning as in Theorem 6.3.1.

Proposition 7.2.4 *Let $(P, \leq)$ be a poset with $\hat{0}$ and $\hat{1}$. Then:*

- (i) $A_{O(P)}(n) = Z(P; n+1)$ for all $n \in \mathbf{N}$;
- (ii) $A_{S(P)}(n) = c_n(P)$ for all $n \in \mathbf{N}$.

Proof: (i) follows immediately from the combinatorial definition of the zeta polynomial of a poset (see, e. g. , [63], §3.11); (ii) is clear. $\square$

From the above proposition we immediately deduce, in particular, that there exist polynomials $C_P(z)$ and $V_P(z)$ such that

$$\sum_{n \geq 0} A_{O(P)}(n) z^n = \frac{V_P(z)}{(1-z)^{l(P)+1}}, \quad (2)$$

$$\sum_{n \geq 0} A_{S(P)}(n) z^n = C_P(z). \quad (3)$$

where $l(P)$ is the degree of $Z(P; x)$ (which also equals the length of the longest chain of P) (see, e. g. , [63], Prop.3.11.1). We may therefore apply Theorem 7.2.3 to these digraphs to obtain the following necessary and sufficient conditions for their *PF* property.

Theorem 7.2.5 *Let $(P, \leq)$ be a finite poset and let $V_P(z)$ and $C_P(z)$ be the two polynomials defined by (2) and (3) above. Then we have that:*

- (i) $O(P)$ is a *PF* digraph if and only if $V_P(z)$ has only nonpositive real zeros;
- (ii) $S(P)$ is a *PF* digraph if and only if $C_P(z)$ has only nonpositive real zeros.

Proof: By Theorem 7.2.3 it is enough to show that $V_P(z)$ and $(1-z)^{l(P)+1}$ are relatively prime. By part (i) of Proposition 7.2.4, there follows that $A_{O(P)}(n)$ is

a polynomial function of n of degree *exactly* $l(P)$, therefore, from a well known result in the theory of rational generating functions (see, e. g. [63], Corollary 4.3.1) and from (2) we deduce that $V_P(1) \neq 0$ and the thesis follows. $\square$

Comparing now the last two results with Conjectures 3 and 7 (given on pp. 23 and 126, respectively), yields the following two further equivalent formulations of the distributive lattice conjecture.

Conjecture 8 (*The distributive lattice conjecture*): *Let L be a distributive lattice. Then the digraph $O(L)$ is PF .*

Conjecture 9 (*The distributive lattice conjecture*): *Let L be a distributive lattice. Then the digraph $S(L)$ is PF .*

So, the distributive lattice conjecture actually asserts that two vast classes of digraphs are PF . In the remainder of this chapter we will “forget” this fact and we will try instead to study the more general problem of characterizing PF digraphs.

7.3 A general result

From our definitions of PF and PF_r digraphs it is clear that the determinants of the minors of the infinite matrix $A_D \stackrel{\text{def}}{=} (A_D(j - i))_{i,j \in \mathbf{N}}$ are of fundamental importance. We therefore introduce a standard notation for them, namely, for $p \in \mathbf{P}$, $0 \leq i_1 < i_2 < \dots < i_p$ and $0 \leq j_1 < j_2 < \dots < j_p$ we let $A \begin{pmatrix} i_1, \dots, i_p \\ j_1, \dots, j_p \end{pmatrix}$ equal the determinant of the minor of A obtained from the rows $i_1, \dots, i_p$ and the columns $j_1, \dots, j_p$. Therefore,

$$A \begin{pmatrix} i_1, \dots, i_p \\ j_1, \dots, j_p \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A(j_1 - i_1), & \dots, & A(j_p - i_1) \\ \vdots & & \vdots \\ A(j_1 - i_p) & \dots, & A(j_p - i_p) \end{pmatrix}. \quad (4)$$

In this section we try to understand which (if any) minors of A_D have non-

negative determinant for every digraph D . The following result is an elementary consequence of (4).

Proposition 7.3.1 *Let D be a digraph. Then we have that:*

- (i) $A_D \begin{pmatrix} i_1, \dots, i_p \\ j_1, \dots, j_p \end{pmatrix} = 0$ unless $j_r \geq i_r$ for $r = 1, \dots, p$;
- (ii) $A_D \begin{pmatrix} i_1 + r, \dots, i_p + r \\ j_1 + r, \dots, j_p + r \end{pmatrix} = A \begin{pmatrix} i_1, \dots, i_p \\ j_1, \dots, j_p \end{pmatrix}$ for all $r \in \mathbf{N}$, $p \in \mathbf{P}$ and $0 \leq i_1 < i_2 < \dots < i_p$, $0 \leq j_1 < j_2 < \dots < j_p$;
- (iii) $A_D \begin{pmatrix} i_1, \dots, i_p \\ i_1, \dots, i_p \end{pmatrix} = 1$ for all $p \in \mathbf{P}$ and $0 \leq i_1 < i_2 < \dots < i_p$.

Proof: (i) follows immediately from (4) and the observation that if $j_r < i_r$ for some $r \in [p]$ then $j_l < i_k$ for all $l \leq r$ and all $k \geq r$; (ii) and (iii) follow from (4). $\square$

Besides the minors in the preceding proposition there seems to be only one other general class of minors of A_D that have nonnegative determinants for any digraph D . Before we can state and prove this, however, we need to introduce a general class of enumerative statistics and generating functions associated to any digraph D . For any digraph $D = ([v], \mathcal{R})$, ($v \in \mathbf{P}$), and $S \subseteq [n - 1]$ we define sets $\beta_n(S)$ and $\alpha_n(S)$ by

$$\beta_n(S) \stackrel{\text{def}}{=} \{(i_1, \dots, i_n) \in [v]^n : (i_r, i_{r+1}) \notin \mathcal{R} \Leftrightarrow r \in S\}, \quad (5)$$

$$\alpha_n(S) \stackrel{\text{def}}{=} \{(i_1, \dots, i_n) \in [v]^n : (i_r, i_{r+1}) \notin \mathcal{R} \Rightarrow r \in S\}, \quad (6)$$

and generating functions $F(\alpha_n(S))$, $F(\beta_n(S)) \in \mathbf{Z}[x_1, \dots, x_v]$ by

$$F(\alpha_n(S))(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{(i_1, \dots, i_n) \in \alpha_n(S)} x_{i_1} \dots x_{i_n}, \quad (7)$$

$$F(\beta_n(S))(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{(i_1, \dots, i_n) \in \beta_n(S)} x_{i_1} \dots x_{i_n}. \quad (8)$$

where $\mathbf{x} \stackrel{\text{def}}{=} (x_1, \dots, x_v)$. So, in particular, $F(\beta_n(S))(1, \dots, 1) = |\beta_n(S)|$ and $F(\alpha_n(S))(1, \dots, 1) = |\alpha_n(S)|$. For a reason that will become clear later we will use a special notation for $F(\beta_n(\emptyset))(\mathbf{x})$, namely, we let

$$h_n^D(\mathbf{x}) \stackrel{\text{def}}{=} F(\beta_n(\emptyset))(\mathbf{x}), \quad (9)$$

for $n \in \mathbf{P}$, and we set, by convention, $h_0^D(\mathbf{x}) \stackrel{\text{def}}{=} 1$, $h_{-n}^D(\mathbf{x}) \stackrel{\text{def}}{=} 0$ if $n \in \mathbf{P}$. Note that, with these definitions and conventions, we have that

$$h_n^D(1, \dots, 1) = A_D(n),$$

for all $n \in \mathbf{Z}$.

We can now prove the main result of this section, which is known and has appeared, in one form or another, in many places in the literature (see, e. g. ,[26], [63],[19]). We reprove it here for completeness and also to show the place that it occupies in our theory.

Theorem 7.3.2 *Let $D = ([v], \mathcal{R})$ be a digraph. Then*

$$\det \left((h_{j_{k+1}-j_l}^D(\mathbf{x}))_{0 \leq l, k \leq p-1} \right) = F(\beta_{j_p}(\{j_1, \dots, j_{p-1}\}))(\mathbf{x}),$$

for $0 = j_0 < j_1 < j_2 < \dots < j_p$ and $p \in \mathbf{P}$. In particular,

$$A_D \begin{pmatrix} 0, j_1, \dots, j_{p-1} \\ j_1, j_2, \dots, j_p \end{pmatrix} = |\beta_{j_p}(\{j_1, \dots, j_{p-1}\})|. \quad (10)$$

Proof: Let $n \stackrel{\text{def}}{=} j_p$. From the definitions (5), (6), (7) and (8) there follows immediately that

$$F(\alpha_n(S))(\mathbf{x}) = \sum_{T \subseteq S} F(\beta_n(T))(\mathbf{x}), \quad (11)$$

for all $S \subseteq [n-1]$, and by (9) also that

$$F(\alpha_n(S))(\mathbf{x}) = \prod_{i=0}^k h_{s_{i+1}-s_i}^D(\mathbf{x}), \quad (12)$$

if $S = \{s_1, \dots, s_k\}_< \subseteq [n-1]$ (where $s_0 \stackrel{\text{def}}{=} 0$ and $s_{k+1} \stackrel{\text{def}}{=} n$). Let now $T \stackrel{\text{def}}{=} \{j_1, \dots, j_{p-1}\}_< \subseteq [n-1]$, then from (11) and (12), using the principle of

inclusion-exclusion, we deduce that

$$\begin{aligned}
F(\beta_n(T))(\mathbf{x}) &= \sum_{S \subseteq T} (-1)^{|T-S|} F(\alpha_n(S))(\mathbf{x}) \\
&= \sum_{t \geq 0}^{p-1} \sum_{0=i_0 < i_1 < \dots < i_t < i_{t+1}=p} (-1)^{p-1-t} \prod_{l=0}^t h_{j_{i_{l+1}}-j_{i_l}}^D(\mathbf{x}) \\
&= \det \left((h_{j_{k+1}-j_l}^D(\mathbf{x}))_{0 \leq l, k \leq p-1} \right),
\end{aligned}$$

as desired. Now (10) is obtained by letting $x_1 = \dots = x_v = 1$. $\square$

7.4 The connection with the theory of symmetric functions

In this and the next section we use the theory of symmetric functions. We will use the terminology and notations of [38], Chap.I. In particular, for any partition λ , we denote by h_λ (respectively e_λ , p_λ , and s_λ) the *complete homogeneous* (respectively, *elementary*, *power sum*, and *Schur*) symmetric functions.

Let $D = ([v], \mathcal{R})$, ($v \in \mathbf{P}$) be a digraph that will remain fixed throughout this section. For any two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0)$ and $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_p \geq 0)$ into at most p parts we define polynomials $h_\lambda^D(\mathbf{x})$, $s_{\lambda/\mu}^D(\mathbf{x}) \in \mathbf{Z}[\mathbf{x}]$ (where $\mathbf{x} \stackrel{\text{def}}{=} (x_1, \dots, x_v)$) by

$$h_\lambda^D(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^p h_{\lambda_i}^D(\mathbf{x}), \quad (13)$$

$$s_{\lambda/\mu}^D(\mathbf{x}) \stackrel{\text{def}}{=} \det \left[(h_{\lambda_i - \mu_j + j - i}^D(\mathbf{x}))_{1 \leq i, j \leq p} \right], \quad (14)$$

Note that $h_\lambda^D(\mathbf{x})$ and $s_{\lambda/\mu}^D(\mathbf{x})$ are both homogeneous polynomials of degrees $|\lambda|$ and $|\lambda/\mu|$, respectively, and that, by (4), the definition (14) implies that

$$s_{\lambda/\mu}^D(1, \dots, 1) = A \left(\begin{matrix} i_1, \dots, i_p \\ j_1, \dots, j_p \end{matrix} \right), \quad (15)$$

where $j_k \stackrel{\text{def}}{=} \lambda_{p-k+1} + k - 1$ and $i_k \stackrel{\text{def}}{=} \mu_{p-k+1} + k - 1$, for $k = 1, \dots, p$.

Let now Λ denote the ring of symmetric functions in infinitely many variables with coefficients in $\mathbf{Z}$, and denote by Λ^D the subalgebra of $\mathbf{Z}[\mathbf{x}]$ generated by $\{h_n^D(\mathbf{x})\}_{n \in \mathbf{N}}$ over $\mathbf{Z}$. Now, it is well known that $\Lambda \cong \mathbf{Z}[h_0, h_1, h_2, \dots]$ (i. e. , Λ is generated by $\{h_n\}_{n \in \mathbf{N}}$ as a $\mathbf{Z}$ -algebra and the h_n 's are algebraically independent over $\mathbf{Z}$). We can therefore define a surjective ring homomorphism

$$\psi_D : \Lambda \rightarrow \Lambda_D ,$$

by

$$\psi_D(h_i) \stackrel{\text{def}}{=} h_i^D(\mathbf{x}) ,$$

for $i \in \mathbf{N}$. For any symmetric function $a \in \Lambda$ we will often also write $a^D \stackrel{\text{def}}{=} \psi_D(a)$.

It should be noted that, since ψ_D is degree preserving, Λ_D is a graded ring. However, simple examples show that Λ_D is *not*, in general, a λ -ring (see, e. g. , [34], for definitions and further information about λ -rings). It would be interesting to know for which digraphs D Λ_D is a λ -ring.

From the definition of ψ_D and from (13) and (14) it immediately follows that

$$\psi_D(h_\lambda) = h_\lambda^D(\mathbf{x}) , \tag{16}$$

$$\psi_D(s_{\lambda/\mu}) = s_{\lambda/\mu}^D(\mathbf{x}) , \tag{17}$$

for all partitions λ, μ , where, in (17), we have used a well known identity in the theory of symmetric functions (see, e. g. , [38], Chap.I, §5). Equation (17) is a very fundamental property of the homomorphism ψ_D because it allows us to use the powerful theory of symmetric functions to obtain information on the $s_{\lambda/\mu}^D$'s and hence, by (15), also on PF digraphs. One example of this method is the following result which is also the main result of this section.

Theorem 7.4.1 *Let D be a digraph. Then D is PF if and only if $s_\lambda^D(1, \dots, 1) \geq 0$ for all partitions λ .*

Proof: If D is a PF digraph then the thesis follows immediately from (15).

Conversely, suppose that $s_\lambda^D(1, \dots, 1) \geq 0$ for all partitions λ . Now, it is a well known and non-trivial fact in the theory of symmetric functions (see, e. g. ,[38], Chap.I, §7) that

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu},$$

where $c_{\mu\nu}^{\lambda} \in \mathbb{N}$. Applying ψ_D to this identity and then letting $x_1 = \dots = x_{\nu} = 1$ we get that

$$s_{\lambda/\mu}^D(1, \dots, 1) = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}^D(1, \dots, 1),$$

which shows that $s_{\lambda/\mu}^D(1, \dots, 1) \geq 0$ if $s_{\nu}^D(1, \dots, 1) \geq 0$ for all partitions ν . This, by (15), proves the theorem. $\square$

One other consequence of the above method is the following generalization of Theorem 7.3.2.

Theorem 7.4.2 *Let D be a digraph. Then $s_{\lambda/\mu}^D(\mathbf{x}) \in \mathbb{N}[\mathbf{x}]$ (and hence, in particular, $s_{\lambda/\mu}^D(1, \dots, 1) \geq 0$) for all partitions λ, μ such that λ/μ is a border strip.*

Proof: Let ρ/ν be a connected border strip. Then there exist partitions $\lambda = (\lambda_1, \dots, \lambda_p)$ and $\mu = (\mu_1, \dots, \mu_{p-1})$ such that $\rho/\nu = \lambda/\mu$ (as skew shapes) and $\lambda_i = \mu_{i-1} + 1$ for $i = 2, \dots, p$. Now, it is immediate from our definitions and from Theorem 7.3.2 that

$$s_{\lambda/\mu}^D(\mathbf{x}) = F(\beta_{j_p}(\{j_1, \dots, j_{p-1}\}))(\mathbf{x}), \quad (18)$$

where $j_k = \lambda_{p-k+1} + k - 1$ for $k = 1, \dots, p$. Also, it is well known from the theory of symmetric functions that $s_{\rho/\nu} = s_{\lambda/\mu}$ if $\rho/\nu = \lambda/\mu$ as skew shapes. Hence $s_{\rho/\nu}^D(\mathbf{x}) = s_{\lambda/\mu}^D(\mathbf{x}) \in \mathbb{N}[\mathbf{x}]$ by (18). This proves the theorem for a connected border strip. Now, if ρ/ν is any border strip then it is well known (see, e. g.,[38], Chap.I, (5.7)) that

$$s_{\rho/\nu} = \prod_{i=1}^n s_{\theta_i},$$

where $\theta_1, \dots, \theta_n$ are connected border strips. Hence $s_{\rho/\nu}^D(\mathbf{x}) = \prod_{i=1}^n s_{\theta_i}^D(\mathbf{x}) \in \mathbb{N}[\mathbf{x}]$, as desired. $\square$

In general, it would be interesting to know which symmetric functions $a \in \Lambda$ have the property that $\psi_D(a) \in \mathbf{N}[\mathbf{x}]$ for all digraphs D . It is clear from our definitions that h_λ has this property for all partitions λ . We will now show that the same is true for the elementary symmetric function e_λ .

Theorem 7.4.3 *Let D be a digraph. Then $\psi_D(e_\lambda) \in \mathbf{N}[\mathbf{x}]$ for all partitions λ .*

Proof: We have that, for $\lambda = (\lambda_1 \geq \dots \geq \lambda_p \geq 0)$,

$$\begin{aligned} [\psi_D(e_\lambda)](\mathbf{x}) &= \left[\psi_D \left(\prod_{i=1}^p e_{\lambda_i} \right) \right](\mathbf{x}) \\ &= \left[\psi_D \left(\prod_{i=1}^p s_{(1^{\lambda_i})} \right) \right](\mathbf{x}) \\ &= \prod_{i=1}^p [s_{(1^{\lambda_i})}^D(\mathbf{x})] \end{aligned} \quad (19)$$

where we have used some well known identities from the theory of symmetric functions. Since (1^{λ_i}) is a border strip Theorem 7.4.2 implies that $s_{(1^{\lambda_i})}^D(\mathbf{x}) \in \mathbf{N}[\mathbf{x}]$ and the thesis follows from (19). $\square$

We conclude this section with a result that gives an “explicit” evaluation of the minors of A involved in Theorem 7.4.1, but one that seems to be of little practical use.

Proposition 7.4.4 *Let D be a digraph. Then, for all $0 \leq j_1 < j_2 < \dots < j_p$ and $p \in \mathbf{P}$, we have that*

$$A \begin{pmatrix} 0, 1, \dots, p-1 \\ j_1, j_2, \dots, j_p \end{pmatrix} = \sum_{\mu=(\mu_1, \mu_2, \dots)} H_{\mu \lambda} \prod_{i \geq 1} A(\mu_i), \quad (20)$$

where $\lambda \stackrel{\text{def}}{=} (j_p - p + 1, \dots, j_2 - 1, j_1)$ and H is the inverse matrix of the Kostka matrix, (see, e. g. , [98], ChapI, §6, for the definition of the Kostka matrix).

Proof: It is a well known identity in the theory of symmetric functions that

$$s_\lambda = \sum_{\mu} H_{\mu \lambda} h_\mu,$$

for all partitions λ . Applying ψ_D and then setting $x_1 = \dots = x_\nu = 1$ yields

$$s_\lambda^D(1, \dots, 1) = \sum_{\mu} H_{\mu \lambda} h_\mu^D(1, \dots, 1),$$

which is equivalent to (20). $\square$

7.5 The inversion Theorem

In this section we prove that the class of PF digraphs is closed under the complement operation where, for any digraph $D \stackrel{\text{def}}{=} ([v], \mathcal{R})$, ($v \in \mathbf{P}$) the *complement digraph* of D is $\overline{D} \stackrel{\text{def}}{=} ([v], \overline{\mathcal{R}})$ where $\overline{\mathcal{R}} \stackrel{\text{def}}{=} [v]^2 \setminus \mathcal{R}$.

We will denote by ω the automorphism of Λ defined by $\omega(h_\lambda) = e_\lambda$ for all partitions λ (see [38], Chap.I, §2, for further information about ω). The first result in this section describes the relationship between ψ_D , Λ^D and $\psi_{\overline{D}}$, $\Lambda^{\overline{D}}$.

Theorem 7.5.1 *Let D be a digraph and $\overline{D}$ be its complement. Then $\Lambda^D = \Lambda^{\overline{D}}$ and $\psi_D \circ \omega = \psi_{\overline{D}}$.*

Proof: From our definitions and from equation (18) there follows immediately that

$$e_n^D(\mathbf{x}) = F(\beta_n([n-1]))(\mathbf{x}),$$

for all $n \in \mathbf{P}$. By (9) this implies that $e_n^D(\mathbf{x}) = h_n^{\overline{D}}(\mathbf{x})$ for all $n \in \mathbf{P}$ and hence that

$$e_\lambda^D(\mathbf{x}) = h_\lambda^{\overline{D}}(\mathbf{x}) \tag{21}$$

for all partitions λ . This shows that $\Lambda^D \supseteq \Lambda^{\overline{D}}$ and hence, since $\overline{\overline{D}} = D$, that $\Lambda^D = \Lambda^{\overline{D}}$. Furthermore, from (21) we immediately deduce that

$$\begin{aligned} \psi_{\overline{D}}(h_\lambda) &= h_\lambda^{\overline{D}} \\ &= e_\lambda^D \\ &= \psi_D(e_\lambda) \\ &= \psi_D(\omega(h_\lambda)), \end{aligned}$$

and the thesis follows. $\square$

The main result of this section is now almost immediate.

Theorem 7.5.2 *Let D be a digraph and $\overline{D}$ be its complement. Then D is a PF digraph if and only if $\overline{D}$ is a PF digraph.*

Proof: From the preceding theorem we deduce that $s_{\lambda/\mu}^{\overline{D}}(\mathbf{x}) = s_{\lambda'/\mu'}^D(\mathbf{x})$ for all partitions λ, μ . The thesis then follows from (15). $\square$

One other consequence of Theorem 7.5.1 is the following.

Theorem 7.5.3 *Let D be a digraph and $\overline{D}$ be its complement. Then*

$$\left(\sum_{n \geq 0} h_n^D(\mathbf{x}) t^n \right)^{-1} = \sum_{n \geq 0} h_n^{\overline{D}}(\mathbf{x}) (-t)^n, \quad (22)$$

(as formal power series in $\mathbb{Z}[\mathbf{x}][[t]]$). In particular,

$$\left(\sum_{n \geq 0} A_D(n) t^n \right)^{-1} = \sum_{n \geq 0} A_{\overline{D}}(n) (-t)^n.$$

Proof: It is well known (see, e. g. , [38], Chap.I, (2.6)) that

$$\sum_{r=0}^n (-1)^r e_r h_{n-r} = \delta_{n,0}$$

for all $n \in \mathbb{N}$. Applying ψ_D to both sides and using (21) then yields (22). $\square$

We close this section by noting the relationship between the p_λ^D 's and the $p_\lambda^{\overline{D}}$'s.

Proposition 7.5.4 *Let D be a digraph and $\overline{D}$ be its complement. Then $p_\lambda^{\overline{D}} = \epsilon_\lambda p_\lambda^D$ for all partitions λ , where $\epsilon_\lambda \stackrel{\text{def}}{=} (-1)^{|\lambda| - l(\lambda)}$.*

Proof: Apply $\psi_{\overline{D}}$ to p_λ and use Theorem 7.5.1. $\square$

7.6 PF₂ digraphs

In this section we briefly study PF₂ digraphs. We first prove that, for any distributive lattice L , the associated digraph $O(L)$ is PF₂. We then give two simple necessary and sufficient conditions for an arbitrary digraph to be PF₂.

Our first result is an immediate consequence of Propositions 2.5.1 and 7.2.2 and gives an alternative description of PF_2 digraphs.

Proposition 7.6.1 *Let D be a digraph. Then D is a PF_2 digraph if and only if $\{A_D(n)\}_{n \in \mathbf{N}}$ is a log-concave sequence. $\square$*

The next result is, essentially, a special case of Proposition 7.2.4.

Proposition 7.6.2 *Let P be a poset. Then $A_{O(J(P))}(n) = \Omega(P; n+1)$ for all $n \in \mathbf{N}$.*

Proof: By Proposition 7.2.4 we know that $A_{O(J(P))}(n) = Z(J(P); n+1)$ for all $n \in \mathbf{N}$. On the other hand, it is well known (see, e. g. , [63], §3.11) and also easy to see directly that

$$Z(J(P); x) = \Omega(P; x)$$

(as polynomials in $\mathbf{R}[x]$) for any poset P and the thesis follows. $\square$

Putting together Propositions 7.6.1 and 7.6.2 yields the following result.

Proposition 7.6.3 *Let P be a poset. Then $O(J(P))$ is a PF_2 digraph if and only if $\{\Omega(P; n)\}_{n \in \mathbf{P}}$ is a log-concave sequence. $\square$*

The log-concavity of the sequence $\{\Omega(P; n)\}_{n \in \mathbf{P}}$ has been studied in the literature and the combinatorial proof of a stronger statement appears in [14]. However, in the special case of $\{\Omega(P; n)\}_{n \in \mathbf{P}}$ the proof given in [14] actually generalizes to give a (different) stronger result. In order to prove this result in a reasonably concise way it will be convenient to introduce some notation. Let (P, ω) be a finite labeled poset. For any (P, ω) -partition $\sigma : P \rightarrow [s]$ with largest part $\leq s$ we define a subset $M(\sigma)$ of P by

$$M(\sigma) \stackrel{\text{def}}{=} \{x \in P : \sigma(x) = s\}.$$

We also let

$$E \stackrel{\text{def}}{=} \{(x, y) \in P^2 : x \text{ is covered by } y\}.$$

Let now $\tau : P \rightarrow [t]$ be a (P, ω) -partition with largest part $\leq t$ and σ be as

above. Then we define a subset $B(\sigma, \tau) \subseteq E$ by

$$B(\sigma, \tau) \stackrel{\text{def}}{=} \{(x, y) \in E : \omega(x) > \omega(y), \tau(x) > \sigma(y) + 1, \sigma(x) + 1 > \tau(y)\} \cup \\ \{(x, y) \in E : \omega(x) < \omega(y), \tau(x) \geq \sigma(y) + 1, \sigma(x) + 1 \geq \tau(y)\},$$

and we let $R(\sigma, \tau) \stackrel{\text{def}}{=} E \setminus B(\sigma, \tau)$. For any subset S of P we also let $C_{(\sigma, \tau)}(S)$ be the set of all $x \in P$ that are connected by a sequence of edges in $R(\sigma, \tau)$ to some $y \in S$. Finally, for any two functions $f, g : P \rightarrow \mathbf{N}$, we let

$$(f < g) \stackrel{\text{def}}{=} \{x \in P : f(x) < g(x)\}$$

and define $(f \leq g)$ analogously.

The next result gives a fundamental property of the operator $C_{(\sigma, \tau)}$.

Lemma 7.6.4 *Let (P, ω) , σ and τ be as above and let $S \subseteq (\sigma < \tau)$. Then $C_{(\sigma, \tau)}(S) \subseteq (\sigma < \tau)$.*

Proof: It is clearly enough to show that if $y \in (\sigma < \tau)$ and either $(x, y) \in R(\sigma, \tau)$ or $(y, x) \in R(\sigma, \tau)$ then $x \in (\sigma < \tau)$. Suppose that $(x, y) \in R(\sigma, \tau)$ and that $\omega(x) > \omega(y)$, say. Then $\sigma(x) + 1 \leq \tau(y)$ (otherwise we would have $\sigma(y) + 1 \geq \tau(x) > \tau(y) > \sigma(y)$) and therefore $\tau(x) > \sigma(x)$, as desired. The case $\omega(x) < \omega(y)$ is similar and so is the case $(y, x) \in R(\sigma, \tau)$. $\square$

We can now prove the first main result of this section.

Theorem 7.6.5 *Let (P, ω) be a labeled poset. Then the sequence $\{\Omega(P, \omega; n)\}_{n \in \mathbf{P}}$ is log-concave.*

Proof: Let $s \in \mathbf{P}$, $s \geq 2$. We will construct an injection from pairs (σ_1, σ_2) of (P, ω) -partitions with largest part $\leq s-1$ and $s+1$, respectively to pairs (σ_3, σ_4) of (P, ω) -partitions with largest part $\leq s$. Let σ_1 and σ_2 be as above. Then, clearly, $M(\sigma_2) \subseteq (\sigma_1 < \sigma_2)$ and therefore, by Lemma 7.6.4, $C_{(\sigma_1, \sigma_2)}(M(\sigma_2)) \subseteq (\sigma_1 < \sigma_2)$. Hence, in particular, $\sigma_2(x) \geq 2$ for all $x \in C_{(\sigma_1, \sigma_2)}(M(\sigma_2))$. We can therefore define two maps $\sigma_3, \sigma_4 : P \rightarrow [s]$ by

$$(\sigma_3(x), \sigma_4(x)) \stackrel{\text{def}}{=} \begin{cases} (\sigma_2(x) - 1, \sigma_1(x) + 1), & \text{if } x \in C_{(\sigma_1, \sigma_2)}(M(\sigma_2)), \\ (\sigma_1(x), \sigma_2(x)), & \text{otherwise.} \end{cases} \quad (23)$$

Let now $(x, y) \in E$. Then it can be easily verified that

$$\sigma_3(x) > \sigma_3(y) \quad , \quad \sigma_4(x) > \sigma_4(y)$$

if $\omega(x) > \omega(y)$, and that

$$\sigma_3(x) \geq \sigma_3(y) \quad , \quad \sigma_4(x) \geq \sigma_4(y)$$

if $\omega(x) < \omega(y)$. Hence σ_3 and σ_4 are (P, ω) -partitions with largest part $\leq s$.

To prove that the correspondence $(\sigma_1, \sigma_2) \mapsto (\sigma_3, \sigma_4)$ above defined is injective we show that σ_1 and σ_2 are uniquely determined by σ_3 and σ_4 . In fact, it is a routine matter to check that

$$M(\sigma_2) = M(\sigma_3),$$

and that

$$B(\sigma_1, \sigma_2) = B(\sigma_3, \sigma_4),$$

(we leave the details to the reader). But this implies that

$$C_{(\sigma_1, \sigma_2)}(M(\sigma_2)) = C_{(\sigma_3, \sigma_4)}(M(\sigma_3))$$

and hence, by (23), that σ_1 and σ_2 are uniquely determined by σ_3 and σ_4 , as desired. $\square$

Now, it is a well known classical result (see, e. g. ,[5]) that every finite distributive lattice is of the form $J(P)$ for some poset P . Hence, combining Proposition 7.6.3 and Theorem 7.6.5 in the special case that ω is a natural labeling we obtain the second main result of this section.

Theorem 7.6.6 *Let L be a distributive lattice. Then the digraph $O(L)$ is PF_2 . $\square$*

It should here be noted that, however, it is still an open question whether $S(L)$ is a PF_2 digraph for any distributive lattice L .

We close this section by giving two necessary and sufficient conditions for an arbitrary digraph to be PF_2 which might be useful in deciding the above open

problem.

Proposition 7.6.7 *Let D be a digraph. Then the following are equivalent:*

- (i) D is a PF_2 digraph;
- (ii) $|\beta_{2n}(\{n\})| \geq |\beta_{2n}(\{n-1\})|$, for all $n \geq 2$;
- (iii) the sequence $\{|\beta_{2n}(\{i\})|\}_{i=1,\dots,2n-1}$ is symmetric and unimodal, for all $n \geq 2$.

Proof: From (10) we have that

$$|\beta_{2n}(\{i\})| = \begin{vmatrix} A(i) & A(2n) \\ A(0) & A(2n-i) \end{vmatrix} \quad (24)$$

for $i = 1, \dots, 2n-1$. This implies, in particular, that the sequence $\{|\beta_{2n}(\{i\})|\}_{i=1,\dots,2n-1}$ is symmetric (regardless of whether D is PF_2 or not). Furthermore, from (24) there follows that

$$A \begin{pmatrix} 0 & 1 \\ i+1 & 2n-i \end{pmatrix} = |\beta_{2n}(\{i+1\})| - |\beta_{2n}(\{i\})|, \quad (25)$$

for $i = 1, \dots, n-1$, and this shows the equivalence of (i) and (iii). The equivalence of (ii) and (iii) now follows immediately from (25) and Proposition 7.6.1. $\square$

The above proposition thus reduces the problem of proving that a digraph is PF_2 to that of showing that the sequences $\{|\beta_{2n}(\{i\})|\}_{i=1,\dots,2n-1}$ are symmetric and unimodal for all $n \geq 2$ and there are many well known and sophisticated techniques available for doing this (most notably using the representation theory of Lie algebras and superalgebras, see, e. g. , [61], [62] and [47]). One other possible use of Proposition 7.6.7 is to find injections $i_n : \beta_{2n}(\{n-1\}) \hookrightarrow \beta_{2n}(\{n\})$, for all $n \geq 2$, thus giving a combinatorial proof of the PF_2 property for a digraph D . It would be interesting, for example, to find such injections explicitly for the digraphs considered in Theorem 7.6.6. In this case it should, in fact, be possible to “translate” the combinatorial proof of Theorem 7.6.5 (and

hence of Theorem 7.6.6) to obtain such injections.

7.7 Semitransitive digraphs

In this section we present a very interesting class of digraphs that are always PF . This class has been discovered by Gessel and Viennot in [23] and almost all the results that we present here are due to these two authors (even though they are here stated in a different language). Our main contribution is to show how their work fits in the theory of PF digraphs and what implications it has with respect to the distributive lattice conjecture.

Let $v \in \mathbf{P}$ and $D = ([v], \mathcal{R})$ be a digraph. Given a skew shape ρ/ν then a $[v]$ -*diagram of shape* ρ/ν is just a map $T : \rho/\nu \rightarrow [v]$. For $(i, j) \in \rho/\nu$ we will usually write “ $T_{i,j}$ ” instead of “ $T((i, j))$ ”. Given a map T as above we say that T is a D -*tableau of shape* ρ/ν if T has the property that $(T_{i,j}, T_{i,j+1}) \in \mathcal{R}$ (respectively, $(T_{i+1,j}, T_{i,j}) \notin \mathcal{R}$) for all $(i, j) \in \rho/\nu$ such that $(i, j+1) \in \rho/\nu$ (respectively, $(i+1, j) \in \rho/\nu$). We will denote by $D_{\rho/\nu}$ the set of all D -tableaux of shape ρ/ν . To each $[v]$ -diagram $T : \rho/\nu \rightarrow [v]$ of shape ρ/ν we associate the monomial $x^T \stackrel{\text{def}}{=} \prod_{(i,j) \in \rho/\nu} x_{T_{i,j}}$, where $x_1, \dots, x_v$ are independent variables. Equivalently,

$$x^T = x_1^{t_1} \dots x_v^{t_v},$$

where $t_i \stackrel{\text{def}}{=} |T^{-1}(\{i\})|$ for $i = 1, \dots, v$.

If the reader now analyzes carefully the proof of Theorem 7.4.2 and in particular equation (18) he will easily see that the concepts just introduced provide a convenient way to interpret combinatorially the coefficients of $s_{\lambda/\mu}^D(x_1, \dots, x_v)$ when λ/μ is a border strip. This is made precise in the following result whose easy verification we leave to the reader.

Proposition 7.7.1 *Let D be a digraph and λ/μ be a border strip. Then*

$$s_{\lambda/\mu}^D(x_1, \dots, x_v) = \sum_{T \in D_{\lambda/\mu}} x^T. \square$$

The above result makes it natural to ask whether the same combinatorial interpretation holds for an arbitrary skew-shape λ/μ . Easy examples show that this is not true in general, however, it is possible to characterize quite explicitly the class of digraphs for which the above statement is true.

Let $v \in \mathbf{P}$ and $D = ([v], \mathcal{R})$ be a digraph. We say that D is a *semitransitive digraph* if for all $a, b, c, d \in [v]$ with $(a, b), (b, c) \in \mathcal{R}$ we have that either $(a, d) \in \mathcal{R}$ or $(d, c) \in \mathcal{R}$. The following result gives some elementary properties of semitransitive digraphs and follows easily from the definitions.

Proposition 7.7.2 *Let $D = ([v], \mathcal{R})$ ($v \in \mathbf{P}$) be a digraph. Then the following are equivalent:*

- (i) D is semitransitive;
- (ii) $\overline{D}$ is semitransitive, where $\overline{D}$ is the complement digraph of D ;
- (iii) $D^* \stackrel{\text{def}}{=} ([v], \mathcal{R}^*)$ is semitransitive, where $\mathcal{R}^* \stackrel{\text{def}}{=} \{(a, b) : (b, a) \in \mathcal{R}\}. \square$

The next theorem is due to Gessel and Viennot and is one of the main results of [23]. Since its proof is rather long and the exposition given in [23] is very clear we omit to give it here.

Theorem 7.7.3 *Let D be a semitransitive digraph and λ/μ be a skew-shape. Then $s_{\lambda/\mu}^D(x_1, \dots, x_v) = \sum_{T \in D_{\lambda/\mu}} x^T$.*

In particular, the preceding theorem implies the following result.

Theorem 7.7.4 *Every semitransitive digraph is a PF digraph. $\square$*

Theorems 7.7.3 and 7.7.4 above are very elegant and provide a large class of PF digraphs. Unfortunately, however, Theorem 7.7.3 does *not* hold for any larger class of digraphs. In fact, we have the following elementary but surprising result.

Proposition 7.7.5 *Let D be a digraph. Then D is semitransitive if and only*

if $s_{(2,2)}^D(1, \dots, 1) = |D_{(2,2)}|$.

Proof: By an elementary counting argument it is easy to show that

$$|D_{(2,2)}| = s_{(2,2)}^D(1, \dots, 1) + |S|,$$

where $S \stackrel{\text{def}}{=} \{(a, b, c, d) \in [v]^4 : (a, b, c) \in \beta_3(\emptyset), (a, d, c) \in \beta_3([2])\}$. The thesis follows. $\square$

It should be noted that Proposition 7.7.5 often provides a very easy way to check whether a given digraph is semitransitive or not. Of course, our initial motivation for developing the theory of PF digraphs was to be able to study the distributive lattice conjecture in a more general context. It is therefore natural now to ask whether the results presented in this section have any implication for this conjecture. In view of Theorem 7.7.4 and of the statements of the distributive lattice conjecture given on page 150 the first question that presents itself immediately is to try to characterize those distributive lattices L such that either $O(L)$ or $S(L)$ are semitransitive digraphs. Unfortunately, it is very easy to show that, given a *poset* P then $O(P)$ is semitransitive if and only if P is a chain. Since one can easily prove directly that the poset conjecture holds for chains this is not very useful. However, it does not seem so easy to characterize the posets P such that $S(P)$ is semitransitive. In fact, I don't know of any explicit characterization of even the class of distributive lattices that have that property. So, it is possible that Theorem 7.7.4 may yield some new distributive lattices for which the distributive lattice conjecture holds.

Bibliography

- [1] J. C. Ahuja, *Distribution of the sum of independent decapitated negative binomial variables*, Ann. Math. Statist. **42** (1971), 383-384.
- [2] J. C. Ahuja and E. A. Enneking, *Concavity property and a recurrence relation for associated Lah numbers*, Fibonacci Quart. **17** (1979), 158-161.
- [3] T. Ando, *Totally Positive Matrices*, Linear Algebra and its Applications, **90** (1987), 165-219.
- [4] T. M. Apostol, *Calculus*, vol.1, 2nd ed., John Wiley, New York, 1967.
- [5] Garrett Birkhoff, *Lattice Theory*, 3rd ed., American Math. Soc., Providence, 1967.
- [6] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc., **260** (1980), 159-183.
- [7] A. Björner and M. Wachs, *On lexicographically shellable posets*, Trans. Amer. Math. Soc., **277** (1983), 323-341.
- [8] A. Björner, A. Garsia, and R. Stanley, *An introduction to Cohen-Macaulay partially ordered sets*, in I. Rival, ed., Ordered Sets, Reidel, Dordrecht/Boston, 1982, pp.583-615.
- [9] F. Brenti, *Query #389*, Notices Amer. Math. Soc. **35**(1988), 315.
- [10] L. Carlitz, *Note on the numbers of Jordan and Ward*, Duke Math J. **38** (1971), 783-790.
- [11] L. Chihara and D. Stanton, *Zeros of generalized Krawtchouk polynomials*, preprint.
- [12] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht/Boston, 1974.
- [13] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer-Verlag, Graduate Texts in Mathematics, 11, New York, 1978.
- [14] D. E. Daykin, J. W. Daykin, and M. S. Paterson, *On log-concavity for order preserving maps of partial orders*, Discrete Math. **50** (1984), 221-226.

- [15] P. Edelman, *The Zeta Polynomial of a Partially Ordered Set*, Ph.D. Thesis, June 1980, Massachusetts Institute of Technology.
- [16] P. Edelman, *Chain enumeration and non-crossing partitions*, Discrete Math. **31**(1980), 171-180.
- [17] A. Edrei, *On the generating function of totally positive matrices*, II, J. d'Anal. Math., **2** (1952), 104-109.
- [18] A. Edrei, *Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence*, Canad. J. Math., **5** (1953), 86-94.
- [19] I. Gessel, *Generating Functions and Enumeration of Sequences*, Ph.D. Thesis, June 1977, Massachusetts Institute of Technology.
- [20] I. Gessel, *A note on Stirling permutations*, preprint.
- [21] I. Gessel and R. Stanley, *Stirling polynomials*, J. of Combinatorial Theory(A), **24** (1978), 24-33.
- [22] I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. in Math., **58** (1985), 300-321.
- [23] I. Gessel and G. Viennot, *Determinants, paths, and plane partitions*, preprint.
- [24] P. Gilmore and A. J. Hoffman, *A characterization of comparability graphs and interval graphs*, Canad. J. Math. **16** (1967), 539-548.
- [25] H. W. Gould, *A new symmetrical combinatorial identity*, J. of Combinatorial Theory(A), **13** (1972), 278-286.
- [26] I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, John Wiley, New York, 1983.
- [27] P. Hanlon, *Jack symmetric functions and some combinatorial properties of Young symmetrizers*, J. Combinatorial Theory(A) **47**(1988), 37-70.
- [28] F. Harary, *Graph Theory*, Addison Wesley, 1969.
- [29] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.
- [30] O. J. Heilmann and E. H. Lieb, *Theory of monomer-dimer systems*, Comm. Math. Phys. **25** (1970), 190-232.
- [31] H. Jack, *A class of symmetric polynomials with a parameter*, Proc. Royal Soc. Edinburgh (A) **69**(1969), 1-17.

- [32] C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, 1965.
- [33] S. Karlin, *Total Positivity*, vol.1, Stanford University Press, 1968.
- [34] D. Knutson, *λ -rings and the Representation Theory of the Symmetric Group*, Springer, 1973, Lecture Notes in Mathematics, 308.
- [35] G. Kreweras, *Sur une classe de problèmes de dénombrement liés au treillis des partitions des entiers*, Cahiers du BURO, no.6, Institut de Statistique de L'Univ. Paris, 1965.
- [36] D. C. Kurtz, *A note on concavity properties of triangular arrays of numbers*, J. Combinatorial Theory(A), **13** (1972), 135-139.
- [37] B. J. Levine, *Distribution of Zeros of Entire Functions*, Amer. Math. Soc., Translations of Math. Monographs, 5, Providence, 1964.
- [38] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, New York/London, 1979.
- [39] I. G. Macdonald, *Commuting differential operators and zonal spherical functions*, in Algebraic Groups, Utrecht 1986 (A. M. Cohen, et al., eds.), Lecture Notes in Math., # 1271, Springer, Berlin/Heidelberg/New York, 1987, pp. 189-200.
- [40] P. A. MacMahon, *Combinatory Analysis*, Chelsea, New York, 1960. (Originally published in 2 vols. by Cambridge University Press, 1915-1916.
- [41] E. Maló, *Journal de Mathématiques Spéciales*, Paris, Series 4, **4** (1885), 7.
- [42] J. Neggers, *Representations of finite partially ordered sets*, J. Combin. Inform. System Sci., **3** (1978), 113-133.
- [43] R. H. Nevanlinna, *Eindeutige Analytische Funktionen*, 2nd ed., Springer-Verlag, Berlin, 1953.
- [44] R. H. Nevanlinna, *Analytic Functions*, Springer Verlag, Heidelberg, 1970.
- [45] L. M. Osen, *Women in Mathematics*, M.I.T. Press, Cambridge, Massachusetts, 1974.
- [46] G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, vols. I and II, Springer-Verlag, Berlin, Heidelberg, 1976.
- [47] R. Proctor, *Representations of $sl(2, C)$ on posets and the Sperner property*, SIAM J. Algebraic Discrete Methods **3** (1982), 275-280.
- [48] R. Proctor, *A new product identity for bounded plane partitions*, unpublished research announcement dated January 1984.

- [49] R. Proctor, *Bruhat lattices, plane partitions generating functions, and minuscule representations*, Europ. J. Combinatorics, **5** (1984), 331-350.
- [50] J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley, New York, 1958.
- [51] B. Sagan, *Partially Ordered Sets with Hooklengths: an Algorithmic Approach*, Ph.D. Thesis, May 1979, Massachusetts Institute of Technology.
- [52] B. Sagan, *Enumeration of partitions with hooklengths*, Europ. J. Combinatorics **3**(1982), 85-94.
- [53] B. Sagan, *Inductive and injective proofs of log concavity results*, preprint.
- [54] I. J. Schoenberg, *On smoothing operations and their generating functions*, Bull. Amer. Math. Soc., **59** (1953), 199-230.
- [55] I. Schur, *Zwei Sätze über algebraische gleichungen mit laiter reellen wurzeln*, J. reine angew. Math. **144** (1914), 75-88.
- [56] R. Simion, *A multiindexed Sturm sequence of polynomials and unimodality of certain combinatorial sequences*, J. Combinatorial Theory(A), **36** (1984), 15-22.
- [57] R. Stanley, *Ordered Structures and Partitions*, Ph.D. Thesis, January 1971, Harvard University.
- [58] R. Stanley, *Ordered Structures and Partitions*, Memoirs Amer. Math. Soc., no.119, 1972.
- [59] R. Stanley, *Combinatorial reciprocity theorems*, Advances in Math. **14** (1974), 194-253.
- [60] R. Stanley, *Balanced Cohen-Macaulay complexes*, Trans. Amer. Math. Soc. **249** (1979), 139-157.
- [61] R. Stanley, *Unimodal sequences arising from Lie algebras*, in Young Day Proceedings T. V. Narayana, R. M. Mathsen, and J. G. Williams, eds., Dekker, New York/Basel, 1980, pp. 127-136.
- [62] R. Stanley, *Unimodality and Lie superalgebras*, Studies in Applied Math. **72** (1985), 263-281.
- [63] R. Stanley, *Enumerative Combinatorics* , vol.1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.
- [64] R. Stanley, private communication dated November 1986.

- [65] R. Stanley, *Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry*, to appear in Annals of the New York Academy of Sciences.
- [66] R. Stanley, *Some combinatorial properties of Jack symmetric functions* , to appear in Advances in Math.
- [67] G. Szegő, *Orthogonal Polynomials*, rev. ed., Amer. Math. Soc. Coll. Pub., vol.23, New York, 1959.
- [68] D. Wagner, *The partition polynomial of a finite set system*, preprint.
- [69] M. Ward, *The representation of Stirling's numbers and Stirling's polynomials as sums of factorials*, Amer. J. Math. **56** (1934), 87-95.

Appendix

List of counterexamples:

Note: In the list that follows only counterexamples of degree ≥ 3 are given. The counterexamples of degree 2 are relatively easy to find and are therefore omitted.

$$1 + 3 \binom{x}{1} + 3 \binom{x}{2} + \binom{x}{3} = 1 + 7 \binom{x}{1} + 5 \binom{x}{2} + \binom{x}{3} \quad (1)$$

$$2 \binom{x+3}{3} + 2 \binom{x+2}{3} + 2 \binom{x+1}{3} + 36 \binom{x}{3} = 2 + 8(x)_1 + 6(x)_2 + 7(x)_3 \quad (2)$$

$$1 + 2 \binom{x}{1} + 2 \binom{x}{2} + 12 \binom{x}{3} = 1 + 2(x)_1 + (x)_2 + 2(x)_3 \quad (3)$$

$$1 + 2 \binom{x}{1} + 2 \binom{x}{2} + 12 \binom{x}{3} = 1 + 2\langle x \rangle_1 + \langle x \rangle_2 + 2\langle x \rangle_3 \quad (4)$$

$$8 + 4(x)_1 + 2(x)_2 + (x)_3 = 8 + 4 \binom{x}{1} + 4 \binom{x}{2} + 6 \binom{x}{3} \quad (5)$$

$$8 + 4\langle x \rangle_1 + 2\langle x \rangle_2 + \langle x \rangle_3 = 8 + 4 \binom{x}{1} + 4 \binom{x}{2} + 6 \binom{x}{3} \quad (6)$$

$$8 + 4x + 2x^2 + x^3 = 8 + 7 \binom{x}{1} + 10 \binom{x}{2} + 6 \binom{x}{3} \quad (7)$$

$$5 + \sum_{i=1}^4 \binom{x}{i} = 5 + 4 \binom{x}{1} + 6 \binom{x}{2} + 4 \binom{x}{3} + \binom{x}{4} \quad (8)$$

$$16 + 8x + 4x^2 + 2x^3 + x^4 = 16 + 15(x)_1 + 17(x)_2 + 8(x)_3 + (x)_4 \quad (9)$$

$$\sum_{i=0}^4 \binom{x+5-i}{5} + 11 \binom{x}{5} = 1 + 6 \binom{x}{1} + 15 \binom{x}{2} + 20 \binom{x}{3} + 15 \binom{x}{4} + 16 \binom{x}{5} \quad (10)$$

$$\sum_{i=0}^5 4^{5-i} \langle x \rangle_i = 1024 + 696 \binom{x}{1} + 1088 \binom{x}{2} + 1104 \binom{x}{3} + 576 \binom{x}{4} + 120 \binom{x}{5} \quad (11)$$

$$\sum_{i=0}^6 4^{6-i} \langle x \rangle_i = 4096 + 3504(x)_1 + 3976(x)_2 + 1936(x)_3 + 396(x)_4 + 34(x)_5 + (x)_6 \quad (12)$$

$$7 + \sum_{i=1}^6 \binom{x}{i} = 7 + 6(x)_1 + \frac{15}{2}(x)_2 + \frac{10}{3}(x)_3 + \frac{5}{8}(x)_4 + \frac{1}{20}(x)_5 + \frac{1}{720}(x)_6 \quad (13)$$

$$3 + \sum_{i=1}^7 \binom{x}{i} = \frac{1}{7!}(15120 + 13068x + 13132x^2 + \dots) \quad (14)$$

$$\sum_{i=0}^{11} 5^{11-i} \langle x \rangle_i = 48828125 + 20781410x + 26000005x^2 + \dots \quad (15)$$

Table 1
Transition Matrices between the six fundamental bases

$\nearrow$	$\begin{pmatrix} x \\ i \end{pmatrix}$	$(x)_i$	x^i	$\left(\begin{pmatrix} x \\ i \end{pmatrix}\right)$	$\langle x \rangle_i$	$\begin{pmatrix} x+d-i \\ d \end{pmatrix}$
$\begin{pmatrix} x \\ i \end{pmatrix}$	I	F	SF	M	FM	$B^\#$
$(x)_i$	F^{-1}	I	S	MF^{-1}	FMF^{-1}	$B^\#F^{-1}$
x^i	$F^{-1}s$	s	I	$F^{-1}C$	C	$B^\#F^{-1}s$
$\left(\begin{pmatrix} x \\ i \end{pmatrix}\right)$	M^{-1}	FM^{-1}	$C^{-1}F$	I	F	$B^\#M^{-1}$
$\langle x \rangle_i$	$M^{-1}F^{-1}$	$FM^{-1}F^{-1}$	C^{-1}	F^{-1}	I	$B^\#M^{-1}F^{-1}$
$\begin{pmatrix} x+d-i \\ d \end{pmatrix}$	$(B^{-1})^\#$	$F(B^{-1})^\#$	$SF(B^{-1})^\#$	$M(B^{-1})^\#$	$FM(B^{-1})^\#$	I

Table 2
Transition Matrices between the six fundamental bases for $d = 2$

$\nearrow$	$\begin{pmatrix} x \\ i \end{pmatrix}$	$(x)_i$	x^i	$\left(\begin{pmatrix} x \\ i \end{pmatrix}\right)$	$\langle x \rangle_i$	$\begin{pmatrix} x+2-i \\ 2 \end{pmatrix}$
$\begin{pmatrix} x \\ i \end{pmatrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{matrix}$	$\begin{matrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$
$(x)_i$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{matrix}$	$\begin{matrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{matrix}$
x^i	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{matrix}$
$\left(\begin{pmatrix} x \\ i \end{pmatrix}\right)$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$	$\begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{matrix}$
$\langle x \rangle_i$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & \frac{1}{2} \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{matrix}$
$\begin{pmatrix} x+2-i \\ 2 \end{pmatrix}$	$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{matrix}$	$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

Table 3
The polynomials $U_\lambda(q; -1)$:
 $(|\lambda| \leq 6)$

Note: It follows immediately from equations (2) and (3) of section 6.2 that $U_\lambda(q; -1) = U_{\lambda'}(q; -1)$.

λ	$U_\lambda(q; -1)$
1	q
2	$2q^2$
3	$6q^3$
21	$2q^2 + 4q^3$
4	$24q^4$
31	$12q^3 + 12q^4$
2^2	$12q^3 + 12q^4$
5	$120q^5$
41	$72q^4 + 48q^5$
32	$12q^3 + 72q^4 + 36q^5$
31^2	$12q^3 + 72q^4 + 36q^5$
6	$720q^6$
51	$480q^5 + 240q^6$
42	$144q^4 + 432q^5 + 144q^6$
41^2	$144q^4 + 432q^5 + 144q^6$
3^2	$144q^4 + 432q^5 + 144q^6$
321	$12q^3 + 204q^4 + 396q^5 + 108q^6$

Table 4
The associated Lah numbers, $L_k(n, i)$:
 $(1 \leq i \leq n \leq 5)$

$$L_k(1, 1) = k$$

$$L_k(2, 1) = k(k + 1)$$

$$L_k(2, 2) = k^2$$

$$L_k(3, 1) = k(k + 1)(k + 2)$$

$$L_k(3, 2) = 3k^2(k + 1)$$

$$L_k(3, 3) = k^3$$

$$L_k(4, 1) = k(k + 1)(k + 2)(k + 3)$$

$$L_k(4, 2) = k^2(k + 1)(7k + 11)$$

$$L_k(4, 3) = 6k^3(k + 1)$$

$$L_k(4, 4) = k^4$$

$$L_k(5, 1) = k(k + 1)(k + 2)(k + 3)(k + 4)$$

$$L_k(5, 2) = 5k^2(k + 1)(k + 2)(3k + 5)$$

$$L_k(5, 3) = 5k^3(k + 1)(5k + 7)$$

$$L_k(5, 4) = 10k^4(k + 1)$$

$$L_k(5, 5) = k^5$$

Notations

$\Omega(P, \omega; x)$	20	$UNI[v_i(x)]$	35
$e_s(P, \omega)$	20	$\lambda(P)$	35
$W(P, \omega; z)$	20	$\Lambda(P)$	35
$d(\tau, \omega)$	21	$\mathcal{R}$	35
$E(P, \omega; z)$	21	$(x)_i$	36
$e_s(P)$	22	$\langle x \rangle_i$	36
$\Omega(P, x)$	22	$\left(\binom{x}{i}\right)$	36
$W(P, z)$	22	$S(i, j)$	39
$E(P, z)$	22	$s(i, j)$	39
$\mathbf{s}$	22	$c(i, j)$	39
$J(P)$	22	B	39
$w_i(P)$	22	F	39
$c_s(P)$	23	S	39
$\overline{\omega}$	23	s	39
$C(P; z)$	23	C	39
$P \oplus Q$	24	M	39
$\omega_1 \oplus \omega_2$	24	$A^\#$	39
$P + Q$	25	$L(p_i(x), q_i(x))$	39
$w_i(P, \omega)$	27	$\overline{A} \ (\ A \subseteq V_p \)$	62
V_p	27	$\overline{PF}[v_i(x)]$	62
$PF[v_i(x)]$	35	$\overline{PF}_2[v_i(x)]$	62
$PF_2[v_i(x)]$	35	$\overline{UNI}[v_i(x)]$	62
$PF_1[v_i(x)]$	35	$\overline{PF}_1[v_i(x)]$	62

$A_i(x)$	64	$Dig(f)$	129
$L_i(x)$	65	$b(n, k)$	130
$K_i^{(d)}(x)$	67	$P_k^*(n, i)$	131
$PF[M_d]$	70	$L_k^*(n, i)$	131
$PF_2[M_d]$	70	$P_k(n, i)$	132
$UNI[M_d]$	70	$L_k(n, i)$	133
Δ	78	M_k	134
$W_l^t \left[\binom{x+d-i}{d} \right]$	79	Q_k	134
P_λ	102	$B_{k,i}$	135
$s_\lambda(n)$	105	$f_k(x)$	135
$c(i, j)$	105	$S_2(n, k)$	139
$h(i, j)$	105	$c_2(n, k)$	139
$P_{r,s,b}$	109	$\chi(G; x)$	141
$\lambda(r, s, b)$	109	$\chi_2(G; l)$	142
$U_m(P; q)$	112	$\tilde{\mu}(G; x)$	142
$r(P)$	114	$\Delta(P)$	142
$c_i(P, \omega)$	115	$\rho(\mathcal{F}; x)$	143
$a[x]$	120	$p_k(\mathcal{F})$	143
$J_\lambda(x; \alpha)$	122	$A_D(n)$	146
H_λ	122	$O(P)$	148
$J_\lambda[x; \alpha]$	122	$S(P)$	148
$U_\lambda(z; \alpha)$	123	$A \binom{i_1, \dots, i_p}{j_1, \dots, j_p}$	150
$Z(D; x)$	125	$\beta_n(S)$	151
$\mathcal{A}$	126	$\alpha_n(S)$	151
$\mathcal{B}$	126	$F(\beta_n(S))(\mathbf{x})$	151
$T_{k,m}$	127	$F(\alpha_n(S))(\mathbf{x})$	151
$\hat{P}$	128	$h_n^D(\mathbf{x})$	152
T_m^2	128	$h_\lambda^D(\mathbf{x})$	153

$s_{\lambda/\mu}^D(\mathbf{x})$	153
Λ	154
Λ^D	154
ψ_D	154
a^D	154
ω	157
$\overline{\mathcal{R}}$	157
$\overline{D}$	157
$M(\sigma)$	159
E	159
$B(\sigma, \tau)$	160
$R(\sigma, \tau)$	160
$C_{(\sigma, \tau)}(S)$	160
$(f < g)$	160
$(f \leq g)$	160
$D_{\rho/\nu}$	163
x^T	163
D^*	164
$\mathcal{R}^*$	164

Index

- associated Lah number, 133
- associated signless Stirling number of the first kind, 139
- associated Stirling number of the second kind, 139
- board, 131
- column strict, 103
- comparability graph, 142
- complement digraph, 157
- content, 105
- conversion, 39
- descent, 134
- disjoint union, 25
- D-tableau, 163
- eventually vanishing, 31
- Ferrers poset, 102
- functional digraph, 129
- fundamental basis, 36
- Gaussian poset, 112
- interlace, 45
- hook-length, 105
- hook-lengths (in a Gaussian poset), 112
- infinite matrix, 30
- internal zeros, 21
- Jack polynomial, 122
- Jordan number, 139
- k -divisible, 127
- Krawtchouk polynomial, 67
- labeled poset, 19
- labeling, 19
- linear extension, 20
- log-concave, 21
- natural labeling, 19
- non-crossing partition, 127
- non-crossing 2-partition, 128
- order complex, 142
- order digraph, 149
- order polynomial, 20
- ordinal sum, 24
- partition polynomial, 143
- path, 146
- PF sequence, 31
- PF formal power series, 31
- PF_r formal power series, 31
- PF -preserving, 70

PF_2 -preserving, 70
 PF_r sequence, 31
 PF digraph, 146
 PF_r digraph, 146
 PF -set, 35
 PF_2 -set, 35
 PF_1 -set, 35
 (P, ω) -partition, 19
rational formal power series, 87
row generating function, 34
row (of a board), 131
semitransitive, 164
set system, 143
sign sequence, 30
 SR_r matrix, 30
 SR_r sequence, 31
Stirling permutation, 134
Stirling polynomial, 139
strict order digraph, 149
 TP matrix, 30
 TP_r matrix, 30
2-partition, 128
unimodal, 21
Unimodal-set, 35
 UNI -preserving, 70
 $[v]$ -diagram, 163
Ward number, 139