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Characters of (relatively) integrable modules over affine Lie superalgebras

Maria Gorelik[⋆] · **Victor G. Kac**[⋆⋆]

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Abstract. In the paper we consider the problem of computation of characters of relatively integrable irreducible highest weight modules L over finite-dimensional basic Lie superalgebras and over affine Lie superalgebras $\mathfrak{g}$. The problem consists of two parts. First, it is the reduction of the problem to the $\overline{\mathfrak{g}}$-module $F(L)$, where $\overline{\mathfrak{g}}$ is the associated to L integral Lie superalgebra and $F(L)$ is an integrable irreducible highest weight $\overline{\mathfrak{g}}$-module. Second, it is the computation of characters of integrable highest weight modules. There is a general conjecture concerning the first part, which we check in many cases. As for the second part, we prove in many cases the KW-character formula, provided that the KW-condition holds, including almost all finite-dimensional $\mathfrak{g}$-modules when $\mathfrak{g}$ is basic, and all maximally atypical non-critical integrable $\mathfrak{g}$-modules when $\mathfrak{g}$ is affine with non-zero dual Coxeter number.

Keywords and phrases: basic Lie superalgebra, defect, dual Coxeter number, affine Lie superalgebra, odd reflection, integrable highest weight module over basic and affine Lie superalgebra, maximally atypical module, KW-condition, Enright functor, relatively integrable module, character formula

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M. Gorelik
Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
(e-mail: maria.gorelik@weizmann.ac.il)

V.G. Kac
Department of Mathematics, 2-178, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA
(e-mail: kac@math.mit.edu)

Contents

1. Introduction

Given a symmetrizable Kac–Moody algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$ and an irreducible non-critical highest weight $\mathfrak{g}$-module $L = L(\lambda)$, one constructs the associated integral Kac–Moody algebra $\overline{\mathfrak{g}}^\lambda$ as follows. Let $\Delta_{re} \subset \mathfrak{h}^*$ be the set of real roots of $\mathfrak{g}$ and let

$$\Delta_{re}(\lambda) = \{\alpha \in \Delta_{re} \mid 2(\lambda + \rho, \alpha)/(\alpha, \alpha) \in \mathbb{Z}\}$$

be the set of integral real roots. Then $\Delta_{re}(\lambda)$ is the set of real roots of a Kac–Moody algebra $\overline{\mathfrak{g}}^\lambda$ with the same Cartan subalgebra $\mathfrak{h}$. An important result of representation theory is the following relation between the characters of highest weight $\mathfrak{g}$-module L and the (non-critical) highest weight $\overline{\mathfrak{g}}^\lambda$-module $\overline{L} = \overline{L}(\lambda + \rho - \overline{\rho})$:

$$Re^\rho \operatorname{ch} L(\lambda) = \overline{R} e^{\overline{\rho}} \operatorname{ch} \overline{L}(\lambda + \rho - \overline{\rho}), \tag{1}$$

where R and $\overline{R}$ denote the Weyl denominators, and ρ and $\overline{\rho}$ denote the Weyl vectors (see [F], [KT1], [KT2] and references there).

In the case when the $\overline{\mathfrak{g}}^\lambda$-module $\overline{L}(\lambda + \rho - \overline{\rho})$ is integrable, its character is given by the Weyl–Kac character formula [K2], hence (1) gives an explicit formula for $\operatorname{ch} L(\lambda)$.

A $\mathfrak{g}$-module is called *relatively integrable* if the $\overline{\mathfrak{g}}^\lambda$-module $\overline{L}(\lambda + \rho - \overline{\rho})$ is integrable; and it is called *admissible* if, in addition, the $\mathbb{Q}$-span of the set of roots of $\overline{\mathfrak{g}}^\lambda$ coincides with $\mathbb{Q}\Delta$. In particular, if $\lambda = \overline{\rho} - \rho$, we obtain from (1) that the character is given by a product:

$$\operatorname{ch} L(\overline{\rho} - \rho) = e^{\overline{\rho} - \rho} R^{-1} \overline{R}. \tag{2}$$

For example, if $\mathfrak{g}$ is an affine Lie algebra with symmetric Cartan matrix, then there exist admissible λ of rational level k, provided that $k+h^\vee \geq h^\vee/u$, where $h^\vee$ is the dual Coxeter number and u is the denominator of k [KW2]. In this case the character $\operatorname{ch} L(\lambda)$, suitably normalized, is a ratio of theta functions, which is a modular function.

The main problems discussed in this paper are whether for a finite-dimensional basic Lie superalgebra or an associated (untwisted or twisted) affine Lie superalgebra $\mathfrak{g}$, similar results hold. This is a class of Lie superalgebras, which is the closest to symmetrizable Kac–Moody algebras. Of course, there are also Kac–Moody superalgebras, associated to symmetrizable generalized Cartan matrix, which have no real isotropic roots. For them the Weyl–Kac character formula is proved in the same way as in Lie algebra case, and the relation (1) can be derived using Enright functors [IK]. Therefore we exclude these superalgebras from consideration.

Given an irreducible highest weight $\mathfrak{g}$-module $L = L(\lambda)$, we construct in §7.2 a natural generalization of the set of integral real roots for the Lie superalgebra $\mathfrak{g}$ and the corresponding integral Lie superalgebra $\overline{\mathfrak{g}}^\lambda$ (which is also basic or affine, or a sum of such superalgebras), and we prove formula (1) in some cases. In particular, we prove formula (2), see Corollary 11.2.6. We believe that (1) holds for arbitrary λ, but there are not enough techniques to prove this, mainly due to the lack of translation functors (used in [F] in the Lie algebra case), and the lack of Enright functors, associated to isotropic simple roots ([KT1] and [KT2] use them in the Lie algebra case, where all simple roots are non-isotropic). So far, in full generality formula (1) is proved only for finite-dimensional $\mathfrak{g}$ of type $A(m,n)$, see [CMW].

It would be natural to call an irreducible highest weight module L over the Lie superalgebra $\mathfrak{g}$ integrable if it is integrable as a $\mathfrak{g}_{\overline{0}}$-module. For finite-dimensional $\mathfrak{g}$, this definition is adequate, because it is equivalent to $\dim L < \infty$. However, for affine $\mathfrak{g}$, such non one-dimensional integrable irreducible highest weight modules exist only if the Dynkin diagram of $\mathfrak{g}_{\overline{0}}$ is connected, see [KW4]. For that reason, in the affine case, it is natural to study *π-integrable* modules, where π is a subset of the set of simple roots Π_0 of $\mathfrak{g}_{\overline{0}}$, namely the $\mathfrak{g}$-modules L for which all root spaces $\mathfrak{g}_{-\alpha}, \alpha \in \pi$, act locally nilpotently. The definition of (π-)relative integrability and admissibility of L is the same as in the Lie algebra case.

We call a $\mathfrak{g}$-module *integrable* (both for $\mathfrak{g}$ basic and affine) if L is integrable with respect to the "largest" component of $\mathfrak{g}_{\overline{0}}$. For example, if $\mathfrak{g}$ is the non-twisted affine Lie superalgebra, associated to a simple finite-dimensional Lie superalgebra $\dot{\mathfrak{g}}$ with a non-degenerate Killing form κ (then $\dot{\mathfrak{g}}$ is automatically basic), a $\mathfrak{g}$-module L is called integrable if it is π-integrable for $\pi = \{\alpha \in \Pi_0 \mid \kappa(\alpha,\alpha) > 0\}$. (This conicides with the definition of integrability of the affine Lie algebra modules [K3].) The choice of π in the definition of integrability for an arbitrary (possibly twisted) affine Lie superalgebra is explained in §3.1.3. The study of integrable modules over affine Lie superalgebras is very important for applications to modular invariance of modified characters [KW5]–[KW7]. Note that if $\dim \mathfrak{g} < \infty$, then not only the finite-dimensional $\mathfrak{g}$-modules are integrable, unless $\mathfrak{g}$ is of type A or C.

Let $L(\lambda)$ be either an integrable module over a finite-dimensional basic Lie superalgebra $\mathfrak{g}$, or an integrable module of non-critical level over an affine Lie superalgebra $\mathfrak{g}$, and let Δ be the set of roots of $\mathfrak{g}$. Let $\Delta^{\perp}_{\lambda+\rho}$ be the set of roots of $\mathfrak{g}$, orthogonal to $\lambda + \rho$, and choose a maximal linearly independent subset S in $\Delta^{\perp}_{\lambda+\rho}$, which spans an isotropic subspace. Assume that S satisfies the KW-condition, namely S can be included in a set Π of simple roots of Δ. A natural analogue of the Weyl–Kac character formula for integrable highest weight modules over Kac–Moody algebras is the following KW-formula, proposed in [KW3], §3, for basic $\mathfrak{g}$, and in [KW4], §9, for affine $\mathfrak{g}$:

$$j_\lambda R e^\rho \operatorname{ch} L(\lambda) = \sum_{w\in W'} \operatorname{sgn}(w) w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) \tag{3}$$

for some positive integer j_λ, where W' is a certain subgroup of the Weyl group W.

In §4 of the present paper we prove the KW-formula with $j_\lambda = 1$ and $W' = W(\pi)$, in the case when $\mathbb{Q}S$ is a maximal isotropic subspace in $\mathbb{Q}\Delta$ and the choice of the pair $\Pi \supset S$ is "good", provided that either $\mathfrak{g}$ is basic, or $\mathfrak{g}$ is affine with $h^\vee \neq 0$ or is equal to $A(n,n)^{(1)}$ (in the case $\lambda = 0$ this was proved in [G1], [G2], [R]). The choice of the pair $\Pi \supset S$ is "good" if $(\alpha,\alpha) \geq 0$ for all $\alpha \in \Pi$ and S contains all "branching" nodes of Π (if they exist), as defined in §3.3.1. Using odd reflections, we show that the KW-formula holds for many other (but not all) pairs $\Pi \supset S$. Recall that $h^\vee$ is the half of the eigenvalue of the Casimir operator on basic $\mathfrak{g}$ (which is 0 if and only if $\kappa = 0$), and it is called the *dual Coxeter number* of any twisted affine superalgebra, associated to $\mathfrak{g}$.

Incidentally, using a different character formula for level 1 $\mathfrak{osp}(M,N)^{(1)}$-modules, obtained in [KW4], we thereby derive in §4.5 an interesting identity for mock theta functions.

In §5 we prove the KW-formula for all irreducible finite-dimensional modules (satisfying the KW-condition) over a basic Lie superalgebra $\mathfrak{g}$, except for a few cases when $\mathfrak{g}$ is of type $D(m,n)$[1]. This formula has been previously verified only for $\mathfrak{g}$ of type $A(m,n)$, see [CHR], using the earlier work [S1], [S2], [B], [SZ] on computation of finite-dimensional characters of $\mathfrak{gl}(m,n)$. There have

[1] After this paper has been completed, we learned about the paper of S.-J. Cheng and J.-H. Kwon "Kac–Wakimoto character formula for orthosymplectic Lie superalgebra" [CK], where the KW-formula is established by different methods for all irreducible finite-dimensional $\mathfrak{osp}(m,n)$-modules, satisfying the KW-condition, including the cases we were unable to settle.

been a number of earlier papers, where the KW-formula was verified in the case $\#S = 1$, see [BL], [J1], [J2], [JHKT], [KW3].

In §6 we prove a result similar to (3), see (26), for "strongly integrable" maximally atypical $\mathfrak{g}$-modules in the case when $\mathfrak{g}$ is affine and $h^\vee \neq 0$. There are fewer strongly integrable $\mathfrak{g}$-modules than the integrable ones, but here we do not require that the pair $\Pi \supset S$ is "good". We also prove formula (26) for the non-critical vacuum (but not all strongly integrable) modules over affine superalgebras with $h^\vee = 0$.

In §11 we study another extremal case — when S is empty. Such $\mathfrak{g}$-modules $L = L(\lambda)$ are called typical and it was proven in [K2] that the usual Weyl character formula holds for them if $\dim \mathfrak{g} < \infty$ and $\dim L(\lambda) < \infty$. We prove that formula (3) with $S = \emptyset$ holds if we let W' be the "integral" subgroup $W(L)$ of W, provided that L is relatively integrable. In other words, we prove that in this case both conjectural formulas (1) and (3) hold. We also verify (1) in a few other instances of typical and of relatively integrable modules. As a corollary, we obtain the character formula for all relatively integrable modules over $A(0,n)^{(1)}$ and $C(n)^{(1)}$.

Note, however, that while we expect that (1) always holds, and that (3) holds for all irreducible finite-dimensional modules (satisfying the KW-condition) over basic $\mathfrak{g}$ (*cf.* §5), we do not expect (3) to hold in full generality, except when $\mathbb{Q}S$ is a maximal isotropic subspace of $\mathbb{Q}\Delta$, i.e., the module is maximally atypical.

We also prove formulas (1) and (3) for all admissible $\mathfrak{g}$-modules when $\overline{\mathfrak{g}}^\lambda$ is of small rank, see §9, 11 and 12. In particular, we obtain the character formula for all relatively integrable $B(1,1)^{(1)}$-modules, and also for all admissible $A(1,1)^{(1)}$-modules, associated to integrable vacuum $A(1,1)^{(1)}$-modules.

Our proofs use the ideas from [KT1], [KT2], and [G1], [G2].

In the present paper we prove all character formulas for affine superalgebras $\mathfrak{g}$ (except for those with $h^\vee = 0$) used in [KW5], [KW6] to show that for $\mathfrak{g} = A(1,0)^{(1)}$, $A(1,1)^{(1)}$, and $B(1,1)^{(1)}$ the, modified in the spirit of Zwegers [Z], normalized supercharacters of maximally atypical admissible $\mathfrak{g}$-modules of given level span $SL(2,\mathbb{Z})$-invariant space, and used in [KW7] to show that such modular invariance holds for maximally atypical integrable $\mathfrak{g}$-modules over an arbitrary (non-twisted) affine superalgebra $\mathfrak{g} \neq A(n,n)^{(1)}$.

The results of this paper were reported at the conferences in Uppsala in September 2012, in Rome in December 2012, in Taipei in May 2013, and in Rio de Janeiro in June 2013.

2. Preliminaries

Throughout the paper the base field is $\mathbb{C}$ and $\mathfrak{g}$ is either a basic Lie superalgebra with a non-degenerate invariant bilinear form $(-,-)$, or the associated to it and its finite order automorphism, preserving $(-,-)$, symmetrizable affine Lie superalgebra. Recall that a basic Lie superalgebra $\mathfrak{g}$ is either a simple finite-dimensional Lie algebra or one of the simple finite-dimensional Lie superalgebras $\mathfrak{sl}(m,n)$ $(m \neq n)$, $\mathfrak{psl}(n,n)$ $(n \geq 2)$, $\mathfrak{osp}(m,n)$, $D(2,1,a)$, $F(4)$, $G(3)$ or $\mathfrak{gl}(m,n)$ [K1], and that the associated affine Lie superalgebras are constructed in the same way as in [K3]. Recall that the Killing form κ of $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g} = \mathfrak{sl}(m,n)$ $(m \neq n)$, $\mathfrak{osp}(m,n)$ (m is odd, or m is even and $n \neq m-2 \geq 2$), $F(4)$, $G(3)$; this is equivalent to the property that the dual Coxeter number ($= \frac{1}{2}$ eigenvalue of the Casimir operator on $\mathfrak{g}$) is non-zero. Recall that the dual Coxeter number associated to the Killing form is always a non-negative rational number, see [KW3]. This number is also called the dual Coxeter number of the associated affine superalgebra.

It is well known that for any affine Lie superalgebra the dual Coxeter number is equal to (ρ, δ), where ρ is the Weyl vector and δ is the primitive imaginary root.

The invariant bilinear form extends from the basic Lie superalgebra to the associated affine Lie superalgebra and is denoted again by $(-,-)$.

Recall that one often uses the following notations: $A(m,n) = \mathfrak{sl}(m+1, n+1)$ or $\mathfrak{gl}(m+1, n+1)$ for $m \neq n$, $A(n,n) = \mathfrak{psl}(n,n)$ or $\mathfrak{gl}(n,n)$, $B(m,n) = \mathfrak{osp}(2m+1, 2n)$, $C(n) = \mathfrak{osp}(2,2n)$, $D(m,n) = \mathfrak{osp}(2m,2n)$ $(m > 1)$. The associated affine Lie superalgebra, twisted by an automorphism of $\mathfrak{g}$ of order r, is denoted by $\mathfrak{g}^{(r)}$. We will often write $\Delta = A(m,n)$ to indicate that Δ is the root system of $A(m,n)$, or $\Pi = A(m,n)$ to indicate that Π is a subset of simple roots for the root system of type $A(m,n)$.

Recall that we get all affine Lie superalgebras by picking an automorphism in each connected component of the group of automorphisms of $\mathfrak{g}$. (The affine Lie superalgebra depends only on this connected component; however, unlike in the Lie algebra case, some of the affine Lie superalgebras corresponding to different connected components may be isomorphic.)

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. As in the Lie algebra case, $\mathfrak{g}$ has the root space decomposition with respect to $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots. Denote by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ the subsets of even and odd roots. The restriction of $(-,-)$ to $\mathfrak{h}$ is non-degenerate, hence it induces a bilinear form on $\mathfrak{h}^*$. One can show that $\Delta_{\bar{0}}$ is a union of a finite number of root systems of affine Lie algebras with the same primitive imaginary root δ.

We define $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ if $\alpha \in \Delta$ is a non-isotropic root; for isotropic $\alpha \in \Delta$ we set $\alpha^\vee = \alpha$. Notice that $(\mu, \alpha^\vee)$ does not depend on the normalization of $(-,-)$ if α is non-isotropic.

The Weyl group W of Δ is the subgroup of $GL(\mathfrak{h}^*)$, generated by reflections r_α in non-isotropic roots α, where $r_\alpha \lambda = \lambda - 2(\lambda,\alpha)\alpha/(\alpha,\alpha)$. One knows that W coincides with the Weyl group of $\Delta_{\bar{0}}$ and that $W\Delta = \Delta$ (*cf.* [K3], Chapter 3).

Let $Q := \mathbb{Z}\Delta$ and $Q_{\bar{0}} := \mathbb{Z}\Delta_{\bar{0}}$ be the corresponding root lattices.

2.1. Subsets of positive roots in Δ

Given a real valued additive function χ on Q, which is positive on δ and does not vanish on elements of Δ, we have the corresponding subsets of positive roots Δ^+ and $\Delta^+_{\bar{0}}$ (on which χ is positive).

For different choices of χ the subsets of even positive roots may be different, but they can be transformed to each other by the Weyl group. Throughout the paper we will fix one of them, $\Delta^+_{\bar{0}}$, and consider only the subsets of positive roots Δ^+ in Δ, which contain $\Delta^+_{\bar{0}}$. This choice fixes a triangular decomposition of $\mathfrak{g}$, compatible with the triangular decomposition of $\mathfrak{g}_{\bar{0}}$, corresponding to $\Delta^+_{\bar{0}}$.

Recall that, given a subset of positive roots Δ^+ (containing $\Delta^+_{\bar{0}}$) and an odd simple root $\beta \in \Delta^+$ with $(\beta, \beta) = 0$, we can construct a new subset of positive roots (containing $\Delta^+_{\bar{0}}$) by an *odd reflection* r_β:

$$r_\beta(\Delta^+) = (\Delta^+ \setminus \{\beta\}) \cup \{-\beta\}. \tag{4}$$

2.1.1. **Proposition** *[S4]*

(a) Any two subsets of positive roots in Δ *(containing* $\Delta^+_{\bar{0}}$*) can be obtained from each other by a finite sequence of odd reflections.*

(b) For a simple root $\alpha \in \Delta^+_{\bar{0}}$ *there exists a subset of positive roots, for which* α *or* $\frac{\alpha}{2}$ *is a simple root.*

2.1.2. Let Δ^+ be a subset of positive roots in Δ, and denote by Π the subset of its simple roots; we shall often write $\Delta^+ = \Delta^+(\Pi)$. One has $r_\beta\Delta^+ = \Delta^+(\Pi')$, where

$$\Pi' := \{\alpha \in \Pi \mid \alpha \neq \beta, (\alpha, \beta) = 0\} \cup \{\alpha + \beta \mid \alpha \in \Pi, (\alpha, \beta) \neq 0\} \cup \{-\beta\}.$$

Then we can choose a Weyl vector $\rho_\Pi \in \mathfrak{h}^*$, such that the following two properties hold for each subset Π of simple roots:

(i) $2(\rho_\Pi, \alpha) = (\alpha, \alpha)$, if $\alpha \in \Pi$;
(ii) $\rho_{r_\beta \Pi} = \rho_\Pi + \beta$, if $\beta \in \Pi, (\beta, \beta) = 0$.

Indeed, choose a set of positive roots and let Π be its subset of simple roots; pick an arbitrary Weyl vector ρ_Π, satisfying (i). Define (*cf.* Proposition 2.1.1 (a)):

$$\rho_{r_{\beta_1} \cdots r_{\beta_s} \Pi} := \rho_\Pi + \beta_1 + \cdots + \beta_s.$$

Then (ii) obviously holds and (i) is straightforward to check. Finally, this is well-defined since the equality

$$r_{\beta_1} \cdots r_{\beta_s} \Pi = r_{\gamma_1} \cdots r_{\gamma_t} \Pi \tag{5}$$

forces the equality $\beta_1 + \cdots + \beta_s = \gamma_1 + \cdots + \gamma_t$. Indeed, let us show by induction that

$$\Delta^+(r_{\beta_1} \cdots r_{\beta_s} \Pi) = (\Delta^+(\Pi) \setminus S) \cup (-S),$$

where S is obtained from a multiset $\{\beta_1, \dots, \beta_s\}$ by removing all pairs of opposite roots (i.e., the pairs of the form $(\beta, -\beta)$). Moreover, if $r_{\beta_1} \cdots r_{\beta_s} \Pi$ is defined, then S is a set (each element appears once) and $S \subset \Delta^+(\Pi)$. This can be proven by induction on s. For $s = 1$ this follows from (4); if $r_{\beta_{j+1}} \cdots r_{\beta_1} \Pi$ is defined, then β_{j+1} lies in $r_{\beta_j} \cdots r_{\beta_1} \Pi$ and, in particular, in $\Delta^+(r_{\beta_j} \cdots r_{\beta_1} \Pi) = (\Delta^+(\Pi) \setminus S) \cup (-S)$ by the induction hypothesis. This means that $\beta_{j+1} \not\in S$ and

$$\Delta^+(r_{\beta_{j+1}} \cdots r_{\beta_1} \Pi) = (((\Delta^+(\Pi) \setminus S) \cup (-S)) \setminus \{\beta_{j+1}\}) \cup \{-\beta_{j+1}\}.$$

If $-\beta_{j+1} \not\in S$, then

$$\Delta^+(r_{\beta_{j+1}} \cdots r_{\beta_1} \Pi) = (\Delta^+(\Pi) \setminus (S \cup \{\beta_{j+1}\})) \cup (-(S \cup \{\beta_{j+1}\})),$$

and, if $-\beta_{j+1} \in S$, then

$$\Delta^+(r_{\beta_{j+1}} \cdots r_{\beta_1} \Pi) = (\Delta^+(\Pi) \setminus (S \setminus \{-\beta_{j+1}\})) \cup (-(S \setminus \{-\beta_{j+1}\})),$$

as required.

Now (5) implies

$$(\Delta^+(\Pi) \setminus S) \cup \{-S\} = (\Delta^+(\Pi) \setminus T) \cup \{-T\},$$

where S (resp., T) is the set obtained from the set $\{\beta_1, \dots, \beta_s\}$ (resp., $\{\gamma_1, \dots, \gamma_t\}$) by removing all pairs of opposite roots. This gives $S = T$ so $\beta_1 + \cdots + \beta_s = \gamma_1 + \cdots + \gamma_t$, as required.

2.1.3. Let $Q^+ := \mathbb{Z}_{\geq 0}\Delta^+ (= \mathbb{Z}_{\geq 0}\Pi)$ and $Q^+_{\bar{0}} := \mathbb{Z}_{\geq 0}\Delta^+_{\bar{0}}$, where $\mathbb{Z}_{\geq 0}S$ denote the semigroup of linear combinations of elements from S with coefficients from $\mathbb{Z}_{\geq 0}$. The set Q^+ depends on the set Δ^+ of positive roots, and to emphasize this dependence we shall write $Q^+ = Q^+(\Pi)$, but the set $Q^+_{\bar{0}}$ does not (since we fixed $\Delta^+_{\bar{0}}$).

We consider the corresponding partial orderings on $\mathfrak{h}^*$. The first one is $\nu \geq \mu$ if $\nu - \mu \in Q^+_{\bar{0}}$, and the second one is $\nu \geq_\Pi \mu$ if $\nu - \mu \in \mathbb{Z}_{\geq 0}\Pi$ (it depends on Π). Given $S \subset \mathfrak{h}^*$, an element $\lambda \in S$ is called *maximal* (resp., Π-maximal) if $\lambda \geq \nu$ (resp., $\lambda \geq_\Pi \nu$) for all $\nu \in S$.

Note that the partial ordering $\geq_\Pi$ can be extended to a total ordering $\geq_{\Pi,tot}$ as follows. Fix a total ordering on the basis $\Pi \cup \{\Lambda_0\} = \{\gamma_1, \dots, \gamma_m\}$ of $\mathfrak{h}^*$ and the lexicographic order on $\mathbb{C}$. Then $\nu = \sum_i a_i\gamma_i \geq_{\Pi,tot} \mu = \sum_i b_i\gamma_i$ if $(a_1, \dots, a_m) \geq (b_1, \dots, b_m)$ in the lexicographic order.

2.2. *The algebra $\mathcal{R}$*

We introduce the algebra $\mathcal{R} = \mathcal{R}(\Pi)$ as in [G1], [G2]. The new part is §2.2.1 and §2.2.7–2.2.9.

Let $\mathcal{V}$ be the vector space over $\mathbb{Q}$ of all formal sums (possibly infinite) $Y = \sum_{\nu\in\mathfrak{h}^*} b_\nu e^\nu, b_\nu \in \mathbb{Q}$, and define the support of Y by

$$\operatorname{supp} Y := \{\nu \mid b_\nu \neq 0\}.$$

Let $\mathcal{R}(\Pi)$ be the subspace of $\mathcal{V}$, consisting of finite linear combinations of the elements of the form $\sum_{\nu\in\mathbb{Z}_{\geq 0}\Pi} b_\nu e^{\lambda-\nu}$, where $\lambda \in \mathfrak{h}^*$. The space $\mathcal{R}(\Pi)$ has an obvious structure of a unital commutative algebra, induced by $e^\mu e^\nu = e^{\mu+\nu}, e^0 = 1$. Moreover, $\mathcal{R}(\Pi)$ is a domain. This is clear since for any $Y \in \mathcal{R}(\Pi)$, its support $\operatorname{supp} Y$ has a unique maximal element in the total ordering $\geq_{\Pi,tot}$ and the maximal element in $\operatorname{supp} YY'$ is equal to the sum of maximal elements in $\operatorname{supp} Y$ and in $\operatorname{supp} Y'$.

For each Π define a topology on $\mathcal{V}$ by the set of open neighborhoods $\mathcal{V}_\lambda$, consisting of $Y \in \mathcal{V}$ such that $\operatorname{supp} Y \leq_{\Pi,tot} \lambda$. This makes $\mathcal{R}(\Pi)$ a topological algebra. We idenfity the convergent infinite sums of elements of $\mathcal{R}(\Pi)$ with their limits.

2.2.1. Let

$$\mathcal{V}_{fin} := \{Y \in \mathcal{V} \mid \operatorname{supp} Y \text{ is finite}\}.$$

This is a subalgebra of all algebras $\mathcal{R}(\Pi)$. Hence

$$\mathcal{V}_{fin}\mathcal{R}(\Pi) \subset \mathcal{R}(\Pi). \tag{6}$$

Note also that $\mathcal{V}$ is a $\mathcal{V}_{fin}$-module (but not an algebra). Introduce the equivalence relation $\sim$ on $\mathcal{V}$ by: $X \sim X'$ if there exists $Y \in \mathcal{V}_{fin}$ such that $XY = X'Y$. Note that if $X \in \mathcal{R}(\Pi), X' \in \mathcal{R}(\Pi')$ and $Y \in \mathcal{V}_{fin}$, then $XY = X'Y \in \mathcal{R}(\Pi) \cap \mathcal{R}(\Pi')$ by (6). Since $\mathcal{R}(\Pi)$ is a domain, the equivalence of its two elements $X, X' \in \mathcal{R}(\Pi)$ implies $X = X'$.

2.2.2. Action of the Weyl group. The Weyl group W acts on $\mathscr{V}$ in the obvious way:

$$w\Big(\sum_\nu b_\nu e^\nu\Big) := \sum_\nu b_\nu e^{w\nu}.$$

Obviously, $\mathscr{V}_{fin}$ is W-invariant, but $\mathscr{R} = \mathscr{R}(\Pi)$ is not. For a subgroup $W' \subset W$ introduce the following subalgebra of the algebra $\mathscr{R}$:

$$\mathscr{R}_{W'} := \{Y \in \mathscr{R} \mid wY \in \mathscr{R} \text{ for each } w \in W'\}.$$

2.2.3. Infinite products. A product of the form

$$Y = \prod_{\alpha\in A} (1 + a_\alpha e^{-\alpha})^{d_\alpha}, \tag{7}$$

where $A \subset \Delta$ is such that the set $A \setminus \Delta^+(\Pi)$ is finite, and $a_\alpha \in \mathbb{Q}$, $d_\alpha \in \mathbb{Z}_{\geq 0}$, can be naturally viewed as an element of $\mathscr{R}$. Since $\Delta^+(\Pi) \setminus \Delta^+(\Pi')$ is a finite set (by Proposition 2.1.1 (a)), the element Y lies in all algebras $\mathscr{R}(\Pi')$. Hence the set $\mathscr{Y}$ of all such products is a multiplicative subset of each of the algebras $\mathscr{R}(\Pi)$.

For any $w \in W$ the product

$$wY := \prod_{\alpha\in A} (1 + a_\alpha e^{-w\alpha})^{d_\alpha},$$

is of the above form, since the set $w\Delta_+ \setminus \Delta_+ = -(w\Delta_- \cap \Delta_+)$ is finite. Hence $\mathscr{Y}$ is a W-invariant multiplicative subset of $\mathscr{R}_W$ (for each Π).

Consider the localization $\mathscr{R}_{W'}[\mathscr{Y}^{-1}]$ of the algebra $\mathscr{R}_{W'}$ by the multiplicative subset $\mathscr{Y}$. Let $\varphi_\Pi : \mathscr{R}_{W'}[\mathscr{Y}^{-1}] \to \mathscr{R}$ be an algebra homomorphism, defined by expanding in a geometric progression for $\beta \in \Delta^+$, $a \in \mathbb{Q} \setminus \{0\}$:

$$\varphi_\Pi\Big(\frac{e^\lambda}{1 + ae^{-\beta}}\Big) = e^\lambda(1 - ae^{-\beta} + a^2e^{-2\beta} - \cdots);$$
$$\varphi_\Pi\Big(\frac{e^\lambda}{1 + ae^{\beta}}\Big) = \frac{1}{a}\varphi_\Pi\Big(\frac{e^{\lambda-\beta}}{1 + a^{-1}e^{-\beta}}\Big).$$

This homomorphism defines an embedding of $\mathscr{R}_{W'}[\mathscr{Y}^{-1}]$ in $\mathscr{R}$.

2.2.4. We extend the action of W' from $\mathscr{R}_{W'}$ to $\mathscr{R}_{W'}[\mathscr{Y}^{-1}]$ by setting $w(Y^{-1}X) := (wY)^{-1}(wX)$ for each $X \in \mathscr{R}_{W'}, Y \in \mathscr{Y}$. Let Y be as in (7). Then

$$\operatorname{supp} Y \subset \lambda' - Q^+, \quad \text{where } \lambda' := -\sum_{\{\alpha\in A\setminus\Delta_+ \mid a_\alpha\neq 0\}} d_\alpha\alpha. \tag{8}$$

2.2.5. Let W' be a subgroup of W. For $Y \in \mathscr{R}[\mathscr{Y}^{-1}]$ we say that Y *is* W'-*invariant (resp.,* W'*-skew-invariant)* if $wY = Y$ (resp., $wY = \operatorname{sgn}(w)Y$) for each $w \in W'$.

Note that $Y := \sum_\mu a_\mu e^\mu \in \mathscr{R}$ is a W'-skew-invariant element of $\mathscr{R}_{W'}$ if and only if $a_{w\mu} = \operatorname{sgn}(w)a_\mu$. In particular, if Y is a W'-skew-invariant element of $\mathscr{R}_{W'}$, then $W' \operatorname{supp}(Y) = \operatorname{supp}(Y)$.

We will use the following fact: if $Y \in \mathscr{R}_{W'}$ is W'-skew-invariant and $\rho + \operatorname{supp} Y$ consists of non-critical weights, then $\operatorname{supp} Y$ is the union of regular W'-orbits, where regularity means that the elements of this orbit have trivial stabilizers. This is an immediate corollary of the fact that for an affine Lie algebra the W-orbit of each weight of non-zero level contains either maximal or minimal element and the stabilizer of this element in W is generated by simple reflections; as a result the stabilizer of any weight of non-zero level is generated by reflections. Since for such a reflection r_α we have $r_\alpha Y = -Y$, we have $r_\alpha \lambda = \lambda \Rightarrow \lambda \not\in \operatorname{supp} Y$.

Let $Y := \sum_\mu a_\mu e^\mu$ be any element of $\mathscr{R}_{W'}$. We claim that if $\sum_{w\in W'} \operatorname{sgn}(w)\, w(Y) \in \mathscr{R}$, then $\sum_{w\in W'} \operatorname{sgn}(w)\, w(Y)$ is a W'-skew-invariant element of $\mathscr{R}_{W'}$. Indeed, $\sum_{w\in W'} \operatorname{sgn}(w)\, w(Y) = \sum_\nu b_\nu e^\nu$, where $b_\nu = \sum_{w\in W'} \operatorname{sgn}(w) a_{w\nu}$, so $b_{w\nu} = \operatorname{sgn}(w) b_\nu$, as required.

2.2.6. For each set of simple roots Π' introduce the following products

$$R_{\bar 0} := \prod_{\alpha\in\Delta^+_{\bar 0}} (1-e^{-\alpha}), \quad R(\Pi')_{\bar 1} := \prod_{\alpha\in\Delta_+(\Pi')\cap\Delta_{\bar 1}} (1+e^{-\alpha}).$$

One readily sees (by Proposition 2.1.1 (a)) that $R_{\bar 0}, R(\Pi')_{\bar 1} \in \mathscr{Y}$. We view $R_{\bar 0}, R(\Pi')_{\bar 1}$ and

$$R(\Pi') := \frac{R_{\bar 0}}{R(\Pi')_{\bar 1}}$$

as elements in $\mathscr{R}(\Pi)$, as in §2.2.3. One readily sees that $R(\Pi')e^{\rho_{\Pi'}} \in \mathscr{R}(\Pi)$ does not depend on Π', so we write simply Re^ρ (keeping in mind that this is an element of $\mathscr{R}(\Pi)$ for particular Π). By §2.2.3, all these elements are equivalent (for different Π). Since $R_{\bar 0}, R(\Pi')_{\bar 1} \in \mathscr{Y} \subset \mathscr{R}_W$, the element Re^ρ lies in $\mathscr{R}_W[\mathscr{Y}^{-1}]$. Clearly, $r_\alpha(R(\Pi')e^{\rho_{\Pi'}}) = -R(\Pi')e^{\rho_{\Pi'}}$ for a non-isotropic root $\alpha \in \Pi'$. From Proposition 2.1.1 (b), we conclude that Re^ρ is a W-skew-invariant element of $\mathscr{R}_W[\mathscr{Y}^{-1}]$.

If Π is fixed, we denote by $R_{\bar 1} := R(\Pi)_{\bar 1}$, $R := R(\Pi)$ the corresponding elements in $\mathscr{R}$.

If $\mathfrak{g}$ is finite-dimensional, or one of affine Lie superalgebras $A(0,n)^{(1)}$, $B(0,n)^{(1)}$, $C(n)^{(1)}$, or $A(0,2n-1)^{(2)}$, $C(n+1)^{(2)}$, $A(0,2n)^{(4)}$, then we can introduce a Weyl vector $\rho_{\bar 0}$ satisfying $(\rho_{\bar 0}, \alpha^\vee) = 1$ for each $\alpha \in \Pi_0$. Then $R_{\bar 0}e^{\rho_{\bar 0}}$

is a W-skew-invariant element of $\mathscr{R}_W$, so $R_{\overline{1}}e^{\rho_{\overline{0}}-\rho}$ is a W-invariant element of $\mathscr{R}_W$.

If $\mathfrak{g}$ is an affine Lie superalgebra and $\Delta_{\overline{0}}$ is not connected, then the Weyl vector $\rho_{\overline{0}}$ does not exist. However, for each connected component $\pi \subsetneq \Pi_0$ there exists a Weyl vector ρ_π satisfying $(\rho_\pi, \alpha^\vee) = 1$ for each $\alpha \in \pi$; note that $R_{\overline{0}}e^{\rho_\pi}$ is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}$ and $R_{\overline{1}}e^{\rho_{\overline{0}}-\rho}$ is a $W(\pi)$-invariant element of $\mathscr{R}_{W(\pi)}$.

2.2.7. Poles. For an odd isotropic root $\alpha \in \Pi$ we say that $X \in \mathscr{R}(\Pi)$ has a *pole of order* k at α if k is minimal such that

$$(1 + e^{-\alpha})^k X \in \mathscr{R}(r_\alpha \Pi).$$

For example, R has a pole of order 1 at each odd isotropic root $\alpha \in \Pi$. Another important example appears in the next lemma.

2.2.8. Consider $Y \in \mathscr{Y}$ (see §2.2.3) of the form $Y := \prod_{\beta \in J}(1 + e^{-\beta})$, where $J \subset \Delta$ is a finite set and, for each $\beta \in \Delta_{\overline{1}}$, $J \cap \{\pm\beta\}$ contains at most one element. Recall conventions of §2.2.3 and view Y^{-1} as an element in $\mathscr{R}(\Pi)$, which we denote by $Y^{-1}(\Pi)$.

Let $W' \subset W$ be a subgroup generated by simple reflections and let $\lambda \in \mathfrak{h}^* \setminus \{0\}$ be such that the orbit $W'\lambda$ has a unique maximal element. For each subset $W'' \subset W'$ we introduce the following notation

$$\mathscr{F}_{W''}\Big(\frac{e^\lambda}{\prod_{\beta \in J}(1 + e^{-\beta})}\Big) := \sum_{w \in W''} \operatorname{sgn}(w)\, e^{w\lambda} \prod_{\beta \in J} (1 + e^{-w\beta})^{-1}.$$

From the lemma below it follows that $\mathscr{F}_{W''}(\frac{e^\lambda}{\prod_{\beta \in J}(1+e^{-\beta})})$ lies in $\mathscr{R}(\Pi)$ (i.e., the corresponding partial sums converge in $\mathscr{R}(\Pi)$, *cf.* §2.2) and that these elements are equivalent for different choices of Π.

Lemma. *Let $W' \subset W$ be a subgroup generated by simple reflections; for each $w \in W'$ fix $x_w \in \mathbb{Q}$. Write $W' = W'_f \times W'_{aff}$, where W'_f is finite and W'_{aff} is the product of affine Weyl groups. Let $\lambda \in \mathfrak{h}^* \setminus \{0\}$ be such that $(\lambda, \alpha^\vee) \in \mathbb{Z}$ for each $\alpha \in \Pi_0$ such that $r_\alpha \in W'$, and that $(\lambda, \delta)/(\alpha, \alpha) \geq 0$ for each $\alpha \in \Pi_0$ such that $r_\alpha \in W'_{aff}$.*

(a) For each Π the element

$$X(\Pi) := \sum_{w \in W'} x_w e^{w\lambda} (wY)^{-1}(\Pi)$$

lies in $\mathscr{R}(\Pi)$.

(b) *All elements $X(\Pi)$ are equivalent (with respect to the relation introduced in §2.2.1).*

(c) *For each odd isotropic root $\alpha \in \Pi$ the element $X(\Pi)$ has a pole of order at most one at α. Moreover, $X(\Pi)$ has a pole of order zero at α if $W'(J) \cap \{\pm\alpha\} = \emptyset$.*

Proof. The assumptions on λ imply that the orbit $W'\lambda$ contains a unique maximal element. We may (and will) assume that λ is maximal in its orbit that is $\lambda - w\lambda \in \mathbb{Z}_{\geq 0}\Pi_0$ for any $w \in W'$.

For a fixed set of simple roots Π, we denote by $\mathrm{ht}_\Pi\, \mu$, the height of $\mu = \sum_{\alpha\in\Pi} k_\alpha \alpha$, the number $\mathrm{ht}_\Pi\, \mu = \sum_{\alpha\in\Pi} k_\alpha$.

Note that $X(\Pi)$ is an infinite sum of elements in $\mathscr{R}(\Pi)$; for (a) we have to show that the partial sums converge (*cf.* §2.2). From (8) we obtain

$$\mathrm{supp}(e^{w\lambda}(wY)^{-1}(\Pi)) \subset w\lambda - \mathbb{Z}_{\geq 0}\Pi.$$

In order to prove that $X(\Pi) \in \mathscr{R}(\Pi)$, it is enough to verify that for each r the set

$$H_r(\lambda) := \{w \in W' \mid \mathrm{ht}(\lambda - w\lambda) \leq r\}$$

is finite.

Recall that (see e.g. [K3], Chapter 3) for an affine Lie algebra the stabilizer of any element ν which is maximal in its Weyl group orbit is generated by simple reflections; thus this stabilizer is either finite or coincides with W itself (in this case $\nu = 0$). Hence $Stab_{W'}\lambda$ is finite.

Let $\alpha_1, \ldots, \alpha_r$ be the simple reflections ($\alpha_i \in \Pi_0$) which generate W'. Since λ is maximal in its W'-orbit, the value $(\lambda, \alpha_i^\vee)$ is a non-negative integer. An easy argument (see, for instance, Lemma 1.3.2 in [G2]) shows that for each reduced expression $w = r_{\alpha_{i_1}} \cdots r_{\alpha_{i_r}}$,

$$\mathrm{ht}(\lambda - w\lambda) \geq \#\{j \mid (\lambda, \alpha_{i_j}^\vee) \neq 0\}.$$

Now the fact that H_r is finite follows as in Lemma 2.4.1 (i) in [G2].

(b) Since any two subsets of positive roots are connected by a finite chain of odd reflections, it is enough to verify that

$$X(\Pi)(1 + e^\gamma) = X(r_\gamma\Pi)(1 + e^\gamma) \in \mathscr{R}(\Pi) \cap \mathscr{R}(r_\gamma\Pi). \tag{9}$$

Indeed, by the assumption on J, the intersection $wJ \cap \{\pm\gamma\}$ contains at most one element. If the intersection is empty, then $e^{w\lambda}(wY)^{-1}(\Pi) = e^{w\lambda}(wY)^{-1}(\Pi') \in \mathscr{R}(\Pi) \cap \mathscr{R}(\Pi')$, see §2.2.3. If the intersection $wJ \cap \{\pm\gamma\}$ is non-empty, then $e^{w\lambda}(wY)^{-1}(\Pi)(1 + e^\gamma) = e^{\lambda'}(Y')^{-1}(\Pi)$, where

$$Y' = \prod_{\beta\in wJ\setminus\{\pm\gamma\}} (1 + e^{-\beta})$$

and $\lambda' = w\lambda$ if $-\gamma \in wJ$, $\lambda' = w\lambda + \gamma$ if $\gamma \in wJ$. Since $X(\Pi) \in \mathscr{R}(\Pi)$, $X(r_\gamma \Pi) \in \mathscr{R}(r_\gamma \Pi)$, the sum $X(\Pi)(1+e^\gamma)$ (resp., $X(r_\gamma \Pi)$) is a well-defined element in $\mathscr{R}(\Pi)$ (resp., in $\mathscr{R}(r_\gamma \Pi)$) and since all summands $(1 + e^\gamma)w(\frac{e^\lambda}{1+e^{-\beta}})$ lie in $\mathscr{R}(\Pi) \cap \mathscr{R}(r_\gamma \Pi)$, we obtain (9). This proves (b) and (c). □

Remark. For $\lambda \neq 0$ the conditions

(i) the orbit $W'\lambda$ has a unique maximal element in the $\geq$-ordering;
(ii) $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for each $\alpha \in \Pi_0$ such that $r_\alpha \in W'$;

are equivalent if W' is finite. In the case when W' is infinite, (i) is equivalent to (ii)+(iii), where

(iii) $(\lambda', \delta)/(\alpha, \alpha) > 0$ for each $\alpha \in \Pi_0$ such that $r_\alpha \in W'_{aff}$.

2.2.9. **Lemma.** *Let $\alpha \in \Pi$ be an isotropic root. Assume that $X = \sum x_\mu e^\mu \in \mathscr{R}(\Pi)$, $X' = \sum x'_\mu e^\mu \in \mathscr{R}(r_\alpha \Pi)$ are equivalent and that X has a pole of order ≤ 1 at α. Then for each $\mu \in \mathfrak{h}^*$ one has*

$$\forall k \in \mathbb{Z} \quad x_{\mu+k\alpha} - x'_{\mu+k\alpha} = (-1)^k (x_\mu - x'_\mu). \tag{10}$$

Moreover, $x_\mu = x'_\mu$ if $(\operatorname{supp} X) \cap \{\mu + \mathbb{Z}\alpha\}$ *is finite.*

Proof. Since X has a pole of order ≤ 1 at α, one has $(1 + e^{-\alpha})X \in \mathscr{R}(r_\alpha \Pi)$. Since X, X' are equivalent, the elements $(1 + e^{-\alpha})X, (1 + e^{-\alpha})X' \in \mathscr{R}(r_\alpha \Pi)$ are equivalent and so $(1 + e^{-\alpha})X = (1 + e^{-\alpha})X'$.

Recall that $\mathscr{V}$ is a $\mathscr{V}_{fin}$-module; for $Y = \sum y_\mu e^\mu \in \mathscr{V}$ one has

$$(1 + e^{-\alpha})Y = 0 \implies y_\mu + y_{\mu-\alpha} = 0 \quad \text{for all } \mu.$$

This gives (10).

Finally, note that $x'_{\mu-k\alpha} = 0$ for $k \gg 0$, because $X' \in \mathscr{R}(r_\alpha \Pi)$ and $-\alpha \in r_\alpha \Pi$. If $\operatorname{supp} X \cap \{\mu + \mathbb{Z}\alpha\}$ is finite, then $x_{\mu-k\alpha} = 0$ for $k \gg 0$ and thus $x_\mu - x'_\mu = x_{\mu-k\alpha} - x'_{\mu-k\alpha} = 0$, as required. □

3. Root systems of basic and affine Lie superalgebras

In this section we give some (mostly known) properties of Dynkin diagrams of basic and affine Lie superalgebras which are used in the main text. We call a Dynkin diagram of an indecomposable affine (resp., basic) Lie superalgebra affine (resp., finite) type Dynkin diagram. We identify a set of simple roots Π with the vertices of its Dynkin diagram.

In this section $\mathfrak{g}$ is an indecomposable affine or basic Lie superalgebra with a set of simple roots Π and a symmetrizable Cartan matrix A. We denote by Δ the root system of $\mathfrak{g}$.

Throughout §3–6, unless otherwise stated, we use the following normalization of the invariant bilinear form $(-,-)$. If the dual Coxeter number is non-zero, we normalize the form by the condition $h^\vee \in \mathbb{Q}_{>0}$. If $\mathfrak{g} = D(n+1,n), D(n+1,n)^{(1)}$ or $D(2,1,a), D(2,1,a)^{(1)}, a \in \mathbb{Q}$, we normalize the form by the condition $(\alpha,\alpha) \in \mathbb{Q}_{>0}$ for some $\alpha \in D_{n+1}$ or $\alpha \in D_2 = A_1 \times A_1$; if the dual Coxeter number is zero and $\mathfrak{g} \neq D(n+1,n), D(n+1,n)^{(1)}, D(2,1,a), D(2,1,a)^{(1)}, a \in \mathbb{Q}$, we normalize the form by the condition $(\alpha,\alpha) \in \mathbb{Q}_{>0}$ for some $\alpha \in \Delta$ (note that in this case all connected components of $\Pi_{\overline{0}}$ have the same number of elements).

3.1. *Affine Lie superalgebras*

3.1.1. **Lemma.** *Let Π be a set of simple roots of an indecomposable affine Lie superalgebra and let δ be the minimal imaginary root. Then $\delta = \sum_{\alpha\in\Pi} x_\alpha \alpha$, where each coefficient $x_\alpha \neq 0$.*

Proof. Take $\alpha \in \Pi$ such that $x_\alpha = 0$. Since Δ is affine, $\Delta + r\delta = \Delta$ for some $r > 0$, so $r\delta - \alpha \in \Delta$. One has

$$r\delta - \alpha = \sum_{\beta\in\Pi,\beta\neq\alpha} r x_\beta \beta - \alpha,$$

that is $r x_\beta \leq 0$ for each β. Then $r\delta \in -\Delta^+$, a contradiction. □

3.1.2. Finite parts. For each $\Pi' \subset \Pi$ the set $\mathbb{Z}\Pi' \cap \Delta$ is the set of roots of a Kac–Moody superalgebra with the Cartan matrix A', which is the submatrix of A, corresponding to Π'. Using Lemma 3.1.1, we conclude that any proper subdiagram of a connected Dynkin diagram of affine type is of finite type, i.e., if Π is a set of simple roots of an indecomposable affine Lie superalgebra, then for any proper subset $\Pi' \subset \Pi$ the root system $\mathbb{Z}\Pi' \cap \Delta$ is finite (and is the root system of a certain basic Lie superalgebra).

Let X be an affine Dynkin diagram. We call a connected subdiagram $\dot{X}$, obtained from X by removing one node, a *finite part* of X. By above, $\dot{X}$ is of finite type. We call a root subsystem $\dot{\Delta}$ a finite part of affine root system Δ if $\dot{\Delta}$ admits a set of simple roots $\dot{\Pi}$ which is finite part of a set of simple roots for Δ. The finite parts of affine root systems are described in §13.2.

3.1.3. Definitions. Let $\mathfrak{g}$ be an affine Lie superalgebra with the root system Δ. Let $\dot{\Delta}$ be a finite part of Δ (see §3.1.2). An irreducible *vacuum module* is a module $L(\lambda)$ such that $(\lambda, \dot{\Delta}) = 0$. Note that if Π is a set of simple roots of Δ and $\dot{\Pi}$ is a finite part of Π, and $(\lambda, \dot{\Pi}) = 0$, then $L(\lambda)$ is a vacuum module.

Let $\mathfrak{g}$ be a basic or affine Lie superalgebra. For each subset $\pi \subset \Pi_0$ we say that a $\mathfrak{g}$-module N is *π-integrable* if $\mathfrak{h}$ acts diagonally on N and for each $\alpha \in \pi$ the root spaces $\mathfrak{g}_{\pm\alpha}$ act locally nilpotently on N.

Note that if N is π-integrable, then for each $w \in W(\pi)$ one has $\dim N_\nu = \dim N_{w\nu}$, so $\operatorname{ch} N$ is a $W(\pi)$-invariant element of $\mathscr{V}$, see §2.2 for notation. In particular, if N is a π-integrable irreducible highest weight module, then $\operatorname{ch} N$ is a $W(\pi)$-invariant element of $\mathscr{R}_{W(\pi)}$, see §2.2.2.

Let $\mathfrak{g}$ be an affine Lie superalgebra, let $\dot{\mathfrak{g}}$ be a finite part of $\mathfrak{g}$, and let $\dot{\Pi}_0$ be the subset of simple roots for $\dot{\mathfrak{g}}_{\overline{0}} = \mathfrak{g}_{\overline{0}} \cap \dot{\mathfrak{g}}$. We say that a $\mathfrak{g}$-module N is *integrable* if N is π-integrable for $\pi = \{\alpha \in \Pi_0 \mid (\alpha, \alpha) \in \mathbb{Q}_{>0}\}$.

Note that π is independent of our normalization of $(-,-)$ if $h^\vee \neq 0$, but π changes if we change the sign of $(-,-)$ if $h^\vee = 0$. In all cases, except for $D(2,1,a)^{(1)}$, π is a connected component of Π_0.

3.2.

The sets of simple roots of basic Lie superalgebras which consist of isotropic roots are the following ($n \geq 1$):

$$
(11)\qquad
\begin{array}{ll}
A(n,n) & \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \delta_{n-1} - \varepsilon_n, \varepsilon_n - \delta_n\}, \\
A(n+1,n) & \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \varepsilon_n - \delta_n, \delta_n - \varepsilon_{n+1}\}, \\
D(n,n) & \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \ldots, \delta_n - \varepsilon_n, \delta_n + \varepsilon_n\}, \\
D(n+1,n) & \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \varepsilon_n - \delta_n, \delta_n - \varepsilon_{n+1}, \delta_n + \varepsilon_{n+1}\},
\end{array}
$$

and for $D(2,1,a)$ it is as for $D(2,1)$. The invariant bilinear form (satisfying §3.1.3) can be chosen in such a way that the vectors ε_i, δ_j are mutually orthogonal and $1 = \|\varepsilon_i\|^2 = -\|\delta_j\|^2$, except for the case $D(n,n)$, where $1 = -\|\varepsilon_i\|^2 = \|\delta_j\|^2$.

We claim that the sets of simple roots of indecomposable affine Lie superalgebras which consist of isotropic roots are the following ($n \geq 1$):

$$
(12)\qquad
\begin{array}{ll}
A(n,n)^{(1)} & \{\delta - \varepsilon_1 + \delta_n, \varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \delta_{n-1} - \varepsilon_n, \\
 & \varepsilon_n - \delta_n\}, \\
D(n+1,n)^{(1)} & \{\delta - \varepsilon_1 - \delta_1, \varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \varepsilon_n - \delta_n, \\
 & \delta_n - \varepsilon_{n+1}, \delta_n + \varepsilon_{n+1}\}, \\
A(2n-1,2n-1)^{(2)} & \{\delta - \varepsilon_1 - \delta_1, \varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \varepsilon_n - \delta_n, \\
 & \varepsilon_n + \delta_n\}
\end{array}
$$

and for $D(2,1,a)^{(1)}$ it is as for $D(2,1)^{(1)}$.

This can be explained as follows. If Π consists of isotropic roots and $\Delta(\Pi)$ is affine, then $\mathfrak{g}$ has zero dual Coxeter number and $\mathfrak{g}$ has a finite part which appears in (11). Using the tables in §13.2 we conclude that this holds only for $\mathfrak{g}$ listed in (12). It is easy to see that for these algebras all Dynkin diagrams consisting of isotropic roots are as in (12).

Another result that we are going to use is the following.

3.2.1. **Lemma.** *Let $\beta, \beta' \in \Pi$ be isotropic roots with $(\beta, \beta') \neq 0$. If $(\beta, \alpha) \neq 0$ for some non-isotropic $\alpha \in \Pi$, then $(\beta, \beta')/(\alpha, \alpha)^2 \in \mathbb{Q}_{>0}$.*

Proof. Since $\beta + \beta' \in r_\beta \Pi$, we have $\beta + \beta' \in \Pi_0$. Normalize the form in such a way that $\|\alpha\|^2 = 2$. Then $(\alpha, \beta') \in \mathbb{Z}_{\leq 0}$, $(\alpha, \beta) \in \mathbb{Z}_{<0}$, so $(\alpha, \beta + \beta') \in \mathbb{Z}_{<0}$. Since $\beta + \beta' \in \Pi_0$, we have $(\alpha, (\beta + \beta')^\vee) = 2(\alpha, \beta + \beta')/\|\beta + \beta'\|^2 \in \mathbb{Z}_{<0}$, so $\|\beta + \beta'\|^2 = 2(\beta, \beta') \in \mathbb{Q}_{>0}$. □

3.2.2. Let Π be a connected Dynkin diagram which contains a non-isotropic node, let *Iso* be its subdiagram consisting of isotropic nodes, and let Π' be a connected component of *Iso*. Note that Π' appears in (11). Since Π is connected, Π' contains a node $\beta \in \Pi'$ which is connected to a node in $\Pi \setminus \Pi'$. By Lemma 3.2.1, β can be described as follows. For $A(n+1,n)$, $A(n,n)$, β is one of the ending nodes; for $D(n+1,n)$, $D(n,n)$, $n > 1$, β is the first node. For $D(2,1,a)$, $\beta \in \Pi$ is such that $(\beta, \beta_1)/(\beta, \beta_2) \in \mathbb{Q}_{>0}$, where $\Pi = \{\beta, \beta_1, \beta_2\}$; such β is unique and exists only if $a \in \mathbb{Q}$. In particular, Π' of type $D(2,1,a)$ with $a \notin \mathbb{Q}$ cannot be a connected component of *Iso*.

3.3. Choice of Π, S, π

Let Π be a set of simple roots for Δ which satisfies the following property: $\|\alpha\|^2 \in \mathbb{Q}_{\geq 0}$ for each $\alpha \in \Pi$. Recall that such Π exists for all root systems except for $A(2k,2k)^{(4)}$, $D(k+1,k)^{(2)}$: for each non-twisted (resp., twisted) affine Δ the example of such Π appears in the end of [G2] (resp., [R]), where such Π was used for a proof of the denominator identity.

We consider $\mathfrak{g} \neq D(2,1,a), D(2,1,a)^{(1)}$ with $a \notin \mathbb{Q}$. Set

$$\pi := \{\alpha \in \Pi_0 \mid \|\alpha\|^2 > 0\}.$$

Recall that the *defect* of a finite type root system Δ is the dimension of maximal isotropic subspace in $\mathbb{Q}\Delta$; for $A(m-1,n-1)$, $B(m,n)$, $D(m,n)$ the defect is equal to $\min(m,n)$; for other cases of non Lie algebras it is one. It is well-known that it is equal to the maximal number of mutually orthogonal isotropic simple roots for some choice of Π, a set of simple roots of Δ.

From §13.2 it follows that for affine root system Δ all its finite parts have the same defect and that the maximal number of mutually orthogonal isotropic simple roots for Δ is equal to the defect of its finite part. We call this number the *defect* of the affine superalgebra. A subset $S\subset\Pi$ is called *maximal isotropic* if $\mathbb{Q}S$ is isotropic and $\dim\mathbb{Q}S$ is equal to the defect.

3.3.1. We say that an isotropic node $\beta\in\Pi$ is "*branching*" if the connected component of *Iso* which contains β is of the type $D(k+1,k)$ with $k\geq 1$ and β is the "branching" node in this connected component or if the connected component is a triangle (consisting of isotropic nodes) and β is a node in this triangle which is connected to the rest of the diagram Π (such node is unique). Note that Π contains at most two "branching" nodes.

If Π is not the set of simple roots of $D(2,1,a)$, $D(2,1,a)^{(1)}$, $D(n+1,n)^{(1)}$ consisting of isotropic nodes, we let $S\subset\Pi$ be a maximal subset of mutually orthogonal isotropic simple roots, which contains all "branching" isotropic nodes of Π (if they exist). For example, if $\Pi=D(n+1,n)$ consists of isotropic roots (see (11)), then $S=\{\varepsilon_i-\delta_i\}_{i=1}^n$ is "good". This is what we are called a "good" choice of $S\subset\Pi$ in the introduction. In the exceptional case $\Pi=D(2,1,a)$, consisting of isotropic nodes, we take $S=\{\beta\}$, where $\beta\in\Pi$ is such that $(\beta,\alpha^\vee)>0$ for $\alpha\in\pi$ (such β is unique, since $\pi=A_1\times A_1$).

3.3.2. Remark. Let Π' be a connected component of *Iso*. Using §3.2.2 one easily sees that $\pi':=\pi\cap\Delta(\Pi')$ and $S':=S\cap\Pi'$ is a "good" choice of π,S in the sense of §3.3.1 for Π'.

3.3.3. Examples. For example, for $A(m,n)^{(1)}, m>n$, we have $\pi=\{\delta-\varepsilon_1+\varepsilon_m,\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m\}$ and we have the following "good" choice of Π (satisfying $\|\alpha\|^2\geq 0$ for each $\alpha\in\Pi$):

$$\Pi=\{\delta-\varepsilon_1+\varepsilon_m,\varepsilon_1-\delta_1,\dots,\delta_n-\varepsilon_{n+1},\varepsilon_{n+1}-\varepsilon_{n+2},\dots,\varepsilon_{m-1}-\varepsilon_m\};$$

and for $A(n,n)^{(1)}$ a "good" choice of Π is

$$\Pi:=\{\delta-\varepsilon_1+\delta_n,\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\dots,\varepsilon_n-\delta_n\},$$

so that $\pi=\{\delta-\varepsilon_1+\varepsilon_n,\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{n-1}-\varepsilon_n\}$, with "good" $S=\{\varepsilon_i-\delta_i\}_{i=1}^n$ in both cases.

For $B(2,1)^{(1)}$ $\pi=\{\delta-\varepsilon_1-\varepsilon_2,\varepsilon_1-\varepsilon_2,\varepsilon_2\}$. The only "good" Π is

$$\Pi=\{\delta-\varepsilon_1-\delta_1,\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2\}.$$

Since $\delta_1-\varepsilon_2$ is a "branching" node, only $S=\{\delta_1-\varepsilon_2\}$ is "good".

For $B(2,2)^{(1)}$ $\pi=\{\delta-2\delta_1,\delta_1-\delta_2,2\delta_2\}$. The only "good" Π is

$$\Pi=\{\delta-\varepsilon_1-\delta_1,\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2-\delta_2,\delta_2\}.$$

Note that *Iso* $\cong D(2,2)$, so there are no "branching" nodes; thus any S is good (there are two choices of S: $\{\varepsilon_i-\delta_i\}_{i=1}^2$, $\{\delta-\varepsilon_1-\delta_1, \varepsilon_2-\delta_2\}$).

For $D(n,n)$ one has $\pi=\{\delta_i-\delta_{i+1}\}_{i=1}^{n-1}\cup\{2\delta_n\}$. There are two "good" Π's. For the one, consisting of isotropic roots (see (11)), there are two choices for S: $S=\{\delta_i-\varepsilon_i\}_{i=1}^n$ and $S=\{\delta_i-\varepsilon_i\}_{i=1}^{n-1}\cup\{\delta_n-\varepsilon_n\}$; both choices are "good" (by definition this diagram does not contain a "branching" node). Another "good" Π is

$$\Pi=\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \varepsilon_2-\delta_2, 2\delta_2\}.$$

In this case the only choice of S is $\{\varepsilon_i-\delta_i\}_{i=1}^2$ and it is "good" (Π does not have "branching" nodes).

For $D(3,1)$ one has $\pi=\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_2+\varepsilon_3\}$. We have several "good" Π's. For instance, for

$$\Pi=\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \delta_1+\varepsilon_3\}$$

Iso is a triangle, so the node $\varepsilon_2-\delta_1$ is branching. Thus only $S=\{\varepsilon_2-\delta_1\}$ is "good". Another Π is

$$\Pi=\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_2+\varepsilon_3\},$$

both choices of S: $S=\{\varepsilon_1-\delta_1\}$ and $S=\{\delta_1-\varepsilon_2\}$ are "good".

For $D(4,2)$ one has $\pi=\{\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_3-\varepsilon_4,\varepsilon_3+\varepsilon_4\}$. For the following "good" Π:

$$\Pi=\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \varepsilon_2-\delta_2, \delta_2-\varepsilon_3, \varepsilon_3-\varepsilon_4, \varepsilon_3+\varepsilon_4\},$$

there are two choices for S: $S=\{\varepsilon_i-\delta_i\}_{i=1}^2$ and $S=\{\delta_i-\varepsilon_{i+1}\}_{i=1}^2$; both choices are "good". For the following "good" Π:

$$\Pi=\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \varepsilon_3-\delta_2, \delta_2-\varepsilon_4, \delta_2+\varepsilon_4\},$$

there are three choices for S and only $S=\{\varepsilon_2-\delta_1, \varepsilon_3-\delta_2\}$ is "good", since $\varepsilon_3-\delta_2$ is a "branching" node.

Considering the corresponding affine diagrams for $D(4,2)^{(1)}$, we see that $\delta_1-\varepsilon_2$ becomes "branching" in the first diagram, so only $S=\{\delta_i-\varepsilon_{i+1}\}_{i=1}^2$ remains "good"; in the second case there are no new "branching" points so $S=\{\varepsilon_2-\delta_1, \varepsilon_3-\delta_2\}$ remains "good".

For $D(5,2)^{(1)}$ with a "good" Π:

$$\Pi=\{\delta-\varepsilon_1-\delta_1, \varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_3-\varepsilon_4, \varepsilon_4-\delta_2, \delta_2-\varepsilon_5, \delta_2+\varepsilon_5\}$$

there are two "branching" points $\delta_1-\varepsilon_2, \varepsilon_4-\delta_2$, so only $S=\{\delta_1-\varepsilon_2, \varepsilon_4-\delta_2\}$ is "good".

For $F(4)$ there are six choices of Π, four of them are good and only one of "good" Π's has a "branching" node. For $F(4)^{(1)}$ there are two non-isomorphic "good" Π's: a "kite"-type

$$\Pi = \Big\{\delta-\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \frac{-\delta_1+\varepsilon_1-\varepsilon_2+\varepsilon_3}{2}, \frac{\delta_1-\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}, \frac{\delta_1+\varepsilon_1-\varepsilon_2-\varepsilon_3}{2}\Big\}$$

where the only "good" S is $S=\{(-\delta_1+\varepsilon_1-\varepsilon_2+\varepsilon_3)/2\}$ and a "hill"-type

$$\Pi = \Big\{\delta-\varepsilon_1-\varepsilon_2, \frac{-\delta_1+\varepsilon_1+\varepsilon_2-\varepsilon_3}{2}, \varepsilon_3, \frac{\delta_1-\varepsilon_1+\varepsilon_2-\varepsilon_3}{2}, \varepsilon_1-\varepsilon_2\Big\},$$

where both choices of S are "good".

For $G(3)$ there are three "good" Π's and all S in these Π's are "good". For $G(3)^{(1)}$ there are two "good" Π's:

$$\Pi = \{\delta+\varepsilon_1-\varepsilon_2, -\delta_1+\varepsilon_2, \delta_1-\varepsilon_3, \varepsilon_3\}$$

without "branching" nodes (so both choices of S are "good"), and

$$\Pi = \{\delta+\varepsilon_1-\delta_1, -\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \varepsilon_2-\varepsilon_3\}$$

with $S=\{\delta_1-\varepsilon_2\}$.

3.3.4. **Lemma.** *Let* $\Pi \neq A(2n-1,2n-1)^{(2)}, A(2n,2n)^{(4)}, D(n+1,n)^{(r)}$ $(r=1,2)$ *and let* S,π *be as above.*

(i) For each $\alpha\in\Pi$ *with* $\|\alpha\|^2=0$ *one has* $\alpha\in S$ *or* $\alpha+\beta\in\pi$ *for some* $\beta\in S$.
(ii) If $w\in W(\pi)$ *is such that* $w\rho=\rho$ *and* $wS\subset\Delta^+$, *then* $w=Id$.

Proof. (i) In the light of §3.3.2, it is enough to verify (i) for the case, when Π contains only isotropic roots, that is Π is either $A(n,n)^{(1)}$ or appears in (11). It is easy to check that the claim holds in each case.

(ii) Consider first the case when Π contains only isotropic roots.

Recall that for $S=\{\varepsilon_i-\delta_i\}_{i=1}^n$ for $D(n+1,n)$. For $A(n,n), D(n,n)$, $A(n,n)^{(1)}$ the choice of S is unique up to an automorphism of the Dynkin diagram Π, so we take $S=\{\varepsilon_i-\delta_i\}_{i=1}^n$ for $A(n,n), D(n+1,n), A(n,n)^{(1)}$ and $S=\{\delta_i-\varepsilon_i\}_{i=1}^n$ for $D(n,n)$.

For $A(n,n)$, $A(n,n)^{(1)}$ the group W stabilizes $\sum_{\beta\in S}\beta$. Therefore $wS\subset\Delta^+$ forces $w\beta=\beta$ for each $\beta\in S$, so $w\varepsilon_i=\varepsilon_i, w\delta_i=\delta_i$ for each $i=1,\dots,n$. This gives $w=Id$.

For $D(n,n)$, the group $W(\pi)$ acts by signed permutations on $\{\delta_i\}_{i=1}^n$, and the condition $wS\subset\Delta^+$ gives for each $i=1,\dots,n$ that $w\delta_i=\delta_j$ for $j\leq i$. Thus $w=Id$.

For $D(n+1,n)$, the group $W(\pi)$ acts by signed permutations with even number of sign changes on the set $\{\varepsilon_i\}_{i=1}^{n+1}$, and the condition $wS \subset \Delta^+$ gives for each $i = 1,\dots,n$ that $w\varepsilon_i = \varepsilon_j$ for $j \leq i$. Thus $w = Id$.

For $A(n+1,n)$, the group $W(\pi)$ permutes $\{\varepsilon_i\}_{i=1}^{n+1}$ and S is of the form $\{\varepsilon_i - \delta_i\}_{i=1}^{k} \cup \{\delta_i - \varepsilon_{i+1}\}_{i=k+1}^{n}$ for some $k = 0,\dots,n$. The condition $wS \subset \Delta^+$ gives for each $i = 1,\dots,k$ (resp., $i = k+1,\dots,n+1$) that $w\varepsilon_i = \varepsilon_j$ for $j \leq i$ (resp., $j \geq i$). This implies $w = Id$.

Finally consider the case $D(2,1,a)$. Write $\Pi = \{\beta, \beta_1, \beta_2\}$ with $S = \{\beta\}$. Recall that $\pi = \{\beta+\beta_1, \beta+\beta_2\}$ and $(\beta+\beta_1, \beta+\beta_2) = 0$. Since $r_\beta \Pi = \{-\beta, \beta+\beta_1, \beta+\beta_2\}$, for each $w \in W(\pi)$ one has $w(-\beta) = -\beta + a_1(\beta+\beta_1) + a_2(\beta+\beta_2)$ for some $a_1, a_2 \in \mathbb{Z}_{\geq 0}$ and $a_1 = a_2 = 0$ only for $w = Id$. Hence $w\beta \in \Delta^+$ forces $w = Id$, as required.

Now consider the general case. Let X be the set of connected components of *Iso*; for each connected component Π' choose $\pi' := \Delta(\Pi') \cap \pi$, see §3.3.2. We claim that

$$Stab_{W(\pi)}\rho = \prod_{\pi' \in X} W(\pi'). \tag{13}$$

Indeed, since $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi$, one has $(\rho, \alpha) \geq 0$ for each $\alpha \in \Delta^+$. This gives $(\rho, \alpha^\vee) \geq 0$ for each $\alpha \in \pi$, so the stabilizer of ρ in $W(\pi)$ is generated by the reflections $\{r_\alpha \mid \alpha \in \pi, (\rho,\alpha) = 0\}$. Clearly, $(\rho,\alpha) = 0$ for $\alpha \in \Delta^+$ means that $\alpha \in \mathbb{Z}Iso$. Since $\alpha \in \pi$, we obtain $\alpha \in \pi'$ for some $\Pi' \in X$. This establishes (13).

Now take $w \in Stab_{W(\pi)}\rho$. If $w \neq Id$, then there exists $\Pi' \in X$ such that the projection of w to $W(\pi')$ is not *Id*. Denote this projection by w'. Recall that the set $S \cap \Pi'$ is a maximal isotropic set in Π'. By above, there exists $\beta \in (S \cap \Pi')$ such that $w'\beta \in -\Delta^+$. Clearly, $w\beta = w'\beta$, so $w\beta \in -\Delta^+$ for some $\beta \in S$. This proves (ii). □

4. KW-character formula for maximally atypical modules when $h^\vee \neq 0$

Let $\mathfrak{g}$ be either a basic finite-dimensional Lie superalgebra, except for $D(2,1,a)$ with $a \notin \mathbb{Q}$, or an affine Lie superalgebra with non-zero dual Coxeter number $h^\vee$, or $A(n,n)^{(1)}$, with a subset of simple roots Π such that $\|\alpha\|^2 \in \mathbb{Q}_{\geq 0}$ for each $\alpha \in \Pi$ (see §3 for the normalization of $(-,-)$). Set $\pi := \{\alpha \in \Pi_0 \mid (\alpha,\alpha) > 0\}$.

In this section we prove the KW-formula for the maximally atypical (i.e., $\#S = defect(\mathfrak{g})$) π-integrable $\mathfrak{g}$-modules, which admit a “good” choice of $S \subset \Pi$. These include, in particular, the integrable vacuum modules over the affine Lie superalgebras with non-zero dual Coxeter number and over $A(n,n)^{(1)}$.

4.1. Main result

Except for the case described in the next sentence, we let $S \subset \Pi$ be a maximal set of mutually isotropic orthogonal simple roots, which contains all "branching" isotropic nodes, if they exist (recall that an isotropic node $\beta \in \Pi$ is "branching" if the connected component of *Iso* which contains β is of the type $D(k+1,k)$ with $k \geq 1$ and β is the "branching" node in this connected component, see §3.3.1). If $\Pi = D(2,1,a)$ consists of isotropic roots, we take $S = \{\beta\}$, where $\beta \in \Pi$ is such that $(\beta, \alpha^\vee) > 0$ for $\alpha \in \pi$ (such β is unique, since $\pi = A_1 \times A_1$).

Let $L(\lambda)$ be a non-critical π-integrable $\mathfrak{g}$-module with the property $(\lambda, \beta) = 0$ for each $\beta \in S$. We claim that, for the above "good" choice of S, $\mathrm{ch}\, L$ is given by the KW-formula:

$$Re^{\rho}\, \mathrm{ch}\, L(\lambda) = \sum_{w \in W(\pi)} \mathrm{sgn}(w) \frac{e^{w(\lambda+\rho)}}{\prod_{\beta \in S}(1+e^{-w\beta})}. \tag{14}$$

The condition that $L(\lambda)$ is π-integrable implies $(\lambda, \alpha) \geq 0$ for each $\alpha \in \pi$. Combining with $(\lambda, \beta) = 0$ for $\beta \in S$, we obtain, using Lemma 3.3.4 (i), that $(\lambda, \alpha) \geq 0$ for each $\alpha \in \Pi$. In particular, if $\mathfrak{g} \neq A(n,n)^{(1)}$, the condition that λ is non-critical is superfluous.

If $\mathfrak{g} = A(m,n)^{(1)}$ with $m \neq n$, the conditions on λ are equivalent to $(\lambda, \alpha) \in \mathbb{Z}_{\geq 0}$ for each $\alpha \in \Pi$, $(\lambda, \beta) = 0$ for each $\beta \in S$. For $\mathfrak{g} = A(n,n)^{(1)}$ the additional condition is $\lambda \notin \mathbb{Z}\delta$.

4.2. Other choices of $\Pi \supset S$

Let Π be any set of simple roots containing a maximal set of mutually orthogonal isotropic roots S ($\#S = \mathit{defect}(\mathfrak{g})$), and $L = L(\lambda, \Pi)$ is a π-integrable module with $(\lambda, S) = 0$. Beyond the cases when the pair $\Pi \supset S$ is "good" formula (14) holds in the following cases of non-exceptional $\mathfrak{g}$:

- $A(m,n)$, $B(m,n)$, $A(m,n)^{(1)}$ for any m, n;
- $B(m,n)^{(1)}$, $A(2n, 2m-1)^{(2)}$, $D(m,n)$, $D(m,n)^{(1)}$ for $m \leq n$;
- $A(2m, 2n)^{(4)}$, $D(m+1, n)^{(2)}$ for $m \neq n$;
- $B(m,n)^{(1)}$, $A(2n, 2m-1)^{(2)}$ for $m > n+1$ if the affine root α_0 is such that $\|\alpha_0\|^2 > 0$.

In order to prove this we will show in §4.2.1 below that the $\mathfrak{g}$-module L is isomorphic to the $\mathfrak{g}$-module $L(\lambda, \Pi')$, where Π' contains a maximal set of mutually orthogonal isotropic roots S' such that (Π', S') is good and $(\lambda, S') = 0$.

By the same method one can show that for $G(3)^{(1)}$ formula (14) holds for "good" pairs $S \subset \Pi$, described in §3.3.3, and for $\Pi = \{\delta + \varepsilon_1 - \varepsilon_2, \varepsilon_2 -$

$\varepsilon_3, \varepsilon_3 - \delta_1, \delta_1\}$ (S is unique). In particular, (14) holds for any S if $\|\alpha_0\|^2 > 0$ (see §13.2.4 for the list of all Π's).

In the case $F(4)^{(1)}$ formula (14) holds for the "good" pairs $S \subset \Pi$ described in §3.3.3 (for the "kite"-type Π if S is the "branching" node and for the "hill"-type Π if S is any isotropic node).

4.2.1. If formula (14) holds for the pair (Π, S), then it remains valid for the pair $(r_\alpha\Pi, r_\alpha S)$, where $\alpha \in S$, see Lemmas 2.2.8, 5.7.1. Let us show that using the reflections with respect to the roots in S we can transform the pair $\Pi \supset S$ to a "good" pair.

For each affine $\Delta \neq D(m,n)^{(1)}$ we choose $\dot{\Delta} \neq D(k,l)$ which contains S (see §13.2.1); for $D(m,n)^{(1)}$ we take $\dot{\Delta} = D(m,n)$ containing S. If Δ is finite we set $\dot{\Delta} := \Delta$.

Consider $\dot{\Pi} = \Pi \cap \dot{\Delta}$. It is convenient to use the arc diagrams introduced in [GKMP]; these are dots and crosses diagrams described in §5.6 (dots corresponds to ε_i's and crosses to δ_i's) where a dot and a cross is connected by an arc if the corresponding root lies in S. For $\dot{\Pi} = A(m,n), B(m,n), D(m,n)$, the arc diagram has m dots, n crosses and $\min(m,n)$ arcs; since $S \subset \dot{\Pi}$, all arcs connect neighboring elements and different arcs have no common vertices. Since there are $\min(m,n)$ arcs each cross is connected to a dot if $m \geq n$ (resp., each dot is connected to a cross if $m \leq n$). The odd reflection $(\Pi, S) \to (r_\alpha\Pi, r_\alpha S)$, where $\alpha \in S$, corresponds to the interchanging of the vertices of the arcs corresponding to α; we call this "arc reflection". The condition that $\|\alpha\|^2 \geq 0$ for each $\alpha \in \dot{\Pi}'$ means that the corresponding arc diagram does not contain two neighboring crosses (resp., dots) for $m > n$ (resp., $m \leq n$), and, in addition, for $B(m,n), D(m,n), m > n$ the last symbol is a dot, and for $B(m,n), n \geq m, D(m,n), n > m$, the last symbol is a cross.

If the arc diagram $(\dot{\Pi}, S)$ contains the same number of dots and crosses, then using the arc reflections we can obtain the diagram $(\dot{\Pi}', S')$ with alternating symbols (i.e., dots or crosses) which ends by any symbol; we choose this symbol to be a cross (resp., a dot) for $\Delta \neq A(2n,2n-1)^{(2)}$ (resp., for $\Delta \neq A(2n,2n-1)^{(2)}$). If the arc diagram $(\dot{\Pi}, S)$ contains more crosses than dots we can obtain an arc diagram $(\dot{\Pi}', S')$ without neighboring dots, which starts and ends by crosses; we construct a similar arc diagram $(\dot{\Pi}', S')$ if there are more dots than crosses. Let Π' be the corresponding set of simple roots for Δ ($\Pi' = \dot{\Pi}'$ if Δ is finite and $\Pi' = \dot{\Pi}' \cup \{\alpha_0'\}$ if Δ is affine). Let us show that the pair (Π', S') is "good" if $\Delta \neq B(m,n)^{(1)}$ or $A(2n,2m-1)^{(2)}$ for $m > n+1$. Indeed, by above, $\|\alpha\|^2 > 0$ for each $\alpha \in \dot{\Pi}$. Moreover, $\dot{\Pi}'$ does not have a branching node (this holds for all diagrams $A(m,n)$, $B(m,n)$ and for $D(m,n)$ if the last symbol is cross). Thus $(\dot{\Pi}', \dot{S}')$ is "good".

Now we may assume that Δ is affine. Let ξ_l, ξ_r correspond respectively to the first and to the last symbol in $\dot{\Pi}'$: $\xi_l = \varepsilon_1$ (resp., $\xi_r = \varepsilon_m$) if the first (resp., the last) symbol is a dot and $\xi_l = \delta_1$ (resp., $\xi_r = \delta_n$) otherwise. If $\Delta = A(m,n)^{(1)}$ with $m \geq n$, then $\xi_l = \varepsilon_1$ and $\alpha_0 = \xi_l - \xi_r$, so $\|\alpha_0\|^2 \geq 0$. For $\dot{\Pi} = B(m,n)$ with $n > m$ one has $\xi_l = \delta_1$, so $\alpha_0 = \delta - 2\delta_1$ (resp., $\alpha_0 = \delta - \delta_1$) for $\Delta = B(m,n)^{(1)}$ (resp., for for $\Delta = A(2m,2n)^{(4)}$, $D(m+1,n)^{(2)}$); thus $\|\alpha_0\|^2 > 0$. For $\dot{\Pi} = B(m,n)$ with $m > n$ one has $\xi_l = \varepsilon_1$ and $\alpha_0 = \delta - 2\varepsilon_1$ (resp., $\alpha_0 = \delta - \varepsilon_1$) for $\Delta = A(2m,2n-1)^{(2)}$ (resp., for for $\Delta = A(2m,2n)^{(4)}$, $D(m+1,n)^{(2)}$); thus $\|\alpha_0\|^2 > 0$. In all these cases Π' does not contain branching nodes, so (Π, S) is "good".

For $B(n,n)^{(1)}$, $D(n,n)^{(1)}$, $A(2n,2n-1)^{(2)}$ for the resulting Π' one has *Iso* $= D(n,n)$, which does not contain "branching" nodes; hence (Π', S') is "good".

For $D(m,n)^{(1)}$ with $n > m$ the arc diagram $\dot{\Pi} = D(m,n)$ starts and ends by crosses, so $\alpha'_0 = \delta - 2\delta_1$ and Π' does not have branching nodes. Thus (Π', S') is good.

Consider the remaining cases $B(m,n)^{(1)}$, $A(2n,2m-1)^{(2)}$ with $m > n+1$ with $\|\alpha_0\|^2 > 0$. The condition $\|\alpha_0\|^2 > 0$ means that the first two symbols in the arc diagram of $\dot{\Pi}$ are dots, so using the arc reflections we can obtain a diagram $(\dot{\Pi}', S')$ without neighboring crosses which also starts with two dots and ends by a dot. Then $\|\alpha'_0\|^2 > 0$. The diagram $\dot{\Pi}'$ does not contain branching nodes; by adding a non-isotropic node α'_0 to $\dot{\Pi}$ we do not create new "branching" nodes, so (Π', S') is good.

4.3. Proof of (14) for a good pair $\Pi \supset S$

We rewrite formula (14) in the form

$$Re^{\rho}\,\mathrm{ch}\,L(\lambda) = \sum_{w \in W(\pi)} \mathrm{sgn}(w)\, Y_w, \qquad \text{where } Y_w := w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1+e^{-\beta})}\Big).$$

By Lemma 2.2.8, $\sum_{w \in W(\pi)} \mathrm{sgn}(w)\, Y_w \in \mathscr{R}$. One has

$$\text{(15)} \qquad \mathrm{supp}\, Y_w \subset w(\lambda+\rho) + \Big(\sum_{\{\beta \in S \mid w\beta \in \Delta^-\}} w\beta\Big) - \mathbb{Z}_{\geq 0}\Pi.$$

Since $L(\lambda)$ is π-integrable, $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for each $\alpha \in \pi$. It is easy to deduce from Proposition 2.1.1 that $r_\alpha \rho \in \rho - \Delta$ for each $\alpha \in \pi$. Note that $(\rho, \alpha) \geq 0$ for each $\alpha \in \Delta^+$ (because $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi$), so $r_\alpha \rho \in \rho - \Delta^+$ for each $\alpha \in \pi$. Therefore $w(\lambda+\rho) \in \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi$, so

$$\mathrm{supp}\Big(\sum_{w \in W(\pi)} \mathrm{sgn}(w) Y_w\Big) \subset \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi.$$

Clearly, $\mathrm{supp}(Re^{\rho}\,\mathrm{ch}\,L(\lambda)) \subset \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi$.

Let us show that

(16) $\quad (\lambda+\rho-\mathbb{Z}_{\geq 0}S)\cap \operatorname{supp}(Y_w)=\emptyset \quad \text{for} \quad w\in W(\pi), w\neq Id.$

Indeed, assume that the intersection is non-empty. Then, by (15), $w \in Stab_{W(\pi)}(\lambda+\rho)$ and for each $\beta\in S$ one has either $w\beta\in\Delta^+$ or $-w\beta\in\mathbb{Z}_{\geq 0}S$. If $-w\beta\in\mathbb{Z}_{\geq 0}S$, then $\beta-w\beta\in\mathbb{Z}S$, so $\|\beta-w\beta\|^2=0$; but $\beta-w\beta\in\mathbb{Z}\pi$, thus $\beta=w\beta$. Therefore $w\in Stab_{W(\pi)}(\lambda+\rho)$ and $wS\subset\Delta^+$. By Lemma 3.3.4 (ii), $w=Id$, as required.

We conclude that the coefficient of $e^{\lambda+\rho}$ in $\sum_{w\in W(\pi)}\operatorname{sgn}(w)\,Y_w$ is 1; clearly, the coefficient of $e^{\lambda+\rho}$ in $Re^{\rho}\operatorname{ch}L(\lambda)$ is also 1. Let

$$Z:=Re^{\rho}\operatorname{ch}L(\lambda)-\sum_{w\in W(\pi)}\operatorname{sgn}(w)w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big).$$

Suppose that $Z\neq 0$. By above,

$$\operatorname{supp}(Z)\subset\lambda+\rho-\mathbb{Z}_{\geq 0}\Pi,\ \lambda+\rho\not\in\operatorname{supp}(Z).$$

Recall that the Casimir element acts on a Verma module $M(\mu)$ by a scalar $(\mu,\mu+2\rho)$. The Casimir elements acts on $M(\mu)$ by the same scalar as on $M(\lambda)$ if $\mu+\rho\in\operatorname{supp}(Re^{\rho}\operatorname{ch}L(\lambda))$. Thus $\|\nu\|^2=\|\lambda+\rho\|^2$ for each $\nu\in\operatorname{supp}(Re^{\rho}\operatorname{ch}L(\lambda))$. On the other hand, it is easy to see that $\|\nu\|^2=\|\lambda+\rho\|^2$ for each $\nu\in\operatorname{supp}Y_w$. Thus $\|\nu\|^2=\|\lambda+\rho\|^2$ for each $\nu\in\operatorname{supp}(Z)$.

Let $\lambda+\rho-\mu$ be a maximal element in $\operatorname{supp}(Z)$ with respect to the order $\geq_{\Pi}$ (see §2.1.3). We have

$$\|\lambda+\rho-\mu\|^2=\|\lambda+\rho\|^2,\quad \mu\in\mathbb{Z}_{\geq 0}\Pi,\quad \mu\neq 0.$$

4.3.1. By §2.2.6, $R_{\overline{0}}e^{\rho_\pi}$ is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}$. Since $L(\lambda)$ is π-integrable, $\operatorname{ch}L(\lambda)$ is a $W(\pi)$-invariant element of $\mathscr{R}_{W(\pi)}$, see §3.1.3. Thus

$$R_{\overline{0}}e^{\rho_\pi}\operatorname{ch}L(\lambda)=R_{\overline{1}}e^{\rho_\pi-\rho}(Re^{\rho}\operatorname{ch}L(\lambda))$$

is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}$. By §2.2.6, $R_{\overline{1}}e^{\rho_\pi-\rho}$ is a $W(\pi)$-invariant element of $\mathscr{R}_{W(\pi)}$, so

$$\begin{aligned}&R_{\overline{1}}e^{\rho_\pi-\rho}\sum_{w\in W(\pi)}\operatorname{sgn}(w)\frac{e^{w(\lambda+\rho)}}{\prod_{\beta\in S}(1+e^{-w\beta})}\\ &=\sum_{w\in W(\pi)}\operatorname{sgn}(w)\,e^{w(\lambda+\rho_\pi)}\prod_{\beta\in\Delta^+_{\overline{1}}\setminus S}(1+e^{-w\beta}).\end{aligned}$$

By §2.2.3, $e^{\lambda+\rho_\pi}\prod_{\beta\in\Delta^+_{\bar 1}\setminus S}(1+e^{-\beta})\in\mathcal{R}_{W(\pi)}$, so, by §2.2.5, the sum in the right-hand side is a $W(\pi)$-skew-invariant element of $\mathcal{R}_{W(\pi)}$.

We conclude that $R_{\bar 1}e^{\rho_\pi-\rho}Z$ is a non-zero $W(\pi)$-skew-invariant element of $\mathcal{R}_{W(\pi)}$. Clearly, $\lambda-\mu+\rho_\pi$ is a maximal element in its support. Hence $(\lambda-\mu+\rho_\pi,\alpha^\vee)$ is a positive integer for each $\alpha\in\pi$ (positive, since $\lambda-\mu+\rho_\pi$ is a maximal element in the support of a skew-invariant element of $\mathcal{R}_{W(\pi)}$, and integer, since $(\lambda,\alpha^\vee)\in\mathbb{Z}_{\geq 0}$). Therefore

$$(\lambda-\mu,\alpha)\geq 0 \quad \text{for each } \alpha\in\pi.$$

4.3.2. By Lemma 3.3.4, for any $\gamma\in\Pi$ one has $\gamma\in\pi\cup S$, or $2\gamma\in\pi$, or $\gamma=\alpha-\beta,\alpha\in\pi,\beta\in S$. Define a linear map $p:\mathbb{Z}_{\geq 0}\Pi\to\frac{1}{2}\mathbb{Z}_{\geq 0}\pi$ which is zero on S and the identity on π. Recall that $(\lambda+\rho,S)=(S,S)=0$. We have $2(\lambda+\rho,\mu)=(\mu,\mu)$, so $2(\lambda+\rho,p(\mu))=(2\mu-p(\mu),p(\mu))$, which implies

$$2(\lambda+\rho-\mu,p(\mu))=-\|p(\mu)\|^2.$$

Since $(\rho,\alpha),(\lambda-\mu,\alpha)\geq 0$ for each $\alpha\in\pi$, the left-hand side is non-negative. Hence $\|p(\mu)\|^2\leq 0$. Since $p(\mu)\in\frac{1}{2}\mathbb{Z}_{\geq 0}\pi$, we obtain $p(\mu)=0$ if Δ is finite, and $p(\mu)=s\delta$ if Δ is affine. In the latter case $2(\lambda+\rho,p(\mu))=(2\mu-p(\mu),p(\mu))$ gives $2(\lambda+\rho,s\delta)=0$. Since λ is non-critical, $s=0$, so $p(\mu)=0$. Hence $\mu\in\mathbb{Z}S\setminus\{0\}$.

4.3.3. By (16), for $\mu\in\mathbb{Z}S$, the coefficient of $e^{\lambda+\rho-\mu}$ in Y_w is equal to zero if $w\neq Id$. One readily sees that $L(\lambda)_{\lambda-\nu}=0$ for each $\nu\in\mathbb{Z}S,\nu\neq 0$. Thus the coefficient of $e^{\lambda+\rho-\mu}$ in $Re^\rho\operatorname{ch}L$ is equal to the coefficient of $e^{\lambda+\rho-\mu}$ in $e^{\lambda+\rho}\prod_{\beta\in S}(1+e^{-\beta})^{-1}=Y_{Id}$. Hence the coefficient of $e^{\lambda+\rho-\mu}$ in Z is equal to zero, a contradiction. This gives $Z=0$ and establishes the KW-formula. □

4.4.

Remark. Let $\mathfrak{g}$ be of the type $A(0,n)^{(1)}$ or $C(n)^{(1)}$. Note that $\Delta_{\bar 0}$ is indecomposable, so $\pi=\Pi_0, W(\pi)=W$. Let $L=L(\lambda,\Pi)$ be a π-integrable module. If L is typical, i.e., $(\lambda+\rho,\beta)\neq 0$ for all $\beta\in\Delta_{\bar 1}$, then $\operatorname{ch}L$ is given by the Weyl–Kac character formula $Re^\rho\operatorname{ch}L=\sum_{w\in W}\operatorname{sgn}(w)e^{w(\lambda+\rho)}$. If L is atypical, then there exists Π' such that $L=L(\lambda',\Pi')$ and $(\lambda'+\rho',\beta')=0$ for some $\beta'\in\Pi'$, see the next paragraph. By above, $\operatorname{ch}L$ is given by the KW-character formula $Re^\rho\operatorname{ch}L=\sum_{w\in W}\operatorname{sgn}(w)e^{w(\lambda'+\rho')}/(1+e^{-w\beta'})$. This formula was proven earlier in [S4].

Let us show that $(\lambda+\rho,\beta)=0$ for some $\beta\in\Delta_{\bar 1}$ forces the existence of Π' such that $L=L(\lambda',\Pi')$ and $(\lambda'+\rho',\beta')=0$ for some $\beta'\in\Pi'$. Indeed, it is easy to show that for this $\mathfrak{g}$ for any odd root β there exists Π'' such that

$\beta \in \Pi''$. Recall that Π'' can be obtained from Π by a chain of odd reflections $r_{\beta_1}, \dots, r_{\beta_s}$. Writing $\Pi^0 := \Pi, \lambda^0 := \lambda^i$ and $\Pi^j = r_{\beta_j}\Pi^{j-1}$ (so $\Pi'' = \Pi^{s+1}$), we introduce λ^j by $L = L(\lambda^j, \Pi^j)$. Then $\lambda^j + \rho^j = \lambda^{j-1} + \rho^{j-1}$, (where ρ^j is the Weyl vector for Π^j) if $(\lambda^{j-1} + \rho^{j-1}, \beta_j) \neq 0$. Therefore, if $(\lambda^{j-1} + \rho^{j-1}, \beta_j) \neq 0$ for $j = 1, \dots, s$, then $\lambda^{s+1} + \rho^{s+1} = \lambda + \rho$, and taking $\Pi' := \Pi'' = \Pi^{s+1}$ we obtain $(\lambda' + \rho', \beta) = 0$ and $\beta \in \Pi''$. If $(\lambda^{j-1} + \rho^{j-1}, \beta_j) = 0$ for some j, then we take $\Pi' := \Pi^{j-1}$ and $\beta' := \beta_j$ ($\beta_j \in \Pi^{j-1}$ by the definition of an odd reflection).

4.5. A new identity

In [KW4] a product character formula was obtained for the $\mathfrak{osp}(m,n)^{(1)}$-module $V_1 := L(\Lambda_0)$. The bilinear form in this case is normalized there by $\|\varepsilon_i\|^2 = 1 = -\|\delta_j\|^2$. If $M \geq N + 2$, this normalization coincides with our normalization. On the other hand, we have established in this section the KW-character formula for V_1. Comparing these formulas, we obtain a new identity, see below.

4.5.1. Let $\mathfrak{g} = \mathfrak{osp}(m,n)^{(1)}$, $M \geq N + 2$. Consider a set of simple roots

$$\Pi = \{\delta - (\varepsilon_1 + \delta_1), \varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \dots, \varepsilon_n - \delta_n, \delta_n - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_{n+2}, \dots\}.$$

The set Π has an involution ι which exchanges the first two roots (via $\varepsilon_1 \mapsto \delta - \varepsilon_1$) and fixes the rest. This involution induces an involution ι of $\mathfrak{g}$. Consider the vacuum $\mathfrak{g}$-module V_1 and its twisted by ι module V_1^ι. By [KW4], these are all integrable (i.e., π-integrable) $\mathfrak{g}$-modules of level 1.

One has $V_1 = L(\Lambda_0)$, where $(\Lambda_0, \delta) = 1, (\Lambda_0, \varepsilon_i) = (\Lambda_0, \delta_j) = 0$, and $V_1^\iota = L(\Lambda_0 + \varepsilon_1)$. Formula (14) gives

$$Re^\rho \operatorname{ch} V_1 = \sum_{w \in W(\pi)} \operatorname{sgn}(w) \frac{e^{w(\Lambda_0+\rho)}}{\prod_{i=1}^n (1 + e^{-w(\varepsilon_i+\delta_i)})},$$

$$Re^\rho \operatorname{ch} V_1^\iota = \sum_{w \in W(\pi)} \operatorname{sgn}(w) \frac{e^{w(\Lambda_0+\varepsilon_1+\rho)}}{(1 + e^{-w(\delta+\varepsilon_1+\delta_1)}) \prod_{i=2}^n (1 + e^{-w(\varepsilon_i+\delta_i)})},$$

where $W(\pi)$ is the Weyl group of $\mathfrak{so}_M$ (B_m if $M = 2m + 1$, and D_m if $M = 2m$).

On the other hand, formula (7.5) from [KW4] gives

$$\begin{aligned} &Re^\rho(\operatorname{ch} V_1 \pm \operatorname{ch} V_1^\iota) \\ &= Re^{\Lambda_0+\rho} \prod_{k=1}^\infty \frac{(1 \pm q^{k-1/2})^{p(M)} \prod_{i=1}^m (1 \pm e^{\varepsilon_i} q^{k-1/2})(1 \pm e^{-\varepsilon_i} q^{k-1/2})}{\prod_{j=1}^n (1 \mp e^{\delta_j} q^{k-1/2})(1 \mp e^{-\delta_j} q^{k-1/2})}, \end{aligned}$$

where $q = e^{-\delta}$, $N = 2n$, and either $p(M) = 0, M = 2m, \mathfrak{osp}(m,n) = D(m,n)$, or $p(M) = 1, M = 2m+1, \mathfrak{osp}(m,n) = B(m,n)$.

Comparing the last character formula with the previous two, gives a product formula for some mock theta functions (as defined in [KW5]).

5. KW-character formula for finite-dimensional modules

We say that a highest weight module L satisfies the KW-condition, if for some set of simple roots Π one has $L = L(\lambda, \Pi)$ and Π contains a subset S, which spans a maximal isotropic subspace in $\mathbb{Q}\Delta^{\perp}_{\lambda+\rho}$ (where $\Delta^{\perp}_{\lambda+\rho} = \{\alpha \in \Delta \mid (\lambda+\rho, \alpha) = 0\}$). Sometimes we say that L satisfies the KW-condition for Π (or for (Π, S)).

Note that $\dim \mathbb{Q}\Delta^{\perp}_{\lambda+\rho}$ is the invariant of L (if $L = L(\lambda, \Pi) = L(\lambda', \Pi')$, then $\dim \mathbb{Q}\Delta^{\perp}_{\lambda+\rho} = \dim \mathbb{Q}\Delta^{\perp}_{\lambda'+\rho'}$), see [KW3], Corollary 3.1.

Throughout this section $\mathfrak{g}$ is a basic Lie superalgebra $\mathfrak{g}$ and L is a finite-dimensional irreducible (hence highest weight) $\mathfrak{g}$-module, which satisfies the KW-condition for some (Π, S), and $r := \dim \mathbb{Q}\Delta^{\perp}_{\lambda+\rho}$. Note that $\#S = r \leq$ *defect*$(\mathfrak{g})$. We assume that $r > 0$ (otherwise L is typical and $Re^{\rho} \operatorname{ch} L = \sum_{w\in W} \operatorname{sgn}(w)\, e^{w(\lambda+\rho)}$, by [K2], [KW3]).

Recall that KW-formula (3) has the form

$$j_{\lambda} Re^{\rho} \operatorname{ch} L = \sum_{w \in W} \operatorname{sgn}(w) \frac{e^{w(\lambda+\rho)}}{\prod_{\beta \in S}(1 + e^{-w\beta})}, \tag{17}$$

where $j_{\lambda} \neq 0$. In this section we prove this formula for finite-dimensional modules, satisfying the KW-condition, in all cases except for $\mathfrak{g} = D(m,n)$ with $S = \{\delta_k - \varepsilon_m\}$ or $S = \{\varepsilon_m - \delta_k\}$ with $k < n$. For $C(n)$ the formula was proved in §4.

The coefficient j_{λ} is equal to $r!$ for $A(m,n)$, $2^r\, r!$ for $B(m,n)$, to 1 for $C(n)$, to $2^r\, r!$ or $2^{r-1}\, r!$ for $D(m,n)$, and to 2 for the exceptional Lie superalgebras, *cf.* (19) and §5.2.1 below.

5.1. Outline of the proof

Let us explain the outline of the proof. In §5.2 we deduce (17) from (14) for (Π, S) satisfying the assumptions of §4.1, and then, using Lemma 5.7.1, we deduce (17) for any (Π, S) for the exceptional Lie superalgebras. In §5.4 we establish (17) under the assumption that $(\rho, \alpha^{\vee}) \geq 0$ for all $\alpha \in \Pi_0$ (this assumption holds for some subsets of simple roots, if $\mathfrak{g}$ is $A(m-1, n-1)$ or $D(m,n)$). For $B(m,n)$ we establish (17) for some special subsets of simple roots in §5.5. Then, in §5.7–5.9, we explain why for $A(m-1, n-1)$ (resp.,

$B(m,n)$) the KW-formula for any (Π', S') is equivalent to the KW-formula for (Π, S) as in §5.4 (resp., in §5.5), and why this holds for $D(m,n)$ except for the cases, when $S = \{\delta_k - \varepsilon_m\}$ or $S = \{\varepsilon_m - \delta_k\}$ with $k < n$.

For $D(m,n)$ one of the simplest cases, when we have not established the KW-formula, is $D(3,2)$ with $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \delta_1, \delta_1 - \delta_2, 2\delta_2\}$ and $\lambda = 4\varepsilon_1 + 4\varepsilon_2 - \varepsilon_3 + \delta_1$.

5.2. Case of maximal atypicality

Let $r(= \#S)$ be equal to the defect of $\mathfrak{g}$. Assume that Π, S satisfy the assumptions of §4.1. Take $\pi = \{\alpha \in \Pi_0 \mid \|\alpha\|^2 > 0\}$ for $\Delta \neq D(n+1,n), D(2,1,a)$, $\pi = D_{n+1}$ for $D(n+1,n)$, and $\pi = D_2 = A_1 \times A_1$ for $D(2,1,a)$. Then, by (14),

$$Re^{\rho} \operatorname{ch} L = \mathscr{F}_{W(\pi)}\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big),$$

see §2.2.8 for notation. Write $W = W(\pi) \times W(\Pi_0 \setminus \pi)$. Since L is finite-dimensional, the left-hand side of this formula is W-skew-invariant. Then

$$\begin{aligned} |W(\Pi_0 \setminus \pi)| Re^{\rho} \operatorname{ch} L &= \mathscr{F}_{W(\Pi_0\setminus\pi)}(Re^{\rho} \operatorname{ch} L) \\ &= \mathscr{F}_{W(\Pi_0\setminus\pi)}\mathscr{F}_{W(\pi)}\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) \\ &= \mathscr{F}_{W}\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big). \end{aligned} \tag{18}$$

This establishes KW-formula (17) for this case with

$$j_\lambda = |W(\Pi_0 \setminus \pi)|; \tag{19}$$

note that $W(\Pi_0 \setminus \pi)$ is the smallest factor in the presentation of $W(\Pi_0)$ as the direct product of Coxeter groups.

In particular, for $\lambda = 0$ we obtain the denominator identity (for such Π, S).

Note that π is the "largest part" of Π_0 in the following sense: $\Pi_0 \setminus \pi$ is a connected component of Π_0 with the property $|W(\pi)| \geq |W(\Pi_0 \setminus \pi)|$ (for $A(n,n), B(n,n), D(2,1,a)$ the choice of π is not unique).

5.2.1. The coefficient j_λ for a non-exceptional Lie superalgebra can be obtained as follows: it is not hard to show (see §5.8) that Π contains a connected subdiagram Π' of defect r with the property $(\lambda, \alpha) = 0$ for $\alpha \in \Pi'$; moreover, Π' is of "the same type" as Π (if $\Pi = A(m,n)$, then $\Pi' = A(m',n')$, etc.). Write $W(\Pi'_0) = W_1 \times W_2$, where W_1, W_2 are the Weyl groups of the components of Π'_0 (connected components if $\Pi' \neq D(2,1)$). If we choose W_2 such that $|W_1| \geq |W_2|$, then $j_\lambda = |W_2|$. If $\Pi = A(m,n)$ (resp., $B(m,n), D(m,n)$), then

$\Pi' = A(m',n')$ (resp., $B(m',n'), D(m',n')$), with $r = \min(m',n')$. Therefore for $A(m,n)$ (resp., for $B(m,n)$) one has $j_\lambda = |W(A_r)| = r!$ (resp., $j_\lambda = 2^r\, r!$). For $D(m,n)$ one has either $\Pi' = D(m',r)$ for $m' > r$ and $j_\lambda = |W(C_r)| = 2^r\, r!$, or $\Pi' = D(r,n')$ for $n' \geq r$ and $j_\lambda = |W(D_r)| = 2^{r-1}\, r!$.

5.2.2. Now let $\mathfrak{g}$ be an exceptional Lie superalgebra and let $L = L(\lambda,\Pi)$ be a finite-dimensional $\mathfrak{g}$-module, satisfying the KW-conditionfor (Π, S). We claim that the KW-formula holds and $j_\lambda = 2$. Note that $\mathfrak{g}$ has defect one, so $r = 1$, that is $S = \{\beta\}$ for some $\beta \in \Pi$.

Indeed, by above, if Π, S satisfy the assumptions of §4.1, then the KW-formula holds (this was also proved previously, see [KW2]) and $j_\lambda = |W(\Pi_0 \setminus \pi)| = 2$. Assume that $\mathfrak{g} \neq D(2,1,a)$ with $a \not\in \mathbb{Q}$ and Π, S do not satisfy the assumptions of §4.1. It is easy to check that in this case β is the only isotropic root in Π and that $(r_\beta \Pi, S = \{-\beta\})$ satisfy the assumptions of §4.1. In particular, the KW-formula holds for $(r_\beta \Pi, S' = \{-\beta\})$. By Lemma 5.7.1, this implies the KW-formula for $(\Pi, \{\beta\})$.

Now let $\mathfrak{g} = D(2,1,a)$ for irrational a (this case is not covered by §4). It is easy to see that, in this case, the trivial module is the only finite-dimensional atypical module. KW-formula for the trivial module is the denominator identity, which, clearly, does not depend on a; it holds for rational a, hence it holds in general.

This establishes KW-formula (17) with $j_\lambda = 2$ for the exceptional Lie superalgebras.

5.3.

Denote by j_λ the coefficient of $e^{\lambda+\rho}$ in $\mathcal{F}_W(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})})$. Set

$$Z := j_\lambda R e^\rho \operatorname{ch} L - \mathcal{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big).$$

The KW-formula is equivalent to $j_\lambda \neq 0$ and $Z = 0$.

If $Z \neq 0$, we denote by λ' a maximal element in $\operatorname{supp} Z$. The arguments of §4.3.1 show that

$$(\lambda' - \rho, \alpha^\vee) \geq 0 \quad \text{for each } \alpha \in \Pi_0. \tag{20}$$

5.3.1. **Lemma.** $\operatorname{supp} Z \subset W(\lambda + \rho - \mathbb{Z}S)$.

Proof. Clearly, $\operatorname{supp}(\frac{e^{w(\lambda+\rho)}}{\prod_{\beta\in S}(1+e^{-\beta})}) \subset W(\lambda+\rho-\mathbb{Z}S)$.

Let us show that $\operatorname{supp} Re^\rho \operatorname{ch} L \subset W(\lambda+\rho-\mathbb{Z}S)$.

In the light of Proposition 7.3.1 (in §7), it is enough to verify that for $y \in W, \mu \in \mathbb{Z}S$ and an isotropic root β, if $(y(\lambda+\rho-\mu), \beta) = 0$, then $y(\lambda+\rho-\mu) - \beta = y'(\lambda+\rho-\mu')$ for some $y' \in W, \mu' \in \mathbb{Z}S$.

We start with the case $y = Id$, so $(\lambda+\rho-\mu, \beta) = 0$. If $(S, \beta) = 0$, then $(\lambda+\rho, \beta) = 0$ and, by the KW-condition, $\beta \in \mathbb{Z}S$. Therefore the claim holds for $y' = Id, \mu' = \mu+\beta$. Now let $(S, \beta) \neq 0$, that is $(\beta, \beta') \neq 0$ for some $\beta' \in S$. Since $\mathfrak{g}_{\pm\beta}$ generate a copy of $\mathfrak{sl}(1,1) \subset \mathfrak{g}$, either $\beta+\beta'$ or $\beta-\beta'$ is a root, i.e., $\alpha := \beta - x\beta' \in \Delta_{\bar{0}}$ for $x = 1$ or $x = -1$. Since $\lambda+\rho-\mu$ is orthogonal to S and to β', it is orthogonal to α. One has $r_\alpha\beta = x\beta'$, so

$$\lambda+\rho-\mu-\beta = \lambda+\rho-\mu-xr_\alpha\beta' = r_\alpha(\lambda+\rho-\mu-x\beta').$$

Thus the claim holds for $y' = r_\alpha, \mu' = \mu + x\beta$.

Now take an arbitrary $y \in W$. Then $(y(\lambda+\rho-\mu), \beta) = 0$ implies $(\lambda+\rho-\mu, y^{-1}\beta) = 0$. By above, $\lambda+\rho-\mu-y^{-1}\beta = w(\lambda+\rho-\mu')$ for some $w \in W, \mu' \in \mathbb{Z}S$. Then $y(\lambda+\rho-\mu)-\beta = yw(\lambda+\rho-\mu')$, as required. □

5.4. Case $A(m-1,n-1)$ and $D(m,n), r > 1$, or $D(m,n), S = \{\beta\}$, where $\beta = \pm(\varepsilon_m - \delta_n)$

In Corollary 5.8.2 below we show that the KW-formula for any (Π', S') is equivalent to the KW-formula for (Π, S) such that $(\rho, \alpha^\vee) \geq 0$ for each $\alpha \in \Pi_0$. For such (Π, S) we prove the formula below, proving thereby the KW-formula for (Π', S').

5.4.1. Let $\mathfrak{g}$ be $A(m-1,n-1)$ or $D(m,n)$. We shall assume that

$$\forall \alpha \in \Pi_0 \quad (\rho, \alpha^\vee) \geq 0. \tag{21}$$

We will prove the KW-formula under this assumption, by showing that $j_\lambda \neq 0$ and $Z = 0$.

5.4.2. Since L is finite-dimensional, $(\lambda, \alpha^\vee) \geq 0$ for each $\alpha \in \Pi_0$. Assumption (21) implies that $\lambda+\rho$ is maximal in its W-orbit and $Stab_W(\lambda+\rho) = W(\Pi_0')$, where

$$\Pi_0' := \{\alpha \in \Pi_0 \mid (\lambda+\rho, \alpha) = 0\} = \{\alpha \in \Pi_0 \mid (\lambda, \alpha) = (\rho, \alpha) = 0\}.$$

Take $\alpha \in \Pi_0'$. Since Δ is not exceptional, $(\rho, \alpha) = 0$ implies $\alpha = \beta + \beta'$, where $\beta, \beta' \in \Pi$ are isotropic, see Lemma 5.6.4. In this case, $(\beta, \alpha^\vee) = (\beta', \alpha^\vee) = 1$. If $(\lambda, \beta) \neq 0$, then $L = L(\lambda, \Pi) = L(\lambda-\beta, r_\beta\Pi)$ and $(\lambda, \alpha) = 0$ gives $(\lambda-\beta, \alpha^\vee) = -1$, which is impossible, since L is finite-dimensional. Therefore $(\lambda, \beta) = 0$; similarly, $(\lambda, \beta') = 0$. We conclude that Π_0' is spanned by Π', where

$$\Pi' := \{\alpha \in \Pi \mid (\lambda, \alpha) = (\rho, \alpha) = 0\}$$

(more precisely, Π_0' is a set of simple roots for $\Delta(\Pi')_{\bar{0}}$).

5.4.3. Since $\lambda + \rho$ is maximal in its W-orbit, using (15) we obtain

$$\operatorname{supp} Z \subset \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi.$$

Suppose that $Z \neq 0$ and let λ' be a maximal element in $\operatorname{supp} Z$. Then $\lambda' = \lambda + \rho - \nu'$ for some $\nu' \in \mathbb{Z}_{\geq 0}\Pi$.

Let us show that $\nu' \in \mathbb{Z}S$. By Lemma 5.3.1, $\lambda' := w(\lambda + \rho - \mu)$ with $w \in W, \mu \in \mathbb{Z}S$. Combining (20) and (21), we get $(\lambda', \alpha^\vee) \geq 0$ for each $\alpha \in \Pi_0$. Thus $\lambda' = w(\lambda + \rho - \mu)$ is maximal in its W-orbit. Since $\lambda + \rho - \mu$ lies in this orbit, we have $\lambda + \rho - \mu = \lambda' - \nu$ for some $\nu \in \mathbb{Z}_{\geq 0}\Pi_0$. Thus $\mu = \nu + \nu'$, where $\mu \in \mathbb{Z}S$, $\nu \in \mathbb{Z}_{\geq 0}\Pi_0$ and $\nu' \in \mathbb{Z}_{\geq 0}\Pi$. Since S is a set of mutually orthogonal isotropic roots, $\nu = 0$ and $\nu' = \mu$, as required.

5.4.4. Denote by $P : \mathcal{V} \to \mathcal{V}$ the projection sending $\sum_{\nu \in \mathfrak{h}^*} a_\nu e^{\lambda+\rho-\nu}$ to $\sum_{\nu \in \mathbb{Z}S} a_\nu e^{\lambda+\rho-\nu}$. Since $\lambda' \in \lambda + \rho - \mathbb{Z}S$, it is enough to verify that $P(Z) = 0$.

If $w \notin W(\Pi_0')$, then $w(\lambda + \rho) = (\lambda + \rho) - \gamma$, where $\gamma \in \mathbb{Z}_{\geq 0}\Pi_0$, $\gamma \neq 0$. By (15), this implies $P(Y_w) = 0$. Hence

$$P\Big(\mathcal{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})}\Big)\Big) = P\Big(\mathcal{F}_{W(\Pi_0')}\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})}\Big)\Big).$$

Since $(\rho, \alpha) = 0$ for each $\alpha \in \Pi'$, Π' consists of isotropic roots. Clearly, S is a maximal set of mutually orthogonal isotropic roots in Π' (otherwise, $(S, \beta) = 0$ for some $\beta \in \Pi'$, which contradicts the KW-condition). In particular, from the denominator identity ((18) for $\lambda = 0$) for Π' one has

$$\mathcal{F}_{W(\Pi_0')}\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})}\Big) = j(\Pi')R(\Pi')e^{\lambda+\rho},$$

where $j(\Pi') \neq 0$.

Since $\Pi' \subset \Pi$ is orthogonal to λ, the highest weight vector in $L(\lambda)$ generates the trivial module over the corresponding to Π' subalgebra, and so $L(\lambda)_{\lambda-\nu} = 0$ for each $\nu \in \mathbb{Z}\Pi', \nu \neq 0$. Therefore $P(e^\rho \operatorname{ch} L) = 1$. Since $S \subset \Pi$ we have

$$P(Re^\rho \operatorname{ch} L) = P(R(\Pi')e^{\lambda+\rho}).$$

Then

$$\begin{aligned} P(Z) =& P\Big(j_\lambda Re^\rho \operatorname{ch} L - \mathcal{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})}\Big)\Big) \\ =& (j_\lambda - j(\Pi'))P(R(\Pi')e^{\lambda+\rho}). \end{aligned}$$

The coefficient of $e^{\lambda+\rho}$ in the left-hand side is zero and in the right-hand side is $j_\lambda - j(\Pi')$. Hence $j_\lambda = j_{\Pi'} \neq 0$ and $P(Z) = 0$, as required. This completes the proof of the KW-formula under the assumption (21).

5.5. *Case* $B(m,n)$

In this case there is no Π such that $(\rho,\alpha^\vee)\geq 0$ for each $\alpha\in\Pi_0$.

We will show below that the KW-formula for any (Π',S') is equivalent to the KW-formula for (Π,S) such that $(\rho,\alpha^\vee)\geq 0$ for all except one root in Π_0 (which is either ε_m or $2\delta_n$). Moreover, by (25), $\{\alpha\in\Delta\mid(\lambda+\rho,\alpha)=0\}$ is spanned by

$$\Pi^0:=\{\alpha\in\Pi\mid(\lambda+\rho,\alpha)=0\}.$$

Since $(\lambda,\alpha^\vee)\geq 0$ for each $\alpha\in\Pi_0$, Π^0 consists of isotropic roots.

5.5.1. Consider the case when $\alpha_{m+n}=\varepsilon_m$ (for $\alpha_{m+n}=\delta_n$ we interchange ε's and δ's). Then $\alpha_{m+n-1}=\delta_n-\varepsilon_m$.

Normalize $(-,-)$ by $\|\varepsilon_i\|^2=1$. Then $(\lambda,\varepsilon_i)\geq 0\geq(\lambda,\delta_j)$ for each i,j. Since $r>1$, the KW-condition implies that $(\lambda,\varepsilon_n)=(\lambda,\delta_m)=0$ and that Π^0 is connected (and contains $\delta_n-\varepsilon_m$). Then

$$\Pi':=\Pi^0\cup\{\alpha_{m+n}\}$$

is a connected subdiagram containing α_{m+n}. Recall that S is a maximal set of mutually orthogonal isotropic roots in Π^0, so Π' is $B(t,r)$ for $t=r$ or $t=r+1$, and $(\lambda,\varepsilon_{m-i})=(\lambda,\delta_{n-j})=0$ for $0\leq i\leq r-1$.

Write $W(\Pi'_0)=W_+\times W_-$, where W_- is the group of signed permutations of $\{\delta_{n-i}\}_{i=0}^{r-1}$ and W_+ is the group of signed permutations of $\{\varepsilon_{m-i}\}_{i=0}^{t-1}$. We denote by S_t,S_r the subgroups of unsigned permutations in the corresponding groups, and by w_- the longest element in W_- ($w_-\delta_i=-\delta_i$ for $i>n-r$).

5.5.2. Since $(\lambda+\rho,\alpha^\vee)\geq 0$ for $\alpha\in\Pi_0\setminus\{2\delta_n\}$, $\lambda+\rho$ is maximal in $W(B_m)\times S_n$-orbit (where S_n is the subgroup of unsigned permutations in $W(C_n)\subset W$).

One has

$$(\lambda+\rho,\varepsilon_{m-i})=(\lambda+\rho,\delta_{n-j})=\frac{1}{2},$$

so $w_-(\lambda+\rho)$ is maximal in its W-orbit. Since Δ^0 is spanned by Π^0, we have

$$(22)\qquad \begin{aligned}(\lambda+\rho,\delta_i)&=\frac{1}{2}\quad\text{for } n-r<i\leq n;\\ (\lambda+\rho,\delta_j)&<-\frac{1}{2}\quad\text{for } 1\leq i\leq n-r.\end{aligned}$$

In particular, $Stab_W(\lambda+\rho)$ is $S_t\times S_r\subset W(\Pi'_0)$.

5.5.3. Let $Z := 2^r r! Re^{\rho} \operatorname{ch} L - \mathcal{F}_W(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})})$.

Let us show that

$$\operatorname{supp} Z \subset \lambda + \rho - (\mathbb{Z}_{\geq 0}\Pi \setminus \{0\}). \tag{23}$$

Let ρ' be the Weyl vector of Π'. Note that Π', S satisfy the assumptions of §4.1, so (18) for $B(t, r)$ gives

$$\mathcal{F}_{W(\Pi'_0)}\Big(\frac{e^{\rho'}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) = 2^r r!\, \mathcal{F}_{W_+}\Big(\frac{e^{\rho'}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big).$$

The term in the left-hand side is obviously $W(\Pi'_0)$-skew-invariant, so

$$\mathcal{F}_{W(\Pi'_0)}\Big(\frac{e^{\rho'}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) = \operatorname{sgn}(w_-) 2^r r! \mathcal{F}_{W_+}\Big(w_-\Big(\frac{e^{\rho'}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big).$$

Recall that $\operatorname{sgn}(w_-) = 1$. Since λ and $\rho - \rho'$ are $W(\Pi'_0)$-invariant, using the denominator identity (see (18)) for $B(t, r)$, we obtain

$$\begin{aligned}
&\mathcal{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\\
&= \mathcal{F}_{W/W(\Pi'_0)}\Big(e^{\lambda+\rho-\rho'} \mathcal{F}_{W(\Pi'_0)}\Big(\frac{e^{\rho'}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big)\\
&= 2^r r!\, \mathcal{F}_{W/W(\Pi'_0)}\Big(e^{\lambda+\rho-\rho'} \mathcal{F}_{W_+}\Big(w_-\Big(\frac{e^{\rho'}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big)\Big)\\
&= 2^r r!\, \mathcal{F}_{(W/W(\Pi'_0))\times W_+}\Big(w_-\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big),
\end{aligned} \tag{24}$$

where $W/W(\Pi'_0)$ is a set of coset representatives.

Hence

$$\mathcal{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) = 2^r r! \sum_{w\in(W/W(\Pi'_0))\times W_+} \operatorname{sgn}(ww_-)\, Y_{ww_-},$$

where $Y_w := \frac{e^{w(\lambda+\rho)}}{\prod_{\beta\in S}(1+e^{-w\beta})}$.

Write $W = W(B_m) \times W(C_n)$. Then $(W/W(\Pi'_0)) \times W_+ = (W(C_n)/W_-) \times W(B_m)$. Consider the Bruhat order $\geq$ on $W(C_n)$. The following claim can be easily proven by induction on the Bruhat order:

$$\forall y \in W(C_n)\ \exists z \in W_- \quad \text{such that}\ \ yz \geq w_-.$$

Using this claim we choose the set of representatives in $W(C_n)/W_-$ consisting of the elements which are larger than w_- (with respect to the Bruhat order).

Since $w_-(\lambda+\rho)$ is maximal in its W-orbit, we get $ww_-(\lambda+\rho)\leq\lambda+\rho$ for each $w\in(W(C_n)/W_-)\times W(B_m)$.

Take $w\in(W(C_n)/W_-)\times W(B_m)$. Using (15) we obtain

$$\operatorname{supp} Y_{ww_-}\subset\lambda+\rho-\mathbb{Z}_{\geq 0}\Pi,$$

so $\operatorname{supp} Z\subset\lambda+\rho-\mathbb{Z}_{\geq 0}\Pi$. Moreover, since $Stab_W(\lambda+\rho)$ is equal to $S_t\times S_r$, we conclude that $\lambda+\rho\in\operatorname{supp} Y_w$ forces $w=w_-x$ for $x\in W_+$. Combining (24) and (18) for $B(t,r)$, we conclude that the coefficient of $e^{\lambda+\rho}$ in $\mathscr{F}_W(e^{\lambda+\rho}\prod_{\beta\in S}(1+e^{-\beta}))$ is equal to $2^r r!$. Hence $\lambda+\rho\not\in\operatorname{supp} Z$. This establishes (23).

5.5.4. Suppose that $Z\neq 0$. Let $\lambda':=\lambda+\rho-\nu'$ be maximal in $\operatorname{supp} Z$. By (23), $\nu'\in\mathbb{Z}_{\geq 0}\Pi$. By Lemma 5.3.1, $\lambda'=w(\lambda+\rho-\mu)$ for $w\in W,\mu\in\mathbb{Z}S$.

Recall that $(\lambda'-\rho,\alpha^\vee)\geq 0$ for $\alpha\in\Pi_0$. One has $\|\delta_i\|^2=-1$ and $(\rho,\delta_n)=\frac{1}{2}$; therefore $(\lambda',\delta_i)\leq\frac{1}{2}$ for each i and the maximal element in W-orbit of λ' is of the form

$$\lambda'':=\lambda'+\sum_{\{i|(\lambda'-\rho,\delta_i)=0\}}\delta_i.$$

Since $(\lambda'-\rho,\delta_j-\delta_{j+1})\geq 0$, the set $D:=\{\delta_i\mid(\lambda'-\rho,\delta_i)=0\}$ is either empty or of the form $\{\delta_j,\delta_{j+1},\dots,\delta_n\}$. Assume that $\delta_{n-r}\in D$. Then $\delta_i\in D$ for $n-r\leq i\leq n$; for such i one has $(\rho,\delta_i)=\pm\frac{1}{2}$, so $(\lambda',\delta_i)=\pm\frac{1}{2}$. Therefore $(\lambda+\rho-\mu,w\delta_i)=\pm\frac{1}{2}$. Recall that for $j\leq n-r$ one has $(\mu,\delta_j)=0$ and $(\lambda+\rho,\delta_j)<-\frac{1}{2}$ by (22). Hence for each $i=n-r,\dots,n$ one has $w\delta_i=\pm\delta_j$, where $n-r<j\leq n$, a contradiction. We conclude that

$$\lambda''=\lambda'+\sum_{i=s}^{n}\delta_i,\qquad\text{where } n-r<s\leq n.$$

Since λ'' is maximal in $W\lambda'=W(\lambda+\rho-\mu)$, we have

$$\lambda+\rho-\mu=\lambda''-\nu=\lambda'+\sum_{i=s}^{n}\delta_i=\lambda+\rho-\nu'+\sum_{i=s}^{n}\delta_i,$$

for some $\nu\in\mathbb{Z}_{\geq 0}\Pi$. Then

$$\nu+\nu'=\mu+\sum_{i=j}^{n}\delta_i\in\mathbb{Z}_{\geq 0}\Pi'$$

and $\nu,\nu'\in\mathbb{Z}_{\geq 0}\Pi$. Hence $\nu,\nu'\in\mathbb{Z}_{\geq 0}\Pi'$, that is $\lambda'-\rho=\lambda-\nu'$ for $\nu'\in\mathbb{Z}_{\geq 0}\Pi'$.

Recall that Π' is $B(t,r)$. Therefore $\mathbb{Q}\Pi'=\mathbb{Q}\Pi'_0$. For each $\alpha\in\Pi'_0$ one has $(\lambda'-\rho,\alpha^\vee)\geq 0$; since $(\lambda,\alpha)=0$ we obtain $(\nu',\alpha^\vee)\leq 0$. By Theorem 4.3

in [K3], if A is a Cartan matrix of a semisimple Lie algebra and v is a vector with rational coordinates, then $Av \geq 0$ implies $v > 0$ or $v = 0$. Therefore $\nu' \in -\mathbb{Q}_{\geq 0}\Delta^+(\Pi')_{\bar{0}}$. Since $\nu' \in \mathbb{Z}_{\geq 0}\Pi'$, $\nu' = 0$, which contradicts (23).

This proves the KW-formula if Π is such that $(\rho, \alpha^\vee) \geq 0$ for all except one root in Π_0.

5.6. Dots and crosses diagrams

Let $\mathfrak{g}$ be $A(m-1, n-1) = \mathfrak{gl}(m, n)$, $B(m, n)$ or $D(m, n)$.

For a set of simple roots Π we denote by *Iso* the subdiagram of Dynkin diagram consisting of isotropic nodes. We call a set of r mutually orthogonal isotropic simple roots *dense* if S is contained in a connected subdiagram consisting of $2r-1$ isotropic roots.

We will show that if L satisfies the KW-condition for some pair, then $L = L(\lambda, \Pi)$ satisfies the KW-condition for a pair (Π, S), such that *Iso* is connected and S is dense (plus some additional conditions for $B(m, n)$ and $D(m, n)$), and the KW-formula for the former pair is equivalent to the KW-formula for (Π, S).

5.6.1. In this section we encode subsets of simple roots for the root system $\Delta(\mathfrak{g})$ by diagrams described in [GKMP]. We recall this construction below.

Recall that the standard basis of $\mathfrak{h}^*$ consists of ε_i with $i = 1, \dots, m$ and δ_j with $j = 1, \dots, n$, which are mutually orthogonal. We can normalize the form $(-,-)$ in such a way that $\|\varepsilon_i\|^2 = 1$ for each i and $\|\delta_j\|^2 = -1$ for each j.

Set
$$\mathscr{E} := \{\varepsilon_i\}_{i=1}^m, \quad \mathscr{D} := \{\delta_j\}_{j=1}^n, \quad \mathscr{B} = \mathscr{E} \cup \mathscr{D}.$$

We call two elements $v_1, v_2 \in \mathscr{B}$ *elements of the same type* if $\|v_1\|^2 = \|v_2\|^2$ (i.e., $\{v_1, v_2\} \subset \mathscr{E}$ or $\{v_1, v_2\} \subset \mathscr{D}$) and *elements of different types* otherwise.

Fix a total order $>$ on $\mathscr{B} = \{\xi_1 > \cdots > \xi_{m+n}\}$ and define the corresponding set of simple roots $\Pi(\mathscr{B}, >)$ as follows:

$\mathfrak{g}$	$\Pi(\mathscr{B}, >)$
$A(m-1, n-1)$	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1}$
$B(m, n)$	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1} \cup \{\xi_{m+n}\}$
$D(m, n)$	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1} \cup \{2\xi_{m+n}\}$ if $\xi_{m+n} \in \mathscr{D}$
	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1} \cup \{\xi_{m+n-1} + \xi_{m+n}\}$ if $\xi_{m+n} \in \mathscr{E}$

We encode a subset Π of simple roots for the root system $\Delta(\mathfrak{g})$ by the ordered set $\mathscr{B}$, which is pictorially represented by an ordered sequence of dots and crosses, the former corresponding to vertices in $\mathscr{E}$ and the latter to vertices in $\mathscr{D}$.

For instance, the sequence $\cdot \ \cdot \times \times$ encodes $\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\delta_2\}$ for $A(1,1)$, $\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\delta_2, \delta_2\}$ for $B(2,2)$ and $\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\delta_2, 2\delta_2\}$ for $D(2,2)$.

For each $u, v \in \mathscr{B}$, $u-v$ is a root. We call u, v the *ends* of $\alpha = u - v$. The root $u-v$ is isotropic if u, v are of different types and is simple if u, v are neighbors.

A simple odd reflection r_{v-w} with $v, w \in \mathscr{B}$ corresponds to the switch of consecutive vertices v, w in the ordered sequence (v and w should be of different types).

For $v, w \in \mathscr{B}$ denote by $[v, w]$ the (ordered) set of elements of $\mathscr{B}$ lying between v and w, namely, if $v > w$, then $[v, w] = \{u \in \mathscr{B} \mid v \geq u \geq w\}$ and by $]v, w[$ the set $[v, w] \setminus \{v, w\}$.

Let $[v, w]$ be any interval and $|[v, w] \cap \mathscr{E}| = k,\ |[v, w] \cap \mathscr{D}| = l$. We denote by $\Pi([u, v])$ the corresponding set of simple roots of $A(k, l)$-type: if $[v, w] = \{u_1 > u_2 > \cdots > u_s\}$, then $\Pi([u, v]) := \{u_1 - u_2, \ldots, u_{s-1} - u_s\}$. Note that each permutation of dots and crosses in $[v, w]$ correspond to a sequence of odd reflections and thus to a choice of another set of simple roots in $A(k, l)$.

Let $W_{[v,w]}$ be the Weyl group of $\Pi([u, v])$ (the subgroup of W consisting of (non-signed) permutations of $[v, w] \cap \mathscr{E}$ and of $[v, w] \cap \mathscr{D}$), so $W_{[v,w]} \cong S_k \times S_l$.

We say that $[v, w]$ is *balanced* if $\Pi([v, w])$ has a maximal possible number of mutually orthogonal isotropic roots; in other words, if $[v, w]$ consists of k_1 dots and k_2 crosses, then $[v, w]$ is balanced if it contains $\min(k_1, k_2)$ disjoint pairs consisting of neighboring vertices of different types.

Any $\nu \in \mathfrak{h}^*$ can be written in the form $\nu = \sum_{u \in \mathscr{B}} y_u u$ for some scalars y_u; we define the *restriction of* ν to $[v, w]$ by the formula

$$\nu_{[u,v]} := \sum_{u \in [v,w]} y_u u.$$

We say that $\nu_{[u,v]}$ is *trivial* if $(\nu, \alpha) = 0$ for all $\alpha \in \Pi([v, w])$; it means that $A(k, l)$-module of highest weight $\nu_{[u,v]}$ is one-dimensional (where $\Pi([v, w])$ is a set of simple roots for $A(k, l)$).

A set S of mutually orthogonal simple isotropic roots is represented by a set of disjoint pairs consisting of neighboring vertices of different types. It can be encoded by the set of these vertices, which we denote by $\operatorname{supp} S$. Note that S is dense if $\operatorname{supp} S \subset \mathscr{B}$ form an interval consisting of the elements of alternating types.

5.6.2. For the cases $A(m-1, n-1)$ and $B(m, n)$, using the description of Borel subalgebras in [K1], it is not difficult to show that any set of positive roots for $\mathfrak{g}$ (satisfying $\Delta^+(\Pi) \cap \Delta_{\bar{0}} = \Delta^+_{\bar{0}}$) is of the form $\Pi(\mathscr{B}, >)$ for some total order $>$ on $\mathscr{B}$.

Consider the case $D(m,n)$. Recall that $D(m,n)$ has an autormorphism ι_D (induced by a Dynkin diagram automorphism of D_m) which preserves $\mathfrak{h}$ and satisfies $\iota_D(\delta_i) = \delta_i$ for $1 \le i \le n$, $\iota_D(\varepsilon_i) = \varepsilon_i$ for $1 \le i \le m-1$ and $\iota_D(\varepsilon_m) = -\varepsilon_m$. Any set of positive roots for $D(m,n)$ is of the form $\Pi(\mathscr{B}, >)$ or $\iota_D(\Pi(\mathscr{B}, >))$ for some total order $>$ on $\mathscr{B}$. If $\delta_n + \varepsilon_m \in S$, then $\delta_n \pm \varepsilon_m \in \Pi$ and $\iota_D(\Pi) = \Pi$. Thus either (Π, S) or $(\iota(\Pi), \iota(S))$ are such that $\Pi = \Pi(\mathscr{B}, >)$ and S is of the form $S = \{u_i - v_i\}_{i=1}^r$, $u_i, v_i \in \mathscr{B}$. Therefore, using ι_D, we can (and will) assume that $\Pi = \Pi(\mathscr{B}, >)$ and S is of this form.

5.6.3. Summarizing, we consider $\Pi = \Pi(\mathscr{B}, >)$, which is encoded by a sequence of m dots and n crosses, and $S \subset \Pi$ is encoded by $\operatorname{supp} S$, which is a subset of $\mathscr{B}$, consisting of r pairs of neihboring dots and crosses.

5.6.4. **Lemma.** *If $\alpha \in \Pi_0$ is such that $(\rho, \alpha) = 0$, then $\alpha = \beta + \beta'$, where $\beta, \beta' \in \Pi$ are isotropic.*

Proof. Mark each element of $u \in \mathscr{B}$ by the number (ρ, u). If $u > v$ are neighboring in $\mathscr{B}$, then $(\rho, u) = (\rho, v)$ if u, v are of different types and $(\rho, u) - (\rho, v) = \|u\|^2$ otherwise.

If $u - v \in \Pi_0$ for $u, v \in \mathscr{B}$, then u, v are of the same type and there are no other elements of these type between them. This means that $(\rho, u) - (\rho, v) = (1-t)\|u\|^2$, where t is the number of elements in $]u, v[$.

Take $\alpha \in \Pi_0$ such that $(\rho, \alpha) = 0$.

If $\alpha = u - v$, then, by above, $]u, v[$ consists of one element, say w, and u, w are of different types. Hence $u - w, v - w$ are isotropic simple roots and $u - v = (u - w) + (w - v)$ as required. This establishes the claim for $A(m,n)$.

For $B(m,n)$ the last mark is equal to $\pm 1/2$, so all marks are not integral. If $\alpha \ne u - v$ for $u, v \in \mathscr{B}$, then $\alpha = u$ or $\alpha = 2u$ for some $u \in \mathscr{B}$, and thus $(\rho, \alpha) \ne 0$.

For $D(m,n)$ the last mark is zero if the last element is a dot and is -1 otherwise. If $\alpha \ne u - v$ for $u, v \in \mathscr{B}$, then $\alpha = 2\delta_n$, or $\alpha = \varepsilon_{m-1} + \varepsilon_m$. If $\alpha = 2\delta_n$, then (since δ_n is represented by the last cross), $(\rho, \delta_n) = t'1$, where t' is the number of dots after the last cross. Hence $(\rho, \alpha) = 0$ forces $t' = 1$, that is $\delta_n \pm \varepsilon_m \in \Pi$, which gives $2\delta_n = (\delta_n - \varepsilon_m) + (\delta_n + \varepsilon_m)$, as required. Consider the remaining case $\alpha = \varepsilon_{m-1} + \varepsilon_m$. By above, $(\rho, \varepsilon_{m-1} - \varepsilon_m) = 1 - t$, where t is the number of elements in $]\varepsilon_{m-1}, \varepsilon_m[$. Note that ε_m is the last dot in $\mathscr{B}$, so $(\rho, \varepsilon_m) = -t'$, where t' is the number of crosses after ε_m. Then $0 = (\rho, \varepsilon_{m-1} + \varepsilon_m) = 1 - t - 2t'$, so $t' = 0$ and $t = 1$. This means that $\varepsilon_{m-1} - \delta_n, \delta_n \pm \varepsilon_m$ are isotropic simple roots, and $\varepsilon_{m-1} + \varepsilon_m = (\varepsilon_{m-1} - \delta_n) + (\delta_n + \varepsilon_m)$ is the required presentation. □

5.7. Equivalence of the KW-formulas

Let $L = L(\lambda, \Pi)$ be a finite-dimensional $\mathfrak{g}$-module, satisfying the KW-condition for (Π, S).

In Lemma 5.7.1 we show that if $\beta \in \Pi$ is isotropic and $(\beta, S) = 0$, then L satisfies the KW-condition for $(r_\beta \Pi, r_\beta S)$ and the corresponding KW-formulas are equivalent (one has $r_\beta S = S$ if $\beta \notin S$ and $r_\beta S = (S \setminus \{\beta\}) \cup \{-\beta\}$ if $\beta \in S$). In particular, we can permute dots and crosses in any interval $[u, v]$ such that $[u, v] \cap \operatorname{supp} S = \emptyset$; then L satisfies the KW-condition for the resulting Π' and the original S, and the corresponding KW-formulas are equivalent.

In Lemma 5.7.2 we assume that $(\mathscr{B}, >)$ contains an interval $[u, v]$ such that $\operatorname{supp} S \subset [u, v]$ and $(\lambda, u - u') = 0$ for each $u' \in [u, v]$. Let the ordered set $(\mathscr{B},' >)$ be obtained from $(\mathscr{B}, >)$ by some permutation of dots and crosses in $[u, v]$ in such a way that with the new ordering this interval is balanced (with a maximal set of mutually orthogonal isotropic roots S'). In this case $L = L(\lambda, \Pi')$ and L satisfies the KW-condition for (Π', S'), where $\Pi' = \Pi(\mathscr{B},' >)$; moreover, the corresponding KW-formulas are equivalent.

5.7.1. **Lemma.** *Let $L = L(\lambda, \Pi)$ be a finite-dimensional $\mathfrak{g}$-module which satisfies the KW-condition for (Π, S). Let $\beta \in \Pi$ be an isotropic root orthogonal to all elements of S.*

(i) The KW-condition holds for $(\Pi', S') := (r_\beta \Pi, r_\beta S)$.
(ii) Let $L = L(\lambda', \Pi')$ and ρ' be the Weyl vector for Π'.

Then

$$\mathscr{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) \in \mathscr{R}(\Pi), \quad \mathscr{F}_W\Big(\frac{e^{\lambda'+\rho'}}{\prod_{\beta\in S'}(1+e^{-\beta})}\Big) \in \mathscr{R}(\Pi')$$

are equivalent.

In particular, formulas (17) for (S, Π) and (S', Π') are equivalent.

Proof. The proof of (i) is straightforward. For (ii) set

$$X := \mathscr{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) \in \mathscr{R}(\Pi),$$
$$X' := \mathscr{F}_W\Big(\frac{e^{\lambda'+\rho'}}{\prod_{\beta\in S'}(1+e^{-\beta})}\Big) \in \mathscr{R}(\Pi').$$

Note that X, X' are finite sums; for each isotropic root $\beta_1 \in \Pi$ (resp., $\beta_1 \in \Pi'$), X (resp., X') has a pole of order ≤ 1 at β_1.

If $\beta \notin S$, then $S = S'$; moreover, the KW-condition implies that $(\lambda + \rho, \beta) \neq 0$, so $\lambda + \rho = \lambda' + \rho'$. Hence $X, X'$are the expansions of the same element in $\mathscr{R}(\Pi)$ and in $\mathscr{R}(\Pi')$, so they are equivalent.

If $\beta \in S$, then $\lambda = \lambda'$ and so

$$\frac{e^{\lambda'+\rho'}}{\prod_{\beta\in S'}(1+e^{-\beta})} = \frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}.$$

Then again X, X' are the expansions of the same element and they are equivalent. □

5.7.2. **Lemma.** *Let $L = L(\lambda, \Pi)$ satisfy the KW-condition for (Π, S) and let $\Pi = \Pi(\mathscr{B}, >)$. Assume that $[u, v] \in (\mathscr{B}, >)$ is such that* $\operatorname{supp} S \subset [u, v]$ *and $\lambda_{[u,v]}$ is trivial. Let the ordered set $(\mathscr{B},' >)$ be obtained from $(\mathscr{B}, >)$ by permuting some dots and crosses in $[u, v]$: we denote the resulting interval in $(\mathscr{B},' >)$ by $[u', v']$ ($[u, v] = [u', v']$ as non-ordered sets). Then*

(i) $L = L(\lambda, \Pi')$, where $\Pi' := \Pi(\mathscr{B},' >)$.
(ii) If $[u', v']$ is balanced, then L satisfies the KW-condition for (Π', S'), where S' is a maximal set of mutually orthogonal isotropic roots in $\Pi([u', v'])$.

Moreover, the KW-formula for (Π, S) is equivalent to the KW-formula for (Π', S').

Proof. Let $\Pi([u, v]) = A(k, l)$, where $k \leq l$. Then (i) follows from the fact the one-dimensional $A(k, l)$-module has the same highest weight for any choice of simple roots.

For (ii) assume that $[u', v']$ is balanced and S' is a maximal set of mutually orthogonal isotropic roots in $\Pi([u', v'])$. Then S' contains k elements. Since L satisfies the KW-condition, $[u, v]$ is balanced and S contains k elements. Since $(\lambda, \alpha) = 0$ for each $\alpha \in \Pi([u, v])$, one has $(\lambda, \alpha) = 0$ for each $\alpha \in S'$. Hence L satisfies the KW-condition for (Π', S').

It remains to show that the KW-formulas are equivalent.

Denote by $W/W_{[u,v]}$ a set of coset representatives. Note that $\lambda - \lambda_{[u,v]}$ and $\rho - \rho_{[u,v]}$ are $W([u, v])$-stable. Since the denominator identity for $A(k, l)$ holds for $(\Pi([u, v]), S)$ and for $(\Pi([u', v']), S')$, we have

$$\begin{aligned}
&\mathscr{F}_W\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\\
=&\mathscr{F}_{W/W_{[u,v]}}\mathscr{F}_{W_{[u,v]}}\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\\
=&\mathscr{F}_{W/W_{[u,v]}}\Big(e^{\lambda-\lambda_{[u,v]}+\rho-\rho_{[u,v]}}\mathscr{F}_{W_{[u,v]}}\Big(\frac{e^{\lambda_{[u,v]}+\rho_{[u,v]}}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big)\\
=&\mathscr{F}_{W/W_{[u,v]}}\Big(e^{\lambda-\lambda_{[u,v]}+\rho-\rho_{[u,v]}}\mathscr{F}_{W_{[u,v]}}\Big(\frac{e^{\lambda_{[u,v]}+\rho''}}{\prod_{\beta\in S'}(1+e^{-\beta})}\Big)\Big)
\end{aligned}$$

$$=\mathscr{F}_W\Big(\frac{e^{\lambda+\rho-\rho_{[u,v]}+\rho''}}{\prod_{\beta\in S'}(1+e^{-\beta})}\Big),$$

where ρ'' is the Weyl vector for $\Pi([u',v'])$. Recall that $\Pi([u',v'])$ can be obtained from $\Pi([u,v])$ by a sequence of odd reflections $r_{\beta_1}\dots r_{\beta_t}$. Then $\rho''=\rho_{[u,v]}+\sum_{i=1}^t\beta_i$. Since Π' is obtained from Π by the same sequence of odd reflections, we have $\rho'=\rho-\rho_{[u,v]}+\rho''$ is the Weyl vector for Π'.

We conclude that $\mathscr{F}_W(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})})\in\mathscr{R}(\Pi)$ and $\mathscr{F}_W(\frac{e^{\lambda+\rho'}}{\prod_{\beta\in S'}(1+e^{-\beta})})\in\mathscr{R}(\Pi')$ are equivalent elements. The assertion follows. □

5.8. *Properties of* Π, S

For each $u\in\mathscr{B}$ set $y_u:=(\lambda,u)$.

Let $u>v\in\mathscr{B}$ and $\|u\|^2=\|v\|^2$. Then $u-v\in\Pi_0$ and, since $L(\lambda,\Pi)$ is finite-dimensional, $(\lambda,(u-v)^\vee)=(\lambda,u-v)\|u\|^2\geq 0$. This gives $y_u\|u\|^2\geq y_v\|v\|^2$. Moreover, since the irreducible $A(1,0)$-module of the highest weight $a\varepsilon_1-b\delta_1+a\varepsilon_2$ is finite-dimensional only for $b=a$, we obtain that $y_u=y_v$ forces $y_w=y_u$ for each $w\in[u,v]$ (y_w is constant for $w\in[u,v]$).

Let u_S (resp., v_S) be the largest (resp., smallest) element in $\operatorname{supp} S$. Take u',v' such that $u_S-u',v_S-v'\in S$. Recall that u_S,u' and v_S,v' are neighbors of different types and $y_{u_S}=y_{u'}, y_{v_S}=y_{v'}$. By above, y_w is constant for $w\in[u_S,v_S]$. Since $[u_S,v_S]$ contains elements of different types, the set $\{w\in\mathscr{B}\mid y_w=y_{u_S}\}$ is an interval (containing $[u_S,v_S]$). Denote this interval by $[u,v]$ ($u\geq u_S>v_S\geq v$). Then $(\lambda,\alpha)=0$ for each $\alpha\in\Pi([u,v])$. In particular, $[u,v]$ is balanced and $S\subset\Pi([u,v])$ is a maximal set of mutually orthogonal isotropic roots.

For $B(m,n)$, finite-dimensionality of L implies $y_w\|w\|^2\geq 0$ for each $w\in\mathscr{B}$. This gives $y_u=0$ and thus $y_w=0$ for each $w<u$. Hence v is the minimal element in $\mathscr{B}$ and $\lambda_{[u,v]}=0$.

5.8.1. For $D(m,n)$, finite-dimensionality of L implies $y_w\|w\|^2\geq 0$ for each $w\in\mathscr{B}\setminus\{\varepsilon_m\}$ and $|y_{\varepsilon_i}|\geq|y_{\varepsilon_m}|$ for $i<m$. In particular, if $(\lambda,u_S)\neq 0$, then $r=1$ $S=\{\delta_k-\varepsilon_m\}$ or $S=\{\varepsilon_m-\delta_k\}$. If $(\lambda,u_S)=0$, then $(\lambda,u)=0$ for each $u\in\operatorname{supp} S$, and, as for $B(m,n)$, we obtain that v is the minimal element in $\mathscr{B}$ and $\lambda_{[u,v]}=0$. Note that $[u,v]$ contains at least r elements of each type, so $\varepsilon_m,\delta_n\in[u,v]$.

We assume for the rest of §5 that *either* $(\lambda,u_S)=0$ *or* $S=\{a(\delta_n-\varepsilon_m)\}$, *where* $a=\pm1$. Thus we exclude the case $(\lambda,u_S)\neq 0$ and $S=\{a(\delta_k-\varepsilon_m)\}$ with $k<n$.

5.8.2. **Lemma.** *Let the* $\mathfrak{g}$*-module* $L = L(\lambda, \Pi)$ *satisfy the KW-condition for* (Π, S)*, where Iso is connected and* S *is dense. Set* $\lambda^0 := (\lambda + \rho, u_S)$.

- *(i) One has* $(\lambda + \rho, \alpha^\vee) \geq 0$ *if* $\alpha \in \Pi_0$ *and* $\alpha = u - v$ *for* $u, v \in \mathscr{B}$*;*
- *(ii) The multiset* $\{(\lambda + \rho, u)\}_{u \in \mathscr{B}}$ *contains* $2r$ *or* $2r + 1$ *copies of* λ^0 *and all other elements are distinct. The set* $\{x \in \mathscr{B} \mid (\lambda + \rho, x) = \lambda^0\}$ *form an interval* $[u_0, v_0]$*, which contains* $[u_S, v_S]$.

Remark. Note that $[u_S, v_S]$ contains $2r$ elements and $[u_0, v_0]$ contains $2r$ or $2r + 1$ elements.

Proof. Take $u \in [u_S, v_S]$. From §5.8 one has $(\lambda, u) = (\lambda, u_S)$; since S is dense one has $(\rho, u) = (\rho, u_S)$. Therefore $(\lambda + \rho, u) = (\lambda + \rho, u_S)$.

Let $u, v \in \mathscr{B}$ be of the same type and $u > v$. Since *Iso* is connected, $(\rho, (u - v)^\vee) \geq 0$ and if $(\rho, (u - v)^\vee) = 0$, then (ρ, w) is constant for $w \in [u, v]$ (so, $[u, v]$ consists of the elements of alternating types). By §5.8 the same holds for λ. This proves (i) and, moreover, shows that $(\lambda + \rho, u) = (\lambda + \rho, v)$ implies that λ and ρ are constant on $[u, v]$. If $[u, v] \setminus [u_S, v_S]$ contains more than one element, then it contains two neghboring elements u', v' of different types. However, $(\lambda + \rho, u' - v') = 0$ and $(u' - v', S) = 0$, which contradicts the KW-condition. Hence $[u, v] \setminus [u_S, v_S]$ contains at most one element (in particular, u or v lies in $[u_S, v_S]$). This proves (ii). □

Corollary. *Let* $L = L(\lambda, \Pi)$ *satisfy the KW-condition for* (Π, S)*, where Iso is connected and* S *is dense. If* Π *is* $A(m - 1, n - 1)$*, or if* Π *is* $D(m, n)$ *with* $\delta_n - \varepsilon_m \in \Pi$*, then*

$$(\rho, \alpha^\vee) \geq 0 \quad \textit{for each } \alpha \in \Pi_0.$$

5.9. Choice of (Π, S)

Finally, we show that for $A(m, n)$, $B(m, n)$ and $D(m, n)$, if a finite-dimensional $\mathfrak{g}$-module L satisfies the KW-condition for $(\tilde{\Pi}, \tilde{S})$, and, for $D(m, n)$ the assumption of §5.8.1 is fulfilled, then the KW-formula is equivalent to the KW-formula for (Π, S), where *Iso* is connected, S is dense, and, for $D(m, n)$, $\delta_n - \varepsilon_m \in \Pi$.

5.9.1. Let $[u, v]$ be the interval constructed in the second paragraph in §5.8. Using Lemma 5.7.2 we can rearrange dots and crosses in $[u, v]$ such that the resulting interval $[u', v']$, $u' > v'$, is balanced. We do this in such a way that the interval has first a segment of the elements of same type and then a segment of elements of alternating types; for $D(m, n)$ we choose the minimal element to be

ε_m. We choose S' such that $\operatorname{supp} S'$ consists of the last $2r$ elements in $[u', v']$; then S' is dense and $v' \in \operatorname{supp} S'$.

Using Lemma 5.7.1 we now permute dots and crosses in the intervals $\{w \in \mathscr{B} \mid w > u'\}$ and $\{w \in \mathscr{B} \mid w < v'\}$; the second interval is empty for $B(m, n)$ and for $D(m, n)$ (if the assumption in §5.8.1 holds). We do this in such a way that in the resulting order we have $\mathscr{B} = [u_0, u_+] \cup [u_+, v_+] \cup [v_+, v_0]$, where $u_0 \geq u_+ \geq u' > v' \geq v_+ \geq v_0$, the interval $[u_0, u_+]$ (resp., $[v_+, v_0]$) consists of the elements of the same type, and the interval $[u_+, v_+]$ consists of the elements of alternating types.

5.9.2. Consider the resulting ordering $(\mathscr{B}, >)$ and set $\Pi := \Pi(\mathscr{B}, >)$, $S := S'$. Let ρ be the corresponding Weyl vector and λ be the highest weight of L.

From Lemmas 5.7.2, 5.7.1 we conclude that the KW-formula is equivalent to the KW-formula for (Π, S).

Since $\mathscr{B} = [u_0, u_+] \cup [u_+, v_+] \cup [v_+, v_0]$ as above, we have obtained Π, S for which *Iso* is connected and S is dense, which completes the proof for $A(m, n)$.

In addition, for $D(m, n)$ we have obtained that ε_m is minimal in $\mathscr{B}$ and $\delta_n - \varepsilon_m \in S$. Therefore Π contains $\delta_n \pm \varepsilon_m$ and $S = \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=0}^{r-1}$. This completes the proof for $D(m, n)$.

5.9.3. For $B(m, n)$ we have obtained $S = \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=0}^{r-1}$ or $S = \{\varepsilon_{m-i} - \delta_{n-i}\}_{i=0}^{r-1}$. Retain notations of Lemma 5.8.2. Recall that $v_S = v_0$ is minimal in $\mathscr{B}$ and $(\lambda, v) = 0$ for $v \in [u_S, v_S]$. We will show that (Π, S) can be chosen in such a way that *Iso* is connected, S is dense and for $u, v \in \mathscr{B}$ we have

$$(25) \quad |(\lambda+\rho, u)| = |(\lambda+\rho, v)| \implies (\lambda+\rho, u) = (\lambda+\rho, v) = (\lambda+\rho, \alpha_{m+n}).$$

Since v_S is minimal, $\mathscr{B}$ is of the following form: $\xi_1 > \cdots > \xi_k$ are of the same type and $\xi_k > \xi_{k+1} > \cdots > \xi_{m+n} = v_S$ are of alternating types (i.e., *Iso* is connected and contains α_{m+n-1}). This implies $(\rho, u - v) = 0$ if $\|u\|^2 = \|v\|^2 = -\|\xi_1\|^2$.

Set $x_u := (\lambda + \rho, u)$. Normalize $(-,-)$ by the condition $\|v_S\|^2 = 1$. Then $\lambda^0 = \frac{1}{2}$.

Let us show that $|x_u| = |x_v|$ forces u or v in $\operatorname{supp} S$.

Indeed, if $x_u = x_v$, then, by Lemma 5.8.2, u or v is in $\operatorname{supp} S$. If $|x_u| = |x_v|$ and u, v are of different types, then KW-condition forces u or v in $\operatorname{supp} S$. Consider the remaining case, when $x_u = -x_v$, u, v are of the same type and $u \notin \operatorname{supp} S$. If $\|u\|^2 = \|v\|^2 = 1$, then $x_u, x_v \geq \frac{1}{2}$, so $x_u \neq -x_v$. If $\|u\|^2 = \|v\|^2 = -1$, then $x_u, x_v \leq \frac{1}{2}$, so $x_u = -x_v$ forces $|x_u| = |x_v| = \frac{1}{2}$. Then x_u or x_v is $\frac{1}{2}$. By Lemma 5.8.2, u or v is in $[u_0, v_0]$ and $[u_0, v_0] \setminus [u_S, v_S]$ is either empty, or is $\{u_0\}$ and $\|u_0\|^2 = -\|u_S\|^2 = \|v_S\| = 1$. Thus u or v is in $[u_S, v_S] = \operatorname{supp} S$, as required.

We conclude that the multiset $\{|x_u|\}$ contains t copies of $\frac{1}{2}$, where $t = 2r$ or $t = 2r + 1$, and all other elements are distinct. Set $B := \{u \in \mathscr{B} \mid |x_u| = \frac{1}{2}\}$. Clearly, $[u_0, v_0] \subset B$ and (25) holds if $[u_0, v_0] = B$. In particular, (25) holds if $[u_0, v_0]$ contains $2r + 1$ elements.

Consider the case when (25) does not hold. Since $[u_0, v_0]$ contains $[u_S, v_S]$ which has $2r$ elements, this means that $[u_0, v_0] = [u_S, v_S]$ and $B = [u_S, v_S] \coprod \{u\}$, where $x_u = -\frac{1}{2}$. Then $\|u\|^2 = -1$, so u, u_S are of the same type. If $w \in [u, u_S]$ is of the same type as u, then $x_w \in [-\frac{1}{2}, \frac{1}{2}]$ (since $x_{u_S} = \frac{1}{2} = -x_u$), so x_w is $\pm\frac{1}{2}$, that is $w \in B$. This means that either $w = u$ or $w = u_S$. Hence $]u, u_S[$ does not contain elements of the same type as u, u_S; that is either $]u, u_S[= \emptyset$ or $]u, u_S[= \{v\}$ with $\|u\|^2 = -\|v\|^2$. If $]u, u_S[= \{v\}$, we make the reflection with respect to $u - v$ and obtain a new ordered set $(\mathscr{B},' >)$; one has $L = L(\lambda', \Pi')$ $(\Pi' = \Pi(\mathscr{B},' >))$, where λ' is such that $\lambda' + \rho' = \lambda + \rho$, so $(\lambda' + \rho', u) = -\frac{1}{2}$ and in $\mathscr{B}'$ one has $]u, u_S[= \emptyset$. Thus, in both cases (for $\mathscr{B}$ if $]u, u_S[= \emptyset$, and for $\mathscr{B}'$ otherwise) we have $]u, u_S[= \emptyset$. Then $[u, v_0]$ contains $2r+1$ elements ($r+1$ elements of type u_S and r elements of type v_0). Note that the restriction of λ (resp., λ') to $[u, v_0]$ is zero weight, since $(\rho, u - u_S) = -1$.

Using Lemma 5.7.2 we can rearrange dots and crosses in $[u, v_0]$ in an alternating way; the resulting interval is $[u, v']$, where $\|v'\|^2 = \|u\|^2 = -1$ and v' is minimal in the new order on $\mathscr{B}$. Since $\lambda_{[u,v_0]} = 0$, this rearrangement preserves λ (i.e., λ is the highest weight of L with respect to the new set of simple roots). Using Lemma 5.7.1, we rearrange dots and crosses in the rest of $\mathscr{B}$ (in $[u_0, u[$) in such a way that in the resulting order we, again, have first several elements of the same type and then a segment of elements of alternating types. Let Π'' be the new set of simple roots, ρ'' be the new Weyl vector and λ'' is such that $L = L(\lambda'', \Pi'')$. Then $\lambda''_{[u,v']} = 0$, so $(\lambda'' + \rho'', w) = -\frac{1}{2}$ for each $w \in [u, v']$. Note that *Iso* in Π'' is connected; take S'' such that supp S'' consists of last $2r$ elements (i.e., supp $S' =]u, v']$). Then S'' is dense. Since $[u, v']$ contains $2r+1$ elements and $(\lambda'' + \rho'', w) = -\frac{1}{2}$ for each $w \in [u, v_0]$, (25) holds for (Π'', S'').

6. KW-character formula for strongly integrable maximally atypical modules when $h^\vee \neq 0$ and for integrable vacuum modules when $h^\vee = 0$

In this section $\mathfrak{g}$ is a symmetrizable affine Lie superalgebra (with arbitrary $h^\vee$). Let Δ be the root system of $\mathfrak{g}$ and let $\dot{\Delta}$ be a finite part of Δ.

We say that a subset of simple roots Π for Δ is compatible with $\dot{\Delta}$ if $\dot{\Pi} = \dot{\Delta} \cap \Pi$ is a subset of simple roots for $\dot{\Delta}$ (in other words, $\Pi \setminus \dot{\Delta}$ contains only one root).

Let $\dot{\Delta}$ be non-exceptional and let $L = L(\lambda, \Pi)$ be a non-critical $\pi \cup \dot{\Pi}_0$-integrable $\mathfrak{g}$-module of maximal atypicality, where Π is compatible with $\dot{\Delta}$. We call such $\mathfrak{g}$-module *strongly integrable* (*cf.* [KW4]). Assume that L satisfies the KW-condition for Π, S with $S \subset \dot{\Pi}$. We shall prove the following KW-character formula:

$$(26)\qquad Re^{\rho}\,\mathrm{ch}\,L(\lambda) = j_{\lambda}^{-1} \sum_{w\in W(\dot{\Pi}_0\cup\pi)} \mathrm{sgn}(w)\, w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big),$$

where j_λ is the number of elements in the "smallest" factor of $W(\dot{\Pi})$, see (19) and $\pi := \{\alpha \in \Pi_0 \mid \|\alpha\|^2 > 0\}$.

We shall also prove this formula for the non-critical $C_n^{(1)}$-integrable vacuum $D(n+1,n)^{(1)}$-modules and for the non-critical integrable vacuum $D(2,1,a)^{(1)}$-modules.

6.1. Vacuum modules over $\mathfrak{g} = D(2,1,a)^{(1)}$

Recall that $a \neq 0, -1$. One has $D(2,1,a)_{\overline{0}} = A_1 \times A_1 \times A_1$; if we denote the root in ith copy of A_1 by $2\varepsilon_i$, then $\|2\varepsilon_1\|^2 : \|2\varepsilon_2\|^2 : \|2\varepsilon_3\|^2 = 1 : a : (-a-1)$. (Recall the definition of a vacuum module in §3.1.3.)

Let $L(\lambda)$ be a π-integrable vacuum module for some $\pi \subset \Pi_0$ and $k := (\lambda, \delta)$. If $\pi \setminus \dot{\Pi}_0$ contains one root, then $\pi = \{\delta - 2\varepsilon_r, \varepsilon_r\}$ and $L(\lambda)$ is π-integrable if and only if $2k/\|2\varepsilon_r\|^2 \in \mathbb{Z}_{\geq 0}$. If $\pi \setminus \dot{\Pi}_0$ contains two roots, then $\pi = \{\delta - 2\varepsilon_r, \delta - 2\varepsilon_q, 2\varepsilon_r, 2\varepsilon_q\}$, and $L(\lambda)$ is π-integrable if and only if $2k/\|2\varepsilon_r\|^2,\ 2k/\|2\varepsilon_q\|^2 \in \mathbb{Z}_{\geq 0}$; in particular, if $k \neq 0$, then $\|2\varepsilon_r\|^2/\|2\varepsilon_q\|^2 \in \mathbb{Q}_{>0}$, so $a \in \mathbb{Q}$. If $\pi \setminus \dot{\Pi}_0$ contains three roots, then $\pi = \Pi_0$ and $k = 0$.

We consider a non-critical module $L(\lambda)$, that is $k \neq 0$. We see that $L(\lambda)$ can be $A_1^{(1)}$-integrable for any copy $A_1^{(1)}$ in Π_0, but it is $A_1^{(1)} \times A_1^{(1)}$-integrable only if $a \in \mathbb{Q}$ and the roots of π have positive integral square length for some normalization of $(-,-)$.

Let $\pi = \{\alpha \in \Pi_0 \mid \|\alpha\|^2 \in \mathbb{Q}_{>0}\}$ for some normalization of $(-,-)$. If $a \notin \mathbb{Q}$, then π can be any copy of $A_1^{(1)}$. If $a \in \mathbb{Q}$, then either $\pi = A_1^{(1)}$, which corresponds to the longest root (the absolute value of $\|2\varepsilon_i\|^2$ is maximal; this is $2\varepsilon_1$ if $-1 < a < 0$), or $\pi = A_1^{(1)} \times A_1^{(1)}$ ($\dot{\pi} = \{2\varepsilon_2, 2\varepsilon_3\}$ if $-1 < a < 0$).

We fix Π which consists of isotropic roots:

$$\Pi = \{\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3, -\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2 + \varepsilon_3\}.$$

6.1.1. Recall that $j_\lambda = 2$ and set

$$Z := j_\lambda Re^{\rho}\,\mathrm{ch}\,L - \sum_{w\in W(\pi\cup\dot{\Pi}_0)} \mathrm{sgn}(w)\, w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big).$$

Suppose that $Z \neq 0$.

The $\pi \cup \dot{\Pi}_0$-integrability of $L(\lambda)$ gives $(\lambda, \alpha^\vee) \geq 0$ for each $\alpha \in \pi \cup \dot{\Pi}_0$. Since $\rho = 0$, $\lambda + \rho = \lambda$ is maximal in its $W(\pi \cup \dot{\Pi}_0)$-orbit, so $\operatorname{supp} Z \subset \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi$. Let $\lambda - \mu$ be a maximal element in $\operatorname{supp} Z$ ($\mu \in \mathbb{Z}_{\geq 0}\Pi$). The arguments of §4.3.1 show that

$$(27)\quad 2(\lambda + \rho, \mu) = (\mu, \mu) \text{ and } (\lambda - \mu, \alpha^\vee) \geq 0 \quad \text{for each } \alpha \in \pi \cup \dot{\Pi}_0.$$

The coefficient of e^λ in Z is equal to the coefficient of e^λ in

$$j_\lambda e^\lambda - \sum_{\{w \in W(\pi \cup \dot{\Pi}_0) | w\lambda = \lambda\}} \operatorname{sgn}(w)\, w\Big(\frac{e^\lambda}{\prod_{\beta \in S}(1 + e^{-\beta})}\Big).$$

Since λ is maximal in its $W(\pi \cup \dot{\Pi}_0)$-orbit, the stabilizer of λ in $W(\pi \cup \dot{\Pi}_0)$ is equal to $W(\dot{\Pi}_0)$. The KW-formula for $D(2,1,a)$ implies that the coefficient of e^λ is zero. Hence $\mu \neq 0$.

Since $(\lambda, \dot{\Pi}_0) = 0$, (27) gives $(\mu, \alpha^\vee) \leq 0$ for each $\alpha \in \dot{\Pi}_0$. Therefore $\mu = j\delta - \sum_{i=1}^3 e_i \varepsilon_i$, where $e_i \geq 0$ for each i. One readily sees that $\mu \in \Pi$ forces $2j - e_i - e_s \geq 0$ for each $\{i, s\} \subset \{1,2,3\}$.

6.1.2. Consider the case $a \notin \mathbb{Q}$; without loss of generality we assume $\pi = \{\delta - 2\varepsilon_1, 2\varepsilon_1\}$ and normalize the form $(-,-)$ by $\|\varepsilon_1\|^2 = 1$.

Since $2(\lambda, \mu) = (\mu, \mu)$, we have $e_2 = e_3 = 0$, $2jk = e_1^2$. Moreover, $(\lambda - \mu, (\delta - 2\varepsilon_1)^\vee) \geq 0$ gives $k \geq 2e_1$. Since $2j \geq e_1 \geq 0$, we obtain $j = e_1 = 0$, that is $\mu = 0$, a contradiction.

6.1.3. Assume that $a \in \mathbb{Q}$.

For the case $\pi = A_1^{(1)}$, without loss of generality we assume $\pi = \{\delta - 2\varepsilon_1, 2\varepsilon_1\}$ and write $\mu = j(\delta - 2\varepsilon_1) + (2j - e_1)\varepsilon_1 - e_2\varepsilon_2 - e_3\varepsilon_3$.

For the case $\pi = A_1^{(1)} \times A_1^{(1)}$, without loss of generality we assume $\pi = \{\delta - 2\varepsilon_i, 2\varepsilon_i\}_{i=1,2}$ and write $\mu = e_1/2(\delta - 2\varepsilon_1) + (j - e_1/2)(\delta - 2\varepsilon_2) + (2j - e_1 - e_2)\varepsilon_2 - e_3\varepsilon_3$.

In both cases we obtain $\mu = \sum_{\alpha \in X} x_\alpha \alpha$, where $X \subset \pi \cup \dot{\Pi}_0$ and $x_\alpha \|\alpha\|^2 \geq 0$ for each $\alpha \in X$ (recall that $\|\alpha\|^2 > 0$ for $\alpha \in \pi$). Since $2(\lambda, \mu) = (\mu, \mu)$ we have $(\lambda, \mu) + (\lambda - \mu, \mu) = 0$, that is

$$\sum_{\alpha \in X} x_\alpha((\lambda, \alpha) + (\lambda - \mu, \alpha)) = 0.$$

For each $\alpha \in \dot{\Pi}_0 \cup \pi$ one has $(\lambda, \alpha)\|\alpha\|^2 \geq 0$ and $(\lambda - \mu, \alpha)\|\alpha\|^2 \geq 0$ (by (27)). Therefore for each $\alpha \in X$ we have $x_\alpha(\lambda, \alpha), x_\alpha(\lambda - \mu, \alpha) \geq 0$. Hence for each $\alpha \in X$ we obtain $x_\alpha(\lambda, \alpha) = x_\alpha(\lambda - \mu, \alpha) = 0$, that is $x_\alpha(\mu, \alpha) = 0$.

Write $\mu = \mu' + \mu''$, where $\mu' \in \mathbb{Q}\pi$ and $\mu'' \in \mathbb{Q}(\dot{\Pi}_0 \setminus \pi)$. One has $\|\mu'\|^2 = (\mu, \mu') = \sum_{\alpha \in X \cap \pi} x_\alpha(\mu, \alpha) = 0$. Similarly, $\|\mu''\|^2 = 0$. Since $(-,-)$ is non-negatively (resp., negatively) definite on $\mathbb{Q}\pi$ (resp., on $\mathbb{Q}(\dot{\Pi}_0 \setminus \pi)$), we get $\mu'' = 0$ and $\mu = \mu' \in \mathbb{Q}\delta$. Then $2(\lambda, \mu) = (\mu, \mu)$ gives $\mu = 0$, a contradiction.

6.1.4. Remark. The following denominator identity for $D(2, 1, a)^{(1)}$ was proven in [GR]:

$$Re^{\rho} = \prod_{n=1}^{\infty}(1 - e^{-n\delta}) \sum_{t \in T} t(\dot{R}e^{\rho}),$$

where T is the translation group of $A_1^{(1)} \subset D(2, 1, a)^{(1)}_{\bar{0}} = A_1^{(1)} \times A_1^{(1)} \times A_1^{(1)}$. The corresponding embedding $A_1 \subset D(2, 1, a)_{\bar{0}} = A_1 \times A_1 \times A_1$ is not specified in [GR] and we take an opportunity to correct this. This embedding is the same as described above, namely, any embedding if $a \notin \mathbb{Q}$, and the copy with the maximal absolute value of the square length of the root if $a \in \mathbb{Q}$. This choice is necessary for the proof of Proposition 2.3.2 [GR], where it is used that a non-zero linear combination of the two remaining even roots has non-zero square length.

6.2. Other forms of (3) for $\dot{\Delta} \neq D(2, 1, a)$

Let $L = L(\lambda, \Pi)$ be a non-critical integrable $\mathfrak{g}$-module of maximal atypicality, where Π is compatible with $\dot{\Delta}$ and L satisfies the KW-condition for Π, S with $S \subset \dot{\Pi}$, or let L be a non-critical $C_n^{(1)}$-integrable vacuum $D(n+1, n)^{(1)}$-module. In the first case we normalize $(-,-)$ as in §3; in the second case we normalize the form on $D(n+1, n)^{(1)}$ in such a way that $\|\alpha\|^2 = 2$ for some $\alpha \in C_n^{(1)}$. Then

$$\pi := \{\alpha \in \Pi_0 \mid \|\alpha\|^2 > 0\}$$

is a connected component of Π_0, and L is π-integrable.

Let $\dot{\mathfrak{g}}$ be the subalgebra of $\mathfrak{g}$ with the root system $\dot{\Delta}$ and the Cartan algebra $\mathfrak{h}$ (i.e., $\dot{\mathfrak{g}} = (\sum_{\alpha \in \dot{\Delta}} \mathfrak{g}_\alpha) + \mathfrak{h}$) and $\dot{L} = \dot{L}(\lambda, \dot{\Pi})$ be the irreducible $\dot{\mathfrak{g}}$-module of the highest weight λ. Clearly, $\dot{L}$ is a finite-dimensional module satisfying the KW-condition for $\dot{\Pi}, S$.

By §13.2, $\dot{\pi} = \pi \cap \dot{\Pi}_0$ is a finite part of π, i.e., $\dot{\pi}$ is connected and $\pi \setminus \dot{\pi}$ consists of one root, which we denote by $\alpha^{\#}$ (we exclude the case $\Delta = G(3)^{(1)}, \dot{\Delta} = D(2, 1, -3/4)$).

From [K3], 6.5, it follows that in the case when $\alpha^{\#} = j\delta - b\theta$, where $\theta \in \Delta(\dot{\pi}), b \in \mathbb{Q}$, one has $W(\pi) = W(\dot{\pi}) \ltimes T$, where T is a free abelian subgroup of $W(\pi)$. Recall that π is one of the root systems $A_n^{(r)}, B_n^{(1)}, C_n^{(1)}, D_n^{(r)}, G_2^{(1)}$ with $r = 1, 2$. The condition $\alpha^{\#} = j\delta - \theta$ holds for all pairs $(\pi, \dot{\pi})$ (where π is as above and $\dot{\pi}$ is a finite part of π), except for $(B_n^{(1)}, D_n), (A_{2n-1}^{(2)}, D_n)$

and $(G_2^{(1)}, A_2)$. The last case is not possible, since $\pi = G_2^{(1)}$ appears only for $\Delta = G(3)^{(1)}$ and in this case $\dot{\Delta} = G(3)$, $\dot{\pi} = G_2$, by above.

Assume that the pair $(\pi, \dot{\pi})$ is not $(B_n^{(1)}, D_n)$ or $(A_{2n-1}^{(2)}, D_n)$. Then, by above, $W(\pi) = W(\dot{\pi}) \ltimes T$ and so $W(\pi') = W(\dot{\Pi}_0) \ltimes T$, since $\pi' \setminus \pi$ is a connected component of $\dot{\Pi}_0$. Let $\dot{\rho}$ be the Weyl vector for $\dot{\Delta} = \Delta(\dot{\Pi})$. Notice that $(\rho - \dot{\rho}, \dot{\Pi}) = (\lambda, \dot{\Pi}) = 0$, so $\lambda + \rho - \dot{\rho}$ is $W(\dot{\Pi}_0)$-invariant. Then (3) can be rewritten as

$$Re^{\rho}\,\mathrm{ch}\,L(\lambda) = \sum_{t\in T} t\Big(e^{\lambda+\rho-\dot{\rho}} j_{\lambda}^{-1} \sum_{w\in W(\dot{\Pi}_0)} \mathrm{sgn}(w)\, w\Big(\frac{e^{\dot{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big).$$

Using the KW-formula for $\dot{L}$, we get

$$Re^{\rho}\,\mathrm{ch}\,L(\lambda) = \sum_{t\in T} t(\dot{R}e^{\rho}\,\mathrm{ch}\,\dot{L}(\lambda)), \tag{28}$$

where $\dot{R}$ is the Weyl denominator for $\dot{\Pi}$.

For the cases $(B_n^{(1)}, D_n)$, $(A_{2n-1}^{(2)}, D_n)$ we can extend $W(\pi)$ to $W(C_n^{(1)})$ and present $W(C_n^{(1)}) = W(C_n) \ltimes T$ as in [R]; then we obtain (28) for $T \subset W(C_n^{(1)})$.

Note that in (28) there is no S; we do not assume that $\dot{\Pi}$ contains a maximal isotropic subset. More precisely, if the KW-formula holds for some Π, S with $S \subset \dot{\Pi}$, then (28) holds for each set of simple roots Π' compatible with $\dot{\Delta}$. In particular, if L satisfies KW-conditions for Π, S and Π', S', where Π, Π' are compatible with $\dot{\Delta}$ and $S, S' \subset \dot{\Delta}$, then the KW-formulas for Π, S and Π', S' are equivalent.

Note that $\dot{\pi}$ is the "largest part" of $\dot{\Pi}_0$ in the sense of §5.2, except for the following cases: $\Delta = G(3)^{(1)}$ with $\dot{\Delta} = D(2,1,-3/4)$, $\Delta = D(2,1,a)^{(1)}$ with $\pi = A_1^{(1)}$, $\Delta = D(n+1,n)^{(1)}$ with $\pi = C_n^{(1)}$, and $\Delta = A(2n-1,2n-1)^{(2)}$ with $\dot{\pi} = D_n$ (in the case $\Delta = A(2n-1,2n-1)^{(2)}$ one has $\pi = A_{2n-1}^{(2)}$, $\dot{\Delta} = D(n,n)$, and $\dot{\pi}$ can be D_n or C_n). If $\dot{\pi}$ is the "largest part" of $\dot{\Pi}_0$, then, using (18) we can rewrite (26) as

$$Re^{\rho}\,\mathrm{ch}\,L(\lambda) = \sum_{w\in W(\pi)} \mathrm{sgn}(w)\, w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big), \tag{29}$$

cf. (14).

6.3. Case $h^{\vee} \neq 0$ or $\Delta = A(n,n)^{(1)}$ and $\dot{\Delta}$ is not exceptional

In this case $\dot{\pi}$ is the "largest part" of $\dot{\Pi}_0$. Since formulas (29) and (14) are the same, it is enough to show that L satisfies the KW-condition for some Π', S', where Π' is compatible with $\dot{\Delta}$, $S' \subset \dot{\Delta}$, and Π', S' satisfy the assumptions of §4.

In the light of §5.8, one has $(\lambda, \alpha) = 0$ for each $\alpha \in \Pi^0$, where Π^0 is a Dynkin subdiagram of Π and $\Pi^0 = A(k, l)$ (resp., $B(k, l), D(k, l)$) for $\dot{\Pi} = A(m, n)$ (resp., $B(m, n), D(m, n)$) and $\min(k, l) = \#S$ (and $= \min(m, n)$, since λ has maximal atypicality). Let $\mathscr{B}^0 \subset \mathscr{B}$ be the ordered subset corresponding to Π^0 (i.e., $\Pi^0 = (\mathscr{B}^0, >)$). We can rearrange dots and crosses in $\mathscr{B}^0$ in such a way that the last $2\min(k, l)$ elements are of alternating types, and the last element in $\mathscr{B}^0$ is of positive square length (resp., is ε_m) if $\dot{\Delta} = B(m, n)$ (resp., if $\dot{\Delta} = D(m, n)$).

This rearrangement corresponds to a certain sequence of odd reflections (with respect to roots in $\Delta(\Pi^0) \subset \dot{\Delta}$); let Π' be the subset of simple roots obtained from Π by this sequence of odd reflections. Clearly, $\dot{\Pi}' := \dot{\Delta} \cap \Pi'$ is a subset of simple roots for $\dot{\Delta}$ and the corresponding dot-cross diagram contains $\mathscr{B}^0$ with the new order. Thus the dot-cross diagram for $\dot{\Pi}$ contains a segment of $2\min(m, n)$ elements of alternating types; moreover, if $\dot{\Delta} \neq A(m, n)$, then these are the last $2\min(m, n)$ elements and the last element is of positive square length for $B(m, n)$ and is ε_m for $D(m, n)$. Since $\dot{\pi}$ is the "largest part" of $\dot{\Pi}_0$, $\|\alpha\|^2 \geq 0$ for each $\alpha \in \dot{\Pi}'$.

One has $L = L(\lambda, \Pi) = L(\lambda, \Pi')$ and so L satisfies the KW-condition for (Π', S'), where S' is any subset of $\dot{\Pi}'$ which contains $\min(m, n)$ mutually orthogonal isotropic roots. Thus the assumptions of §4 are reduced to the condition $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi'$. Recall that $\|\alpha\|^2 \geq 0$ for each $\alpha \in \dot{\Pi}'$ and that $\Pi' \setminus \dot{\Pi}'$ consists of one root, which we denote by α_0. Hence it remains to verify that $\|\alpha_0\|^2 \geq 0$.

For $\Delta = A(m, n)^{(1)}$, any subset of simple roots is naturally encoded by a cyclic dot-cross diagram, which contains m dots and n crosses. Let $m \geq n$. Since the diagram for Π' contains $2n$ elements of alternating types, it does not contain two neighboring crosses, so $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi'$.

Suppose $\|\alpha_0\|^2 < 0$ and $\Delta \neq A(m, n)^{(1)}$, that is $\dot{\Delta} \neq A(m, n)$. One has $(\alpha_0, \alpha_1) \neq 0$ or $(\alpha_0, \alpha_2) \neq 0$, where α_1, α_2 are the first two roots in $\dot{\Pi}'$. Thus $\|\alpha_i\|^2 = 0$ for $i = 1$ or $i = 2$. By the construction of $\dot{\Pi}'$, the pair α_1, α_2 can be written as $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1$, or $\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2$, or $\varepsilon_1 - \delta_1, \delta_1$ (case $B(1, 1)$), or $\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2$, where $(\varepsilon_i, \varepsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$ and $(\delta_i, \varepsilon_j) = 0$ for $i, j = 1, 2$. Since $\|\alpha_0\|^2 < 0$, one has $\alpha_0 \in \Pi_0$ or $2\alpha_0 \in \Pi_0$, so α_0 or $2\alpha_0$ is a root $\Pi_0 \setminus \{\pi \cup \dot{\Pi}_0)$. Thus $\alpha_0 = \delta - x\delta_1$ for $x \in \{1, 2\}$ or $\alpha_0 = \delta - (\delta_1 + \delta_2)$. Then α_1, α_2 is the pair $\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2$ and $\alpha_0 = \delta - x\delta_1$. By the construction of $\dot{\Pi}'$, $\|\alpha_1\|^2 = 0$ forces $\|\alpha\|^2 = 0$ for each $\alpha \in \dot{\Pi}$ (resp., for each $\alpha \in \dot{\Pi} \setminus \{\varepsilon_m\}$) if $\dot{\Delta} = D(m, n)$ (resp., if $\dot{\Delta} = B(m, n)$). If $\|\alpha\|^2 = 0$ for each $\alpha \in \dot{\Pi}$, then, since $(\rho, \delta) = h^\vee \geq 0$, we get $(\rho, \alpha_0) = \|\alpha_0\|^2/2 \geq 0$, a contradiction. Finally, for $\dot{\Delta} = B(m, n)$ we obtain $\delta = \alpha_0 + x(\sum_{\alpha \in \dot{\Pi}'} \alpha)$, so $(\rho, \delta) = -x^2/2 + x^2/2 = 0$, a contradiction.

6.4. Case $h^\vee = 0$

The remaining cases are $D(n+1,n)^{(r)} (n > 1, r = 1, 2)$, $A(2n-1, 2n-1)^{(2)}$ and $A(2n, 2n)^{(4)}$.

Note that if L is of maximal atypicality, then $\dot{L}$ is a finite-dimensional $\dot{\mathfrak{g}}$-module of maximal atypicality, and §5.8 implies $(\lambda, \dot{\Delta}) = 0$, except for the case $\Delta = D(n+1,n)^{(1)}$ (since for other cases $\dot{\Delta} = D(n,n)$ or $B(n,n)$). Thus if $\Delta \neq D(n+1,n)^{(1)}$, then $L(\lambda, \Pi)$ is a non-critical integrable vacuum module.

We set $\pi'' := \{\alpha \in \dot{\Pi}_0 \mid \|\alpha\|^2 < 0\}$. Then $\pi \cup \dot{\Pi} = \pi \coprod \pi''$. One has $(\lambda, \pi'') = 0$ (it is obvious if L is a vacuum module; otherwise $\Delta = D(n+1,n)^{(1)}, \dot{\Delta} = D(n+1,n), \pi'' = C_n$ and, by §5.8, $(\lambda, \alpha) = 0$ for each $\alpha \in D(n,n) \subset D(n+1,n)$, in particular, for $\alpha \in \pi''$).

We introduce

$$Z := j_\lambda R e^\rho \operatorname{ch} L - \sum_{w \in W(\pi \cup \dot{\Pi})} \operatorname{sgn}(w)\, w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1+e^{-\beta})}\Big).$$

Suppose that $Z \neq 0$. Let λ' be a maximal element in $\operatorname{supp} Z$; we write $\lambda' - \rho = \lambda - \mu$ for $\mu \in \mathbb{Z}\Pi$. The arguments of §4.3.1 show that (27) holds.

6.4.1. Cases $D(n+1|n)^{(1)}, n > 1$, *and* $A(2n-1, 2n-1)^{(2)}$. In these cases we choose Π, which consists of isotropic roots, see (12), and take any S (S is unique up to an automorphism of Π). The proof is similar to the one in §6.1.

For $D(n+1,n)^{(1)}$ one has $\Pi_0 = D_{n+1}^{(1)} \times C_n^{(1)}$, $\dot{\Delta} = D(n+1,n)$. If $L(\lambda)$ is integrable, then $\pi = D_{n+1}^{(1)}$ and $\pi'' = C_n$; if $L(\lambda)$ is a $C_n^{(1)}$-integrable vacuum module, then $\pi = C_n^{(1)}$ and $\pi'' = D_{n+1}$.

For $A(2n-1, 2n-1)^{(2)}$ one has $\Pi_0 = A_{2n-1}^{(2)} \times A_{2n-1}^{(2)}$, $\dot{\Delta} = D(n,n)$. Note that $\dot{\pi} = \pi \cap \dot{\Delta}$ can be D_n or C_n and π'' is C_n or D_n respectively.

Recall that $\rho = 0$. Since $L = L(\lambda, \Pi)$ is $\pi \cup \pi''$-integrable, $\lambda = \lambda + \rho$ is maximal in its $W(\pi) \times W(\pi'')$-orbit, so $\operatorname{supp} Z \subset \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi$. Thus $\mu \in \mathbb{Z}_{\geq 0}\Pi$. Observe that $\mathbb{Q}\Pi = \mathbb{Q}(\pi \cup \pi'')$. Write $\mu = \mu' - \mu''$, where $\mu' \in \mathbb{Q}\pi, \mu'' \in \mathbb{Q}\pi''$. We claim that

$$(30) \qquad \mu' \in \mathbb{Q}_{\geq 0}\pi, \quad \mu'' \in \mathbb{Q}_{\geq 0}\pi''.$$

Indeed, by above, $(\lambda, \pi'') = 0$. Using (27), we get $(\mu'', \alpha^\vee) \leq 0$ for each $\alpha \in \pi''$; Theorem 4.3 in [K3] gives $\mu'' \in \mathbb{Q}_{\geq 0}\pi''$. This implies $\mu'' \in \mathbb{Q}_{\geq 0}\Pi$, so $\mu' = \mu + \mu'' \in \mathbb{Q}_{\geq 0}\Pi$. Thus $\mu' \in \mathbb{Q}\pi \cap \mathbb{Q}_{\geq 0}\Pi$. It remains to verify that

$$(31) \qquad (\mathbb{Q}_{\geq 0}\Pi \cap \mathbb{Q}\pi) \subset \mathbb{Q}_{\geq 0}\pi.$$

Observe that each $\alpha \in \pi$ is a sum of two simple roots $\alpha = \beta(\alpha) + \beta'(\alpha)$ $(\beta(\alpha), \beta'(\alpha) \in \Pi)$ and we can choose $\beta(\alpha)$ in such a way that $\alpha \mapsto \beta(\alpha)$ is an injective map from π to Π (for instance, for $D(n+1,n)^{(1)}$ with $\pi = D_{n+1}^{(1)}$, one has $\delta-\varepsilon_1-\varepsilon_2 = (\delta-\varepsilon_1-\delta_1)+(\delta_1-\varepsilon_2)$, $\varepsilon_i-\varepsilon_{i+1} = (\varepsilon_i-\delta_i)+(\delta_i-\varepsilon_{i+1})$, $\varepsilon_n+\varepsilon_{n+1} = (\varepsilon_n-\delta_n)+(\delta_n-\varepsilon_{n+1})$, so we can take $\beta(\delta-\varepsilon_1-\varepsilon_2) = \delta-\varepsilon_1-\delta_1$, $\beta(\varepsilon_i-\varepsilon_{i+1}) = \delta_i-\varepsilon_{i+1}$, $\beta(\varepsilon_n+\varepsilon_{n+1}) = \delta_n+\varepsilon_{n+1}$), *cf.* (B) in §13.5. Then $\sum_{\alpha\in\pi} a_\alpha\alpha = \sum_{\beta\in\Pi} b_\beta\beta$, where $b_{\beta(\alpha)} = a_\alpha$; this establishes (31) and (30).

In the light of (30) we have

$$\mu = \mu' - \mu'' = \sum_{\alpha\in\pi\cup\pi''} x_\alpha\alpha, \qquad \text{where } x_\alpha\|\alpha\|^2 \geq 0.$$

The formula $2(\lambda+\rho,\mu) = (\mu,\mu)$ gives

$$(32) \qquad 0 = (\lambda,\mu) + (\lambda-\mu,\mu) = \sum_{\alpha\in\pi\cup\pi''} x_\alpha(\lambda,\alpha) + x_\alpha(\lambda-\mu,\alpha).$$

Take $\alpha\in\pi\cup\pi''$. The $\pi\cup\pi''$-integrability of $L(\lambda)$ gives $(\lambda,\alpha)\|\alpha\|^2 \geq 0$. Moreover, $(\lambda-\mu,\alpha)\|\alpha\|^2 \geq 0$, by (27). Thus $x_\alpha(\lambda,\alpha), x_\alpha(\lambda-\mu,\alpha) \geq 0$. Using (32) we obtain $x_\alpha(\lambda,\alpha) = x_\alpha(\lambda-\mu,\alpha) = 0$, that is $x_\alpha(\mu,\alpha) = 0$. One has

$$(\mu',\mu') = (\mu,\mu') = \sum_{\alpha\in\pi} x_\alpha(\mu,\alpha) = 0;$$
$$(\mu'',\mu'') = (\mu,\mu'') = \sum_{\alpha\in\pi''} x_\alpha(\mu,\alpha) = 0.$$

Since $(-,-)$ is negatively definite on $\mathbb{Q}\pi''$ and non-negatively definite on $\mathbb{Q}\pi'$, we obtain $\mu''=0$, $\mu'\in\mathbb{Z}\delta$, that is $\mu = s\delta$. Since L is non-critical, $(\lambda+\rho,\delta)\neq 0$, so the formula $2(\lambda+\rho,\mu) = (\mu,\mu)$ gives $\mu = 0$, that is $\lambda-\mu = \lambda\in\operatorname{supp} Z$.

It remains to verify that $\lambda\not\in\operatorname{supp} Z$. Since $(\lambda,\alpha^\vee)\geq 0$ for $\alpha\in\pi\cup\pi''$, the coefficient of e^λ in Z is equal to the coefficient of e^λ in

$$j_\lambda e^\lambda - \sum_{\{w\in W(\pi\cup\dot{\Pi}_0)|w\lambda=\lambda\}} \operatorname{sgn}(w)\, w\Big(\frac{e^\lambda}{\prod_{\beta\in S}(1+e^{-\beta})}\Big).$$

If $Stab_{W(\pi\cup\pi'')}\lambda\subset W(\dot{\Pi}_0)$, then the KW-formula for $\dot{L}$ implies $\lambda\not\in\operatorname{supp} Z$ as required. Otherwise, $(\lambda,\alpha^\#) = 0$, where $\alpha^\#$ is the "affine root" in π, i.e., $\pi = \dot{\pi}\cup\{\alpha^\#\}$. Since $(\lambda,\alpha^\#) = 0$, L is not a vacuum module (since L is non-critical), so $\mathfrak{g} = D(n+1,n)^{(1)}$ and $(\lambda,\alpha) = 0$ for each $\alpha\in D(n,n)\subset D(n+1,n) = \dot{\Delta}$. Let α_1 be the first root in $\dot{\Pi}$ (i.e., $\alpha_1\in\dot{\Pi}$ and $\alpha_1\not\in D(n,n)$). Then $(\lambda,\alpha_1)\neq 0$. Note that Π admits an involution σ, which interchanges α_0 and α_1 and stabilizes all other simple roots. This involution preserves π,π'' and S. One has $(\sigma(\lambda),\alpha_1) = (\lambda,\alpha_0) = 0$, so $L(\sigma(\lambda),\Pi)$ is a vacuum module and, by above, its character satisfies the KW-formula. This implies the KW-formula for $L(\lambda,\Pi)$. This completes the proof for $D(n+1|n)^{(1)}$, $A(2n-1,2n-1)^{(2)}$.

6.4.2. Cases $D(n+1,n)^{(2)}, A(2n,2n)^{(4)}$. In these cases $\dot{\Delta} = B(n,n)$ and we choose $\dot{\Pi}$ in such a way that $\|\alpha\|^2 \geq 0$ for $\alpha \in \dot{\Pi}$ (recall that $\|\alpha\|^2 > 0$ for $\alpha \in \pi$). The Dynkin diagram is

$$\odot - \otimes - \otimes - \cdots - \otimes - \odot,$$

where both ends are non-isotropic roots; for $A(2n,2n)^{(4)}$ the ends have the same parity and for $D(n+1,n)^{(2)}$ the ends have different parity. One has

$$\dot{\Pi} = \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \ldots, \delta_n - \varepsilon_n, \varepsilon_n\}, \quad \alpha_0 = \delta - \delta_1;$$

we can (and will) normalize the form $(-,-)$ by $\|\varepsilon_i\|^2 = 1 = -\|\delta_i\|^2$. Note that $\dot{\Pi}$ contains $S = \{\delta_i - \varepsilon_i\}_{i=1}^n$. One has

$$\pi = \{a(\delta - \varepsilon_1), \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, a'\varepsilon_n\},$$

where for $\mathfrak{g} = D(n+1,n)^{(2)}$ one has $a = a' = 1$ (resp., $a = a' = 2$) if $\pi = D(n+1)^{(2)}$ (resp., if $\pi = C_n^{(1)}$), and for $\mathfrak{g} = A(2n,2n)^{(4)}$ one has $a = 1, a' = 2$ or $a = 2, a' = 1$. Observe that Δ contains $\delta - \varepsilon_1$ and $\varepsilon_n \in \Delta$ (they are of the same parity for $D(n+1,n)^{(2)}$ and of different parity for $A(2n,2n)^{(4)}$).

Let $k = (\lambda, \delta)$. Recall that $L(\lambda)$ is a non-critical vacuum module, so $k \neq 0$. If $\delta - \varepsilon_1$ is even (resp., odd), then $L(\lambda)$ is π-integrable if and only if $2k \in \mathbb{Z}_{\geq 0}$ (resp., $k \in \mathbb{Z}_{\geq 0}$).

One has $2\rho = \sum_{i=1}^n (\varepsilon_i - \delta_i)$.

Since $\dot{\pi}$ is the "largest part" of $\dot{\Pi}_0$ in the sense of §5.2, (26) can be rewritten as (29). We have $(\rho, \alpha) \geq 0$ for $\alpha \in \dot{\pi}$, $(\rho, \delta - \varepsilon_1) = -1/2$. Since $(\lambda, \delta - \varepsilon_1) = k \geq \frac{1}{2}$, one has $(\lambda + \rho, \alpha) \geq 0$ for each $\alpha \in \pi$. Therefore $\lambda + \rho$ is maximal in its $W(\pi)$-orbit, so $\operatorname{supp} Z \subset \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi$, that is $\mu \in \mathbb{Z}_{\geq 0}\Pi$. Moreover, from Lemma 3.3.4 (ii) we obtain $\mu \neq 0$. Write $\mu = j\delta - \sum_{i=1}^n (e_i\varepsilon_i + d_i\delta_i)$. From (27) we obtain

$$2kj - \sum_{i=1}^n (e_i + d_i) = \sum_{i=1}^n (e_i^2 - d_i^2),$$
$$0 \leq e_n \leq e_{n-1} \leq \cdots \leq e_1 \leq k; \quad 0 \leq d_n \leq d_{n-1} \leq \cdots \leq d_1.$$

Since $\mu \in \mathbb{Z}_{\geq 0}\Pi$ we have $e_i, d_i \in \mathbb{Z}$ and $\sum_{i=1}^n (e_i + d_i) \leq j$. Since $e_i, d_i \geq 0$, the equality $j = 0$ forces $\mu = 0$, which contradicts the above. Let us show that $j = 0$. Since $\sum_{i=1}^n (d_i - d_i^2) \leq 0$, we have $2kj \leq \sum_{i=1}^n (e_i^2 + e_i)$. If $k = 1/2$, then $e_i = 0$ for each i, so $j = 0$. If $k > 1$, then $\sum_{i=1}^n e_i \leq j$ and $e_i \leq k$ imply $2kj \leq kj + j$, that is $j = 0$, as required.

Finally, for $k = 1$ we have $e_i \in \{0, 1\}$ for each i, which implies $2j = \sum_{i=1}^n (2e_i + d_i - d_i^2)$. Combining with $\sum_{i=1}^n (e_i + d_i) \leq j$, we get $d_i = 0$ for each i. This gives $\mu = j\delta - \sum_{i=1}^j \varepsilon_j$ and $j \leq n$. Set $\alpha := \delta_j - \varepsilon_j$. In the light of §2.2.6 and Lemma 2.2.8, $Z = Z(\Pi)$ is equivalent to $Z(r_\alpha \Pi)$ (where $Z(\Pi')$ stands for an element Z defined for Π' and viewed as an element of $\mathscr{R}(\Pi')$). Clearly,

$$\mathrm{supp}(Z(r_\alpha \Pi)) \subset \lambda + \rho_{r_\alpha \Pi} - \mathbb{Z}_{\geq 0}(r_\alpha \Pi) = \lambda + \rho + \alpha - \mathbb{Z}_{\geq 0}(r_\alpha \Pi).$$

Observe that $(\lambda+\rho-\mu, \alpha) \neq 0$, so $\mathrm{supp}(Z) \cap \{\lambda+\rho-\mu+\mathbb{Z}\alpha\} = \{\lambda+\rho-\mu\}$ (since $\|\lambda+\rho-\nu\|^2 = \|\lambda+\rho\|^2$ for each $\nu \in \mathrm{supp}(Z)$). By Lemma 2.2.8, Z has a pole of order ≤ 1 at α, and so, by Lemma 2.2.9, $\lambda + \rho - \mu \in \mathrm{supp}(Z(r_\alpha \Pi))$, that is

$$\lambda + \rho - \mu \in \lambda + \rho + \alpha - \mathbb{Z}_{\geq 0}(r_\alpha \Pi),$$

which gives $\mu + \alpha \in \mathbb{Z}_{\geq 0}(r_\alpha \Pi)$; one readily sees that this does not hold for $\mu = j\delta - \sum_{i=1}^{j} \varepsilon_j$ (if $\mu \neq 0$), a contradiction. This completes the proof.

7. The root system $\Delta(L)$

In this section we exclude $\mathfrak{g}$ of types $D(2,1,a)$ and $D(2,1,a)^{(1)}$ with $a \notin \mathbb{Q}$ from consideration. Then we can (and will) choose a normalization of the bilinear form $(-,-)$, such that $(\alpha, \beta) \in \mathbb{Z}$ for each pair $\alpha, \beta \in \Delta$. Consequently, for each set of simple roots Π one has $2(\rho_\Pi, \alpha) \in \mathbb{Z}$ for each $\alpha \in \Pi$ and thence for each $\alpha \in \Delta$.

For the Lie superalgebras $D(2,1,a)$, $D(2,1,a)^{(1)}$ with non-rational a all the results of this section remain valid if we fix a standard symmetric Cartan matrix for $D(2,1,a)$ as in [K1] and replace $\mathbb{Z}$ by $\mathbb{Z} + \mathbb{Z}a$ in the construction of $\Delta(L)$.

7.1.

We will use the following fact.

Proposition. *If γ is a non-isotropic root and α is a root, then $(\alpha, \gamma^\vee)$ is an integer (resp., even integer) if γ is even (resp., odd).*

Proof. Since $(\gamma, \gamma) \neq 0$, $\|\alpha \pm N\gamma\|^2 \to \infty$ as $N \to \infty$, hence $\mathfrak{g}_{\pm\gamma}$ act locally nilpotently on $\mathfrak{g}$, and thus $(\alpha, \gamma^\vee)$ is an integer (resp., an even integer) if γ is even (resp., odd), using representation theory of A_1 (resp., $B(0,1)$). □

7.2. Definition of $\Delta(L)$

Let Δ^+ be a subset of positive roots in Δ. Consider an irreducible highest weight module $L = L(\lambda, \Delta^+)$ over $\mathfrak{g}$, associated with Δ^+. If β is a simple isotropic root, then L is again an irreducible highest weight module, but associated with the subset of positive roots $r_\beta(\Delta^+)$. Indeed, if $v_\lambda \in L$ is a highest weight vector for Δ^+, then v_λ (resp., $e_{-\beta}v_\lambda$) is a highest weight vector for $r_\beta(\Delta^+)$ if $(\lambda,\beta)=0$ (resp., if $(\lambda,\beta)\neq 0$). Since $(\rho,\beta)=0$, we obtain $L(\lambda,\Delta^+)=L(\lambda',r_\beta\Delta^+)$, where the highest weight λ' of the module $L(\lambda',r_\beta\Delta^+)$ is given by

$$\lambda'=\begin{cases}\lambda-\beta & \text{for } (\lambda+\rho,\beta)\neq 0,\\ \lambda & \text{for } (\lambda+\rho,\beta)=0.\end{cases} \tag{33}$$

Thus the notion of an irreducible highest weight module is independent of the choice of Δ^+ (by Proposition 2.1.1 (a)).

In this paper we consider only non-critical irreducible highest weight modules $L=L(\lambda,\Delta^+)$, i.e., we assume that the highest weight λ satisfies $(\lambda+\rho_\Pi,\delta)\neq 0$. This property is independent of the choice of Δ^+, since, by§2.1.2, one has $\rho_\Pi-\rho_{\Pi'}\in\mathbb{Z}\Delta$ and $L(\lambda,\Pi)=L(\lambda',\Pi')$ forces $\lambda'-\lambda\in\mathbb{Z}\Delta$.

7.2.1. For each $\lambda\in\mathfrak{h}^*$ we introduce the sets

$$\begin{aligned}
&D(\lambda)_{iso}=\{\alpha\in\Delta \mid (\alpha,\alpha)=0,\ (\lambda+\rho,\alpha)=0\},\\
&D(\lambda)_o:=\{\alpha\in\Delta_{\bar 1}\mid (\alpha,\alpha)\neq 0,\ (\lambda+\rho,\alpha^\vee)\in 2\mathbb{Z}+1\},\\
&D(\lambda)_e:=\left\{\alpha\in\Delta_{\bar 0}\,\middle|\,(\alpha,\alpha)\neq 0,\ \frac{\alpha}{2}\notin\Delta_{\bar 1},\ (\lambda+\rho,\alpha^\vee)\in\mathbb{Z}\right\}.
\end{aligned}$$

Let $W_{ess,\lambda}$ be the subgroup of W generated by the reflections $\{r_\alpha \mid \alpha\in D(\lambda)_o\cup D(\lambda)_e\}$. From Proposition 7.1, it follows that

$$\begin{aligned}
&D(\lambda)_o=D(\lambda+\nu)_o,\ D(\lambda)_e=D(\lambda+\nu)_e,\\
&W_{ess,\lambda}=W_{ess,\lambda+\nu}\quad \text{for each } \nu\in Q.
\end{aligned} \tag{34}$$

We introduce the following subset of Δ:

$$\Delta_{ess}(\lambda)=D(\lambda)_o\cup D(\lambda)_e\cup\{2\alpha\mid \alpha\in D(\lambda)_o\}\cup W_{ess,\lambda}D(\lambda)_{iso}.$$

Note that the even roots in $\Delta_{ess}(\lambda)$ are $D(\lambda)_e\cup\{2\alpha\mid\alpha\in D(\lambda)_o\}$ and that $D(\lambda)_o$ (resp., $W_{ess,\lambda}D(\lambda)_{iso}$) is the set of non-isotropic (resp., isotropic) odd roots in $\Delta_{ess}(\lambda)$. The main motivation for this definition is Proposition 7.3.1.

For an irreducible highest weight module $L=L(\lambda,\Pi)$ we set $\Delta_{ess}(L):=\Delta_{ess}(\lambda+\rho)$, $W_{ess}(L):=W_{ess,\lambda+\rho}$. Proposition 7.2.3 shows that $\Delta_{ess}(L)$ is well-defined.

7.2.2. **Lemma.** *For any $\beta \in D(\lambda+\rho)_{iso}$ and $\gamma \in D(\lambda+\rho+\beta)_{iso}$, one has $\gamma \in \Delta_{ess}(\lambda+\rho)$.*

Proof. If $(\beta,\gamma)=0$, then $\gamma \in D(\lambda+\rho)_{iso}$. Assume that $(\beta,\gamma)\neq 0$. Then $\beta+\gamma$ or $\beta-\gamma$ is an even root (this is proven in [S3] for the finite root systems; the non-twisted affine case follows immediately); we denote this root by α. One has

$$(\lambda+\rho,(\beta\pm\gamma)^\vee)=\frac{2(\lambda+\rho,\gamma\pm\beta)}{(\gamma\pm\beta,\gamma\pm\beta)}=\frac{2(\lambda+\rho,\gamma)}{\pm 2(\gamma,\beta)}=\frac{2(-\beta,\gamma)}{\pm 2(\gamma,\beta)}=\pm 1,$$

so $\alpha \in D(\lambda+\rho)_e$ if $\frac{\alpha}{2}\not\in\Delta_{\bar{1}}$.

Consider the case when $\frac{\alpha}{2}\in\Delta_{\bar{1}}$; then $\frac{\alpha}{2}$ is a non-isotropic odd root, so $\mathfrak{g}$ is of the types $B(m,n)$, $G(3)$ or their affinizations. In these cases the roots β,γ are of the form $k_1\delta\pm(\varepsilon_i+\delta_j),\ k_2\delta\pm(\varepsilon_i-\delta_j)$, where $k_1,k_2\in\mathbb{Z}$ and k_1+k_2 is even. (Here and further we use the description of root systems in [K1].) Then either $\alpha=\beta+\gamma$ and $\frac{\beta-\gamma}{2}\in\Delta_{\bar{0}}$, or $\alpha=\beta-\gamma$ and $\frac{\beta+\gamma}{2}\in\Delta_{\bar{0}}$; observe that $\frac{\beta\pm\gamma}{4}\not\in\Delta$. By above,

$$\Big(\lambda+\rho,\Big(\frac{\beta\pm\gamma}{2}\Big)^\vee\Big)=\pm 2$$

so $D(\lambda+\rho)_e$ contains $\frac{\beta-\gamma}{2}$ or $\frac{\beta+\gamma}{2}$.

Since $\gamma=r_{\beta-\gamma}\beta=-r_{\beta+\gamma}\beta$, we obtain $\gamma\in\Delta_{ess}(\lambda+\rho)$, as required. □

7.2.3. **Proposition.** *If Π and Π' are two sets of simple roots and $L(\lambda,\Pi)=L(\lambda',\Pi')$, then $\Delta_{ess}(\lambda+\rho)=\Delta_{ess}(\lambda'+\rho')$.*

Proof. Since any two sets of simple roots are connected by a chain of odd reflections and each odd reflection is invertible, it is enough to show that $\Delta_{ess}(\lambda'+\rho')\subset\Delta_{ess}(\lambda+\rho)$ if $\Pi'=r_\beta\Pi$, where β is an odd isotropic root.

If $\lambda+\rho=\lambda'+\rho'$, then, obviously, $\Delta_{ess}(\lambda+\rho)=\Delta_{ess}(\lambda'+\rho')$. By (33), if $\lambda+\rho\neq\lambda'+\rho'$, then $\lambda'+\rho'=\lambda+\rho+\beta$ and $(\lambda+\rho,\beta)=0$. From Proposition 7.1 it follows that $D(\lambda'+\rho')_e=D(\lambda+\rho)_e$ and $D(\lambda'+\rho')_o=D(\lambda+\rho)_o$; by Lemma 7.2.2, $D(\lambda'+\rho')_{iso}\subset\Delta_{ess}(\lambda+\rho)$. The assertion follows. □

7.2.4. **Proposition.** *One has $W_{ess}(L)\Delta_{ess}(L)=\Delta_{ess}(L)$.*

Proof. It is enough to verify that for $\alpha,\gamma\in D(\lambda)_o\cup D(\lambda)_e$ one has $r_\alpha\gamma\in D(\lambda)_o\cup D(\lambda)_e$. We have

$$(\lambda,(r_\alpha\gamma)^\vee)=(\lambda,\gamma^\vee)-(\lambda,\alpha^\vee)(\alpha,\gamma^\vee).$$

By Proposition 7.1, $(\alpha,\gamma^\vee)$ is an integer (resp., even integer) if γ is even (resp., odd). Thus $r_\alpha\gamma\in D(\lambda)_o\cup D(\lambda)_e$, as required. □

7.3. The properties of $\mathrm{supp}(Re^{\rho}\,\mathrm{ch}\,L)$

Fix a choice of Δ^+, and let $\lambda \in \mathfrak{h}^*$ be a non-critical weight. Recall that λ is called *typical* if $(\lambda+\rho,\beta) \neq 0$ for all odd isotropic roots β, and *atypical* otherwise.

The following proposition is proven in [KK] in the Lie algebra case and in [GK] in the general Lie superalgebra case.

7.3.1. **Proposition.** *If* $\mu \in \mathrm{supp}(Re^{\rho}\,\mathrm{ch}\,L(\lambda))$*, then there exists a chain* $\mu = \mu_r < \mu_{r-1} < \cdots < \mu_0 = \lambda+\rho$*, where either* $\mu_k = r_\gamma \mu_{k-1}$ *for* $\gamma \in (D(\mu_{k-1}+\rho)_o \cup D(\mu_{k-1}+\rho)_e) \cap \Delta^+$ *or* $\mu_k = \mu_{k-1}-\gamma$ *for* $\gamma \in D(\mu_{k-1}+\rho)_{iso} \cap \Delta^+$.

7.3.2. Taking into account (34) and Lemma 7.2.2, we obtain by induction on k that $D(\mu_k+\rho)_e = D(\lambda+\rho)_e$, $D(\mu_k+\rho)_o = D(\lambda+\rho)_o$ and $\Delta_{ess}(\mu_k+\rho) = \Delta_{ess}(\lambda+\rho)$.

This leads to the following useful properties of $\mathrm{supp}(Re^{\rho}\,\mathrm{ch}\,L(\lambda))$.

7.3.3. **Corollary.** *Let* $L = L(\lambda)$ *be a non-critical irreducible highest weight module. For each* $\nu \in \mathrm{supp}(Re^{\rho}\,\mathrm{ch}\,L(\lambda))$ *one has*

(i) $\nu \in \lambda+\rho-\mathbb{Z}_{\geq 0}(\Delta_{ess}(\lambda)\cap\Delta^+)$, $\|\lambda+\rho-\nu\|^2 = \|\lambda+\rho\|^2$.
(ii) If $L(\lambda)$ *is typical, then* $\nu \in W_{ess}(L)(\lambda+\rho)$.

If $L(\lambda)$ *is atypical, then* $(\nu,\beta)=0$ *for some odd isotropic root* β*,*

(iii) $\Delta_{ess}(\nu) = \Delta_{ess}(\lambda+\rho)$.

7.3.4. **Corollary.** *Let* $L = L(\lambda)$ *be a non-critical irreducible highest weight module and* Δ_{ess} *is not irreducible, that is*

$$\Delta_{ess} = \coprod_{i\in X} \Delta^i_{ess}, \qquad \text{where } (\Delta^i_{ess},\Delta^j_{ess}) = 0 \ \text{for all } i \neq j.$$

If $\lambda+\rho-\mu \in \mathrm{supp}(Re^{\rho}\,\mathrm{ch}\,L(\lambda))$*, then* μ *can be written as* $\mu = \sum_{i\in X}\mu_i$ *with the property that for each* $i \in X$ *there exists a chain* $\lambda-\mu^i = \mu_r < \mu_{r-1} < \cdots < \mu_0 = \lambda+\rho$*, where either* $\mu_k = r_\gamma\mu_{k-1}$ *for* $\gamma \in (D(\mu_{k-1}+\rho)_o \cup D(\mu_{k-1}+\rho)_e)\cap\Delta^+\cap\Delta^i_{ess}$ *or* $\mu_k = \mu_{k-1}-\gamma$ *for* $\gamma \in D(\mu_{k-1}+\rho)_{iso}\cap\Delta^+\cap\Delta^i_{ess}$.

In particular, $\mu^i \in \mathbb{Z}_{\geq 0}(\Delta^i_{ess}(\lambda)\cap\Delta^+)$ *and* $\|\lambda+\rho-\mu^i\|^2 = \|\lambda+\rho\|^2$.

7.3.5. A subset E of the set of real roots $\Delta_{re} = \Delta\setminus\mathbb{Z}\delta$ is called a *root subsystem* if the following properties hold:

(i) if $\alpha \in E$, then $-\alpha \in E$;
(ii) if $\alpha \in E$ is not isotropic, then $r_\alpha E = E$;
(iii) if $\alpha \in E$ is isotropic and $\beta \in E$ is such that $(\alpha, \beta) \neq 0$, then either $\beta + \alpha$ or $\beta - \alpha$ lies in E.

Note that any subset E' of Δ_{re} is contained in a unique minimal root subsystem of Δ_{re}. Indeed, in order to construct a minimal root subsystem containing E', we should add to E' an element $-\alpha$ if $\alpha \in E'$, $r_\alpha\beta$ if $\alpha, \beta \in E'$ and α is non-isotropic, and one of the elements $\beta \pm \alpha$ which is in Δ_{re} if α is isotropic with $(\alpha, \beta) \neq 0$ (exactly one of them is in Δ_{re}), and repeat this procedure. It follows from the results of [S3] that if this minimal root subsystem is finite, then it is a root system of a finite-dimensional basic simple Lie superalgebra. If this minimal root subsystem is infinite, then it is the set of real roots of an affine Lie superalgebra, see [Sh].

If $E' = E'_1 \coprod E'_2$ such that $(E'_1, E'_2) = 0$, then the minimal root subsystem of Δ_{re} containing E' is of the form $E_1 \coprod E_2$, where E_1, E_2 are minimal root subsystems containing E'_1, E'_2 respectively.

Let $\Delta(L)$ (resp., $\Delta(\nu)$) be the minimal root subsystem of Δ_{re} containing $\Delta_{ess}(L)$ (resp., $\Delta_{ess}(\nu)$). Denote by $W(L)$ the Weyl group of $\Delta(L)$, i.e., the subgroup of W generated by reflections in non-isotropic roots from $\Delta(L)$.

All results Lemma 7.2.2–Corollary 7.3.3 remain valid if we replace $\Delta_{ess}(L)$ by $\Delta(L)$.

7.4. Examples

Assume that $L := L(\lambda)$ is non-critical.

If $L(\lambda)$ is typical, then $\Delta_{ess}(L) = \Delta(L)$ consists of non-isotropic roots.

If $L(\lambda)$ is finite-dimensional, then $\Delta(L) = \Delta_{ess}(L) = \Delta_{re}$ if L is atypical, and $\Delta(L) = \Delta_{ess}(L) = \{\alpha \in \Delta \mid (\alpha, \alpha) \neq 0\}$ if L is typical.

If $\mathfrak{g} = A(m, n)$ or $A(m, n)^{(1)}$, then $\Delta_{ess}(L) = \Delta(L)$. However it is not true in general, as the following example shows.

7.4.1. Let $\mathfrak{g} = D(m, n)$ and let λ be such that $(\lambda, \varepsilon_i) = (\lambda, \delta_i) = \frac{1}{2}$. Then $\Delta_{ess}(\lambda)_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j; \pm\delta_i \pm \delta_j : i \neq j\}$ and $\Delta_{ess}(\lambda)_{\overline{1}} = \Delta_{\overline{1}}$, so $\Delta_{ess}(\lambda)$ is not a root system; in this case $\Delta(\lambda) = \Delta$.

7.5. The sets $\Pi(L)$

Fix a set of positive roots Δ^+, and let Π be the subset of simple roots. Let $\Delta(L)^+ := \Delta(L) \cap \Delta^+$ and denote the corresponding set of simple roots by

$\Pi(L)$. Denote by $\Pi_0(L)$ the set of simple roots of $\Delta(L) \cap \Delta_{\bar{0}}^+$ (it does not depend on the choice of Π). Note that both $\Pi(L)$ and $\Pi_0(L)$ are linearly dependent, if $\Delta(L)$ has more than one affine component.

7.5.1. **Lemma.** *Let $\alpha \in \Pi(L(\lambda))$ be a non-isotropic root. For each $\nu \in$ $\mathrm{supp}(Re^{\rho}\,\mathrm{ch}\, L)$ one has $r_\alpha \nu \in \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi(L(\lambda))$.*

Proof. We prove the assertion by induction on the length of the chain in Proposition 7.3.1. One has $\nu = \mu_r = \mu_{r-1} - a\gamma$ for $\gamma \in \Delta_{ess}(L(\lambda))^+$, $a \in \mathbb{Z}_{>0}$. If $\gamma = \alpha$, then $\mu_r = r_\alpha \mu_{r-1}$ and $r_\alpha \nu = \mu_{r-1} \in \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi(L(\lambda))$ by Corollary 7.3.3 (i). If $\gamma \neq \alpha$, then $r_\alpha \gamma \in \Delta(L(\lambda))^+$ and thus $r_\alpha \mu_r = r_\alpha \mu_{r-1} - a r_\alpha \gamma$. By the induction hypothesis, $r_\alpha \mu_{r-1} \in \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi(L(\lambda))$, so $r_\alpha \mu_r \in \lambda + \rho - \mathbb{Z}_{\geq 0}\Pi(L(\lambda))$, as required. □

7.5.2. Denote by R_L the analogue of R:

$$R_L = \frac{\prod_{\alpha \in \Delta_{\bar{0}}^+(L)}(1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_{\bar{1}}^+(L)}(1 + e^{-\alpha})}.$$

Note that $R_L \in \mathscr{R}(\Pi)$.

Fix $\rho_L \in \mathfrak{h}^*$ such that $2(\rho_L, \alpha) = (\alpha, \alpha)$ for all $\alpha \in \Pi(L)$.

7.5.3. Remark. Since R_L, ρ_L depend on Π, we will sometimes write $R_{L,\Pi}, \rho_{L,\Pi}$ to prevent a confusion. We choose $\rho_{L,\Pi}$ compatible for all subsets of simple roots Π, proceeding as in §2.1.2. The elements $R_{L,\Pi} e^{\rho_{L,\Pi}}$ are equivalent for all Π.

7.5.4. Let $\beta \in \Pi$ be an odd simple root and let r_β be the corresponding odd reflection. Then

$$r_\beta \Delta^+ \cap \Delta(L) = \begin{cases} \Delta(L)^+ & \text{if } \beta \notin \Pi(L), \\ \Delta(L)^+ \setminus \{\beta\} \cup \{-\beta\} & \text{if } \beta \in \Pi(L). \end{cases}$$

Recall that $r_\beta \Delta^+$ has the set of simple roots $r_\beta \Pi := \{r_\beta \alpha \mid \alpha \in \Pi\}$, where

$$r_\beta \alpha = \begin{cases} -\beta & \text{if } \alpha = \beta, \\ \alpha + \beta & \text{if } (\alpha, \beta) \neq 0, \\ \alpha & \text{otherwise.} \end{cases}$$

As a result, the set of simple roots for $r_\beta \Delta^+ \cap \Delta(L)$ coincides with $\Pi(L)$ if $\beta \notin \Pi(L)$ and is equal to $r_\beta \Pi(L)$, defined by the above formulas, if $\beta \in \Pi(L)$.

7.6. *Character formulas for different choices of* $\Delta^+(\Pi)$

Let L be a irreducible highest weight module. Then $\operatorname{ch} L \in \mathscr{R}(\Pi)$ for each Π. Suppose that $\Delta(L) = \Delta$ and that for some Π the following formula holds in $\mathscr{R}(\Pi)$:

$$Re^{\rho} \operatorname{ch} L = \sum_{w \in W'} x_w w\Big(\frac{e^{\nu}}{\prod_{\beta \in J}(1 + e^{-\beta})}\Big),$$

where $\nu \in \mathfrak{h}^* \setminus \{0\}$, $J \subset \Delta, x_w \in \mathbb{Q}$, and W' satisfies the conditions of §2.2.8. Combining §2.2.6 and §2.2.8, we conclude that the above formula holds in $\mathscr{R}(\Pi')$ for any Π'.

An important special case is when $\nu = \lambda + \rho_{\Pi}, L = L(\lambda, \Pi), J = J(\Pi)$. In this case we can obtain a formula of similar form for certain other subsets of positive roots $\Delta^+(\Pi')$ as follows. Let $\Pi' = r_{\beta}\Pi$ for an odd isotropic root $\beta \in \Pi$. Then $L(\lambda, \Pi) = L(\lambda', \Pi')$, where λ' is given by (33). This implies the formula

$$Re^{\rho} \operatorname{ch} L(\lambda', \Pi') = \sum_{w \in W'} x_w w\Big(\frac{e^{\lambda' + \rho_{\Pi'}}}{\prod_{\beta \in J(\Pi')}(1 + e^{-\beta})}\Big),$$

where $J(\Pi') = J(\Pi)$ if $(\lambda + \rho, \beta) \neq 0$ and $J(\Pi') = (J(\Pi) \setminus \{\beta\}) \cup \{-\beta\}$ if $(\lambda + \rho, \beta) = 0$ and $\beta \in J(\Pi)$ (this method does not work if $(\lambda + \rho, \beta) = 0$ and $\beta \notin J(\Pi)$).

7.6.1. Remark. If $L = L(\lambda, \Pi)$ with $\lambda \neq -\rho$, similar results hold if we substitute W by the integral Weyl group $W(L)$, see §7.3.5: in this case we take W' generated by reflections $r_{\alpha}, \alpha \in I$, where $I \subset \Pi_0(L)$.

7.7. *The map* $F : L \mapsto \overline{L}$

Fix $\Delta^+(\Pi)$ and a non-critical irreducible highest weight module $L = L(\lambda, \Pi)$. Set

$$\overline{\lambda} = \lambda + \rho - \rho_L.$$

Let $\mathfrak{g}'$ be the Kac–Moody superalgebra with the set of real roots $\Delta(L)$ and the set of simple roots $\Pi(L)$. This Kac–Moody superalgebra is of finite or affine type (with a symmetrizable Cartan matrix—its symmetrization is given by the restriction of $(-,-)$ to $\Pi(L)$). Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}'$ and let $\mathfrak{h}''$ be a commutative Lie algebra of dimension $\dim \mathfrak{h} - \dim \mathfrak{h}'$. Consider the Lie superalgebra $\overline{\mathfrak{g}}^{\lambda} := \mathfrak{g}' \times \mathfrak{h}''$ and identify its Cartan subalgebra with $\mathfrak{h}$ in such a way that the simple roots of $\mathfrak{g}'$ are identified with $\Pi(L)$. We denote by $\overline{L}(\overline{\lambda})$ an irreducible highest weight module with highest weight $\overline{\lambda}$ over the Lie superalgebra $\overline{\mathfrak{g}}^{\lambda}$.

Recall that L is non-critical and $\mathfrak{g}'$ is either basic finite-dimensional (if $\Delta(L)$ is finite) or affine. It is easy to see that in the second case $\overline{L}(\overline{\lambda})$ is also non-critical. Indeed, let $\delta' \in \Delta(L)$ be the primitive imaginary root of $\mathfrak{g}'$. Then $\|\delta'\|^2 = 0$, so δ' is an imaginary root in $\Delta_{\overline{0}}$, hence $\delta' = k\delta$, where $k \neq 0$ and $\delta \in \Delta^+_{\overline{0}}$ is a primitive imaginary root. But $(\overline{\lambda} + \rho_L, \delta') = k(\lambda + \rho, \delta) \neq 0$, so $\overline{L}(\overline{\lambda})$ is non-critical.

The following lemma shows that the map $F : L \to \overline{L}$ is a well-defined map of non-critical irreducible highest weight modules.

7.7.1. **Lemma.** *Let Π, Π' be two sets of simple roots. If $L = L(\lambda, \Pi) = L(\lambda', \Pi')$, then*

$$F(L(\lambda, \Pi)) \cong F(L(\lambda', \Pi')).$$

Proof. Recall that Π' can be obtained from Π by a sequence of odd reflections. Let $\beta \in \Pi$ be an isotropic root and $\Pi' = r_\beta \Pi$. Denote by $\Pi(L)$ (resp., by $\Pi'(L)$) the set of simple roots for $\Delta^+ \cap \Delta(L)$ (resp., for $(r_\beta \Delta^+) \cap \Delta(L)$) and choose ρ_L; we may (and will) choose $\rho'_L := \rho_L + \beta$.

Consider the case when $(\lambda + \rho, \beta) \neq 0$. Then $\lambda' + \rho_{r_\beta \Pi} = \lambda + \rho$ so $\overline{\lambda} + \rho_L = \overline{\lambda'} + \rho'_L$.

If $\beta \notin \Delta(L)$, then, by §7.5.4, $\Pi(L) = \Pi'(L)$ and thus $\overline{\lambda'} = \overline{\lambda}$ and

$$F(L(\lambda', r_\beta \Pi)) = \overline{L}(\overline{\lambda}, \Pi(L)) = F(L(\lambda), \Pi).$$

If $\beta \in \Delta(L)$, then, by §7.5.4, $\Pi(L) = r_\beta \Pi'(L)$ and, in particular, we choose $\rho'_L = \rho_L + \beta$. Thus $\overline{\lambda'} = \overline{\lambda} - \beta$ and

$$F(L(\lambda', r_\beta \Pi)) = \overline{L}(\overline{\lambda} - \beta, r_\beta \Pi(L)) \cong \overline{L}(\overline{\lambda}, \Pi(L)) = F(L(\lambda, \Pi)).$$

Consider the case when $(\lambda + \rho, \beta) = 0$. Then $\beta \in \Pi(L)$ and $\rho' = \rho + \beta$, $\rho'_L = \rho_L + \beta$. One has $\lambda' = \lambda$, so $\lambda' + \rho' = \lambda + \rho + \beta$ that is $\overline{\lambda'} + \rho'_L = \overline{\lambda} + \rho_L + \beta$ which gives $\overline{\lambda'} = \overline{\lambda}$. Therefore $F(L(\lambda', r_\beta \Pi)) = \overline{L}(\overline{\lambda'}, r_\beta \Pi(L)) \cong \overline{L}(\overline{\lambda}, \Pi(L)) = F(L(\lambda, \Pi))$. □

7.7.2. In [KRW] it was conjectured that the characters of an admissible $\mathfrak{g}$-module $L := L(\lambda, \Pi)$ and the $\overline{\mathfrak{g}}^\lambda$-module $F(L)$ are related by the formula (*cf.* (1))

$$Re^\rho \operatorname{ch} L = R_L e^{\rho_L} \operatorname{ch} F(L). \tag{35}$$

Since $\Delta(L)^+ \subset \Delta^+(\Pi)$ for each Π, the algebra $\mathscr{R}(\Pi(L))$ can be naturally embedded in $\mathscr{R}(\Pi)$, so we consider the above formula as an equality in $\mathscr{R}(\Pi)$. By §2.2.6 and Lemma 7.7.1 the elements in the left-hand side (resp., the elements in the right-hand side) are equivalent for all subsets of simple roots. Therefore if the formula holds for some Π, it holds for all other choices of Π.

In the next sections we verify formula (35) for certain cases.

8. Linkage $L \sim L'$

Let $\mathfrak{g}$ be a basic or affine Lie superalgebra and $\Delta^+(\Pi)$ be a subset of positive roots in the set of roots Δ. Let Π_0 be the set of simple roots for $\Delta^+_{\bar{0}}$. Define the standard dot action of the Weyl group by $w \cdot \lambda := w(\lambda + \rho) - \rho$.

8.1. Enright functor

Fix $\alpha \in \Pi_0$ and $a \in \mathbb{C}/\mathbb{Z}$. Let $\mathbb{M}(\mathfrak{g}, a)$ (resp., $\mathbb{M}(\mathfrak{g}_{\bar{0}}, a)$) be the category of $\mathfrak{g}$-modules (resp., $\mathfrak{g}_{\bar{0}}$-modules) M with a locally nilpotent action of a root vector e_α, a diagonal action of $\mathfrak{h}$, i.e., $M = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mu$, and such that $(\mu, \alpha^\vee) \equiv a \mod \mathbb{Z}$ if $M_\mu \neq 0$.

For each $\lambda \in \mathfrak{h}^*$ denote by $\mathcal{O}_\lambda$ the full subcategory of the category $\mathcal{O}$, whose objects are $\mathfrak{g}$-modules N satisfying $N_\mu = 0$ if $\mu - \lambda \notin \mathbb{Z}\Delta$. Note that $\mathcal{O}_{r_\alpha\cdot\lambda} = \mathcal{O}_{r_\alpha\lambda}$.

We denote by $\breve{M}(\nu)$ (resp., $\breve{L}(\nu)$) the Verma (resp., irreducible) $\mathfrak{g}_{\bar{0}}$-module with the highest weight ν (since Π_0 is fixed, these modules do not depend on the choice of Π).

We will use the following theorem which can be easily deduced from [KT2], [IK].

8.1.1. **Theorem.** *For each $a \in \mathbb{C}/\mathbb{Z}$ there exists a left exact functor $T_\alpha(a)$: $\mathbb{M}(\mathfrak{g}_{\bar{0}}, a) \to \mathbb{M}(\mathfrak{g}_{\bar{0}}, -a)$ (*Enright functor*) which induces a left exact functor $T_\alpha(a) : \mathbb{M}(\mathfrak{g}, a) \to \mathbb{M}(\mathfrak{g}, -a)$ with the following properties.*

(a) Assume that $\alpha \in \Pi$ or $\frac{\alpha}{2} \in \Pi$, $M(\lambda, \Pi) \in \mathbb{M}(\mathfrak{g}, a)$ and $(\lambda, \alpha^\vee) \notin \mathbb{Z}$. One has

$$T_\alpha(a)(M(\lambda,\Pi)) = M(r_\alpha\cdot\lambda,\Pi), \quad T_\alpha(a)(L(\lambda,\Pi)) = L(r_\alpha\cdot\lambda,\Pi),$$

$$T_\alpha(a)(\breve{M}(\lambda,\Pi)) = \begin{cases} \breve{M}(r_\alpha\cdot\lambda,\Pi) & \text{if } \alpha\in\Pi, \\ \breve{M}\Big(r_\alpha\cdot\lambda - \dfrac{\alpha}{2},\Pi\Big) & \text{if } \dfrac{\alpha}{2}\in\Pi, \end{cases}$$

$$T_\alpha(a)(\breve{L}(\lambda,\Pi)) = \begin{cases} \breve{L}(r_\alpha\cdot\lambda,\Pi) & \text{if } \alpha\in\Pi, \\ \breve{L}\Big(r_\alpha\cdot\lambda - \dfrac{\alpha}{2},\Pi\Big) & \text{if } \dfrac{\alpha}{2}\in\Pi. \end{cases}$$

(b) If $0 \neq a \in \mathbb{C}/\mathbb{Z}$, then $T_\alpha(a)$ is an equivalence of categories $\mathbb{M}(\mathfrak{g}, a) \to \mathbb{M}(\mathfrak{g}, -a)$ and the inverse is given by $T_\alpha(-a)$.

(c) If $\alpha \not\in \Delta(\lambda)$ and $a \equiv (\lambda+\rho,\alpha^\vee)$, then $T_\alpha(a)$ provides an equivalence of categories $\mathcal{O}_\lambda \xrightarrow{\sim} \mathcal{O}_{r_\alpha\lambda}$.

8.2. The linkage $\sim$

Let Θ be the set of functors $T:\mathcal{O}_\lambda \to \mathcal{O}_{\lambda'}$ which can be presented as compositions of Enright functors $T_\alpha(a):\mathcal{O}(\lambda) \xrightarrow{\sim} \mathcal{O}_{r_\alpha\lambda}$ with $\alpha \not\in \Delta(\lambda),\ a=(\lambda,\alpha^\vee)$. By Theorem 8.1.1 (c), each $T\in\Theta$ is an equivalence of categories.

We will say that the highest weight irreducible modules L and L' are *linked* if $L'=T(L)$ for $T\in\Theta$, and denote it by $L\sim L'$.

8.2.1. Let $L\sim L'$ be two linked highest weight irreducible modules and Π,Π' be two subsets of simple roots. Then there exists a finite chain

$$L=L^1=L(\lambda^1,\Pi^1),L^2=L(\lambda^2,\Pi^2),\dots,L^t=L(\lambda^t,\Pi^t)=L',$$

where $\Pi^1=\Pi,\Pi^t=\Pi'$, and for each i we have

$$L^{i+1}=L^i,\Pi^{i+1}=r_\beta\Pi^i$$
$$\text{for some isotropic } \beta\in\Pi^i,\quad \text{or}$$
$$\Pi^{i+1}=\Pi^i \text{ and } L^{i+1}=T(L_i)$$
$$\text{for some } T=T_\alpha(a)\in\Theta:\alpha\in\Pi^i \text{ or } \frac{\alpha}{2}\in\Pi^i.$$

Note that in the first case $\lambda^i+\rho^i=\lambda^{i+1}+\rho^{i+1}$ if $(\lambda^i+\rho^i,\beta)\neq 0$ and $\lambda^i+\rho^i+\beta=\lambda^{i+1}+\rho^{i+1}$ if $(\lambda^i+\rho^i,\beta)=0$, see (33); in particular, one has $\|\lambda^i+\rho^i\|^2=\|\lambda^{i+1}+\rho^{i+1}\|^2$. In the second case ($\Pi^{i+1}=\Pi^i$) one has $\alpha\not\in\Delta(\lambda^i)$ and $\lambda^{i+1}=r_\alpha(\lambda^i+\rho^i)-\rho^i$, where ρ^i is the Weyl vector for Π^i. This implies the following useful property of the linkage:

$$L(\lambda,\Pi)\sim L(\lambda',\Pi') \implies \|\lambda+\rho\|^2=\|\lambda'+\rho'\|^2.$$

8.2.2. Fix Π. Let $L'=T_\alpha(a)(L)$ for some $\alpha\not\in\Delta(L)$ such that α or $\frac{\alpha}{2}$ lies in Π. Write $L=L(\lambda,\Pi),L'=L(\lambda',\Pi)$ and recall that $\lambda=r_\alpha\cdot\lambda'$. Then $\Delta(L')=r_\alpha(\Delta(L))$. The conditions on α imply $r_\alpha(\Delta(L))\cap\Delta^+=\Delta(L)\cap\Delta^+$ so $\Pi(L')=r_\alpha(\Pi(L))$.

Recall the algebra $\overline{\mathfrak{g}}^\lambda$ and the map $L\to F(L)$ introduced in §7.7. Since $\Pi(\lambda')=r_\alpha(\Pi(\lambda))$, there exists a natural isomorphism of the Kac–Moody superalgebras $\iota:[\overline{\mathfrak{g}}^\lambda,\overline{\mathfrak{g}}^\lambda]\xrightarrow{\sim}[\overline{\mathfrak{g}}^\lambda,\overline{\mathfrak{g}}^\lambda]$ with the property $\iota:\overline{\mathfrak{g}}^\lambda_\beta\to\overline{\mathfrak{g}}^\lambda_{r_\alpha\beta}$ for each $\beta\in\Pi(L)$. This isomorphism can be extended to the isomorphism $\iota:\overline{\mathfrak{g}}^\lambda\xrightarrow{\sim}\overline{\mathfrak{g}}^\lambda$ such that $\iota(h)=r_\alpha(h)$ for each $h\in\mathfrak{h}$. Then $\overline{\lambda}(h)=\overline{\lambda'}(\iota(h))$ (see §7.7 for notation), so there exists an isomorphism $F(L)\xrightarrow{\sim}F(L')$ compatible with ι (i.e., $\iota'(av)=\iota(a)v$ for all $a\in\overline{\mathfrak{g}}^\lambda, v\in F(L)$).

8.2.3. **Corollary.** *If $L(\lambda,\Pi) \sim L(\lambda',\Pi')$, then there exists an isomorphism $\iota : \overline{\mathfrak{g}}^{\lambda} \xrightarrow{\sim} \overline{\mathfrak{g}}^{\lambda}$ and an isomorphism $F : F(L) \xrightarrow{\sim} F(L')$ compatible with ι.*

8.2.4. **Proposition.** *If $L \sim L'$ and (35) holds for L, then (35) holds for L'.*

Proof. Recall §7.7.2 that if (35) holds in $\mathscr{R}(\Pi)$, it holds in $\mathscr{R}(\Pi')$ for each Π'.

In the light of §8.2.1 it is enough to consider the case when $L' = T_\alpha(a)(L)$ for some $\alpha \not\in \Delta(L)$ such that $\alpha \in \Pi$ or $\frac{\alpha}{2} \in \Pi$. By §8.2.2, in this case

$$\operatorname{ch} F(L') = r_\alpha(\operatorname{ch} F(L))$$

in $\mathscr{R}(\Pi)$ (both elements lie in $\mathscr{R}(\Pi)$ since $\Pi(L), \Pi(L') = r_\alpha(\Pi(L))$ lie in Δ^+). One has $R_{L'}e^{\rho_{L'}} = r_\alpha(R_L e^{\rho_L})$ so

$$R_{L'}e^{\rho_{L'}} \operatorname{ch} F(L') = r_\alpha(R_L e^{\rho_L} \operatorname{ch} F(L))$$

in $\mathscr{R}(\Pi)$.

Write $L = L(\lambda,\Pi)$. By Theorem 8.1.1, $T_\alpha(a)$ is an equivalence of categories $\mathscr{O}_\lambda \xrightarrow{\sim} \mathscr{O}_{\lambda'}$ and $T_\alpha(a)(M(\nu)) = M(r_\alpha \cdot \nu)$ for each $\nu \in \mathscr{O}_\lambda$. Since $L' = T_\alpha(a)(L)$, we obtain

$$Re^\rho \operatorname{ch} L' = r_\alpha(Re^\rho \operatorname{ch} L)$$

in $\mathscr{R}(\Pi)$. This completes the proof. □

8.3.

We will also use the following simple fact.

Lemma. *For each $w \in W$ there exists $L' \sim L$ such that $\Delta(L') = w\Delta(L)$.*

Proof. Take $\alpha \in \Pi_0$. If $\alpha \in \Delta(L)$, then $r_\alpha\Delta(L) = \Delta(L)$; if $\alpha \not\in \Delta(L)$, then $\Delta(L') = r_\alpha\Delta(L)$ for $L' = T_\alpha(a)(L)$. □

Note that for $\Delta = A(m,n), C(n), A(m,n)^{(1)}, C(n)^{(1)}$ any odd root lies in a set of simple roots. Hence, in these cases, for any $\beta \in \Delta(L)$ we can choose Π such that $\beta \in \Pi$.

If $\Delta \neq A(m,n), C(n), A(m,n)^{(1)}, C(n)^{(1)}$, then W acts transitively on the set of odd isotropic roots, so if $\Delta(L)$ contains odd isotropic roots, then for any odd isotropic root $\beta \in \Delta$ there exists $L' \sim L$ such that $\beta \in \Delta(L')$.

9. Typical case for $\overline{\mathfrak{g}}^{\lambda}$ with even root subsystem of rank ≤ 2

Recall that for an affine Weyl group every orbit on non-zero level has a unique maximal element in the order $\geq$ (if the level is not a negative rational number) or a unique minimal element (if the level is not a positive rational number). We call $\lambda \in \mathfrak{h}^*$ *extremal* if for each connected component π of $\Pi_0(\lambda)$, λ is maximal or minimal in its $W(\pi)$-orbit.

Fix a non-critical level k and denote by W_+ (resp., W_-) the subgroup of $W(\lambda)$ generated by r_α such that $k(\alpha, \alpha) > 0$ (resp., $k(\alpha, \alpha) < 0$). We introduce a partial order on the Weyl group $W(L)$ as follows:

for $x_+, y_+ \in W_+, x_-, y_- \in W_-$ let $x_+x_- \leq_k y_+y_-$ if $x_+ \leq y_+, x_- \geq y_-$ in the Bruhat order on W.

9.1.

Proposition. *If* λ *is extremal, then for each* $y \leq_k w$ *there exists an embedding* $M(w \cdot \lambda) \to M(y \cdot \lambda)$ *and* $\dim \mathrm{Hom}(M(w \cdot \lambda), M(y \cdot \lambda)) = 1$.

Proof. Recall that the Bruhat order is the unique order satisfying $e \geq w$ forces $w = e$, and for each $\alpha \in \Pi_0$ one has:

$$l(r_\alpha w) < l(w) \quad \text{forces} \quad w \geq r_\alpha w;$$
$$w \geq w' \quad \text{forces} \quad r_\alpha w \geq r_\alpha w' \text{ or } w \geq r_\alpha w';$$
$$w \geq w' \quad \text{forces} \quad r_\alpha w \geq r_\alpha w' \text{ or } r_\alpha w \geq w'.$$

We claim that any order $\leq'$ with the properties:

- $l(r_\alpha w) < l(w)$ forces $r_\alpha w \leq' w$, and $y \leq' w$ forces $r_\alpha y \leq' r_\alpha w$ or $r_\alpha y \leq' w$,
- satisfies $y \leq w \;\Rightarrow\; y \leq' w$.

Indeed, the first property implies that $e \leq' x$ for all x. Let us prove the assertion by induction on $l(w)$. If $l(w) = 0$, then $w = y = e$ and thus $y \leq' w$. Now take any w with $l(w) > 0$ and $\alpha \in \Pi_0$ such that $l(r_\alpha w) < l(w)$. Then the property implies $r_\alpha w \leq' w$. One has $y \leq r_\alpha w$ or $r_\alpha y \leq r_\alpha w$. In the first case, the induction hypothesis gives $y \leq' r_\alpha w$, so $r_\alpha w \leq' w$ implies $y \leq' w$, as required. In the case $r_\alpha y \leq r_\alpha w$ the induction hypothesis gives $r_\alpha y \leq' r_\alpha w$ and the second property gives $y \leq' w$ or $y \leq' r_\alpha w \leq' w$. The claim follows.

9.1.1. Next, let us show that for any typical weight λ, $M(r_\alpha \cdot \lambda)$ is a submodule of $M(\lambda)$ and $\dim \mathrm{Hom}(M(r_\alpha \cdot \lambda), M(\lambda)) = 1$ if $r_\alpha \cdot \lambda < \lambda$ and $\alpha \in \Pi_0(\lambda)$.

The proof is by induction on α (with respect to the order given by Π_0). Indeed, if $\alpha \in \Pi_0$ this immediately follows from typicality of $M(\lambda)$. Take $\gamma \in \Pi_0$ such that $r_\gamma \alpha < \alpha$. Since $\alpha \in \Pi_0(\lambda)$ one has $\gamma \notin \Pi_0(\lambda)$ so the

Enright functor T_γ is an equivalence of categories. By induction, $M(r_{r_\gamma\alpha} \cdot (r_\gamma \cdot \lambda))$ is a submodule of $M(r_\gamma \cdot \lambda)$ so $M(r_\alpha \cdot \lambda)$ is a submodule of $M(\lambda)$ (since $r_\alpha = r_\gamma r_{r_\gamma\alpha} r_\gamma$). This proves that $M(r_\alpha \cdot \lambda)$ is a submodule of $M(\lambda)$ for any $\alpha \in \Pi_0(\lambda)$. Moreover, by induction

$$\dim \operatorname{Hom}(M(r_{r_\gamma\alpha} \cdot (r_\gamma \cdot \lambda)), M(r_\gamma \cdot \lambda)) = 1. \tag{36}$$

If $\dim \operatorname{Hom}(M(r_\alpha \cdot \lambda), M(\lambda)) > 1$, then there exists an exact sequence

$$0 \longrightarrow N \longrightarrow M(r_\alpha \cdot \lambda) \oplus M(r_\alpha \cdot \lambda) \longrightarrow M(\lambda)$$

and $N_{r_\alpha \cdot \lambda} = 0$ that is $\operatorname{Hom}(M(r_\alpha \cdot \lambda), N) = 0$. Using the Enright functor we obtain the exact sequence

$$0 \longrightarrow N' \longrightarrow M(r_\gamma r_\alpha \cdot \lambda) \oplus M(r_\gamma r_\alpha \cdot \lambda) \longrightarrow M(r_\gamma \cdot \lambda)$$

with $\operatorname{Hom}(N', M(r_\gamma r_\alpha \cdot \lambda)) = 0$. Since $r_\gamma r_\alpha = r_{r_\gamma\alpha} r_\gamma$, this contradicts (36).

9.1.2. Denote $M(w \cdot \lambda)$ as $M(w)$. Let us show the existence of the embedding $M(w) \subset M(y)$ for $y \leq_k w$.

By §9.1.1, $M(r_\alpha x)$ is a submodule of $M(x)$ if $r_\alpha x >_k x$ for $\alpha \in \Pi_0(\lambda)$.

It remains to show that for $\alpha \in \Pi_0(\lambda)$ $M(w) \subset M(y)$ the module $M(r_\alpha y)$ contains $M(r_\alpha w)$ or $M(w)$.

Indeed, using the Enright functors as in §9.1.1 we can reduce the question to the case when $\alpha \in \Pi_0$. If $M(r_\alpha y)$ contains $M(y)$ it contains $M(w)$ as well. Otherwise, $M(r_\alpha y)$ is a submodule of $M(y)$ and f_α acts locally nilpotently on $M(y)/M(r_\alpha y)$. If $M(r_\alpha w)$ contains $M(w)$, then f_α acts injectively on $L(w \cdot \lambda)$ so $\operatorname{Hom}(L(w \cdot \lambda), M(y)/M(r_\alpha y)) = 0$ and thus $\operatorname{Hom}(M(w), M(r_\alpha y)) = \operatorname{Hom}(M(w), M(y))$, as required. If $M(r_\alpha w)$ is a submodule of $M(w)$, it is also a submodule of $M(y)$ and, by above, $\operatorname{Hom}(M(r_\alpha w), M(r_\alpha y)) = \operatorname{Hom}(M(r_\alpha w), M(y))$.

9.1.3. It remains to verify that $\dim \operatorname{Hom}(M(w), M(y)) \leq 1$ for each y, w.

Write $w = w_- w_+$, $y = y_- y_+$ with $y_-, w_- \in W_-$, $y_+, w_+ \in W_+$. We proceed by induction on $l(w_+) + l(y_-)$.

Assume that $l(w_+) \neq 0$. Then there exists $\alpha \in \Pi(\lambda)$ such that $r_\alpha \in W_+$ and $M(w) \subset M(r_\alpha w)$. Using the Enright functors as in §9.1.1 we can reduce the question to the case when $\alpha \in \Pi_0$. Then, by [IK] Corollary 4.1, $T_\alpha(M(w)) = M(r_\alpha w)$ and $\dim \operatorname{Hom}(M(w), M(y)) \leq \dim \operatorname{Hom}(T_\alpha(M(w)), T_\alpha(M(y)))$. Since $T_\alpha(M(y))$ is either $M(y)$ or $M(r_\alpha y)$, we obtain $\dim \operatorname{Hom}(M(w), M(y)) \leq \dim \operatorname{Hom}(M(r_\alpha w), M(y'))$ for some y'. Arguing like this we obtain $\dim \operatorname{Hom}(M(w), M(y)) \leq \dim \operatorname{Hom}(M(w_-), M(y'))$.

It remains to verify that $\dim \operatorname{Hom}(M(w), M(y)) \leq 1$ for $w \in W_-$. We prove this by induction on $l(w)$.

If $l(w) = 0$, then $w = e$. If $y \notin W_+$, there exists $\alpha \in \Pi(\lambda)$ such that $r_\alpha \in W_-$ and $M(y) \subset M(r_\alpha y)$. Then f_α acts locally nilpotently on $M(y)/M(r_\alpha y)$ and injectively on $M(e)$. Hence $\dim \operatorname{Hom}(M(e), M(y)) \leq \dim \operatorname{Hom}(M(e), M(r_\alpha y))$ and therefore $\dim \operatorname{Hom}(M(e), M(y)) = \dim \operatorname{Hom}(M(e), M(y_+))$. However $y_+\lambda \leq \lambda$, so that we have $\dim \operatorname{Hom}(M(e), M(y_+)) \leq 1$.

Assume that $\dim \operatorname{Hom}(M(w), M(y)) \leq 1$ for all $w \in W_-$ with $l(w) \leq r$. Take $w \in W_-$ with $l(w) = r + 1$. Take $\alpha \in \Pi(\lambda)$ such that $r_\alpha \in W_-$ and $l(r_\alpha w) = r$. Then $M(r_\alpha w) \subset M(w)$ and $\dim \operatorname{Hom}(M(r_\alpha w), M(y)) \leq 1$. As before, using Enright functors we can assume that $\alpha \in \Pi_0$. But then the embedding of $M(r_\alpha w)$ in $M(w)$ is given by the multiplication of the highest weight vector to f_α which is non-zero divisor so $\dim \operatorname{Hom}(M(w), M(y)) \leq 1 \leq \dim \operatorname{Hom}(M(r_\alpha w), M(y)) \leq 1$, as required. This completes the proof. □

9.2.

Now let $L = L(\lambda, \Pi)$ be a $\mathfrak{g}$-module. Assume that $\overline{\mathfrak{g}}^\lambda$ is one of the affine Lie superalgebra with the set of even roots which is the union of affine or finite root systems of rank at most two.

9.2.1. Let λ be a non-critical extremal typical weight. By Proposition 9.1, for $y \leq_k z, y, z \in W(\lambda)$ the module $M(y \cdot \lambda)$ contains a unique singular vector $v(z)$ of weight $z \cdot \lambda$ and this vector gives rise to an embedding $M(z \cdot \lambda) \subset M(y \cdot \lambda)$.

Consider a Kac–Moody algebra $\tilde{\mathfrak{g}}$ with the set of simple roots $\Pi_0(L)$ (and the set of real roots $\Delta_{\overline{0}}(L) \cap \Delta_{re}$); this algebra coincides with $\overline{\mathfrak{g}}_{\lambda,\overline{0}}$ if and only if $\Delta_{\overline{0}}(L)$ is indecomposable. Let $\tilde{\mathfrak{h}}$ be the Cartan subalgebra of this Kac–Moody algebra and let $\rho_{\overline{0}} \in \tilde{\mathfrak{h}}^*$ be a Weyl vector. We introduce the $\cdot$-action of the Weyl group W on $\tilde{\mathfrak{h}}^*$ by the usual formula $w \cdot \mu := w(\mu + \rho_{\overline{0}}) - \rho_{\overline{0}}$. Consider $\lambda_0 \in \tilde{\mathfrak{h}}^*$ satisfying

$$(\lambda_0 + \rho_{\overline{0}}, \alpha) = (\lambda + \rho, \alpha) \quad \text{for each } \alpha \in \Pi_0$$

(if $\Delta_{\overline{0}}$ is indecomposable, then $\lambda_0 + \rho_{\overline{0}} = \lambda + \rho$).

Denote by $\breve{M}(w)$ the Verma module over $\tilde{\mathfrak{g}}$ with the highest weight $w \cdot \lambda_0$. For $y \leq_k w$ the module $\breve{M}(w)$ contains a unique singular vector $v_0(z)$ of weight $z \cdot \lambda_0$ and this vector gives rise to an embedding $M(z \cdot \lambda_0) \subset M(w \cdot \lambda_0)$.

9.2.2. Since $\tilde{\mathfrak{g}}$ is the product of finite-dimensional or affine Lie algebras of rank at most two, any submodule of Verma module $\breve{M}(w)$ over $\tilde{\mathfrak{g}}$ is generated by the singular vectors (and is a sum of the submodules of the form $\breve{M}(y)$, $y \in W(\lambda)$).

Define a map Ψ from the set of submodules of $\breve{M}(w)$ to the set of submodules of $M(w)$ given by $\breve{M}(y) \mapsto M(y)$. It is easy to see that this map is compatible with inclusions.

Let $\breve{M}'(w)$ be the maximal proper submodule of $\breve{M}(w)$. Then $\Psi(\breve{M}')$ is a proper submodule of $M(w)$.

9.2.3. Suppose that for some w the module $M(w)$ has a submodule which is not generated by singular vectors. Then for some $w' \geq_k w$ we have $[M(w) : L(w')] > 1$. Let w, w' be such a pair with the minimal value of $w \cdot \lambda - w' \cdot \lambda$. Then, if N is a submodule of $\Psi(\breve{M}')$ and $\nu \leq w \cdot \lambda - w' \cdot \lambda$, one has

$$p_{w\cdot\lambda-\nu}(\operatorname{ch}\Psi(N)) = p_{w\cdot\lambda_0-\nu}(R_{\bar{1}}\operatorname{ch} N),$$

where $p_\nu(\sum a_\mu e^\mu) := a_\nu$ and $R_{\bar{1}} := \prod_{\beta\in\Delta^+_{\bar{1}}}(1+e^{-\beta})$.

Let $\{M^i(w)\}_{i\geq 0}, \{\breve{M}^i(w)\}_{i\geq 0}$, be the Jantzen filtrations of $M(w)$, $\breve{M}(w)$ respectively. Recall that for each i the modules $M^i(w)/M^{i+1}(w)$, $\breve{M}^i(w)/\breve{M}^{i+1}(w)$ are semisimple. This implies $M(z) \subset M^i(w)$, $\breve{M}(z) \subset \breve{M}^i(w)$ if $z \geq_k w$ and $l(z^{-1}w) \leq i$.

It is easy to see that

$$\breve{M}^i(w) = \sum_{\{z\geq_k w | l(z^{-1}w)\leq i\}} \breve{M}(z).$$

Therefore $\Psi(\breve{M}^i(w)) \subset M^i(w)$. Clearly, $\breve{M}^1 = \breve{M}'$. Therefore for $\nu = w \cdot \lambda - w' \cdot \lambda$ one has

$$\dim M^i(w)_{w\cdot\lambda-\nu} \geq \dim(\Psi(M^i))_{w\cdot\lambda-\nu} = p_{w\cdot\lambda_0-\nu}(R_{\bar{1}}\operatorname{ch}\breve{M}^i)$$

for each $i > 0$. Since $[M(w) : L(w')] > 1$ one has

$$\dim M^1(w)_{w\cdot\lambda-\nu} > \dim(\Psi(M^1))_{w\cdot\lambda-\nu} = p_{w\cdot\lambda_0-\nu}(R_{\bar{1}}\operatorname{ch}\breve{M}^1).$$

However, the Jantzen sum formula implies

$$e^{-w\cdot\lambda}\sum_{i=1}^{\infty}\operatorname{ch} M^i(w) = e^{-w\cdot\lambda_0}R_{\bar{1}}\sum_{i=1}^{\infty}\operatorname{ch}\breve{M}^i(w) \tag{37}$$

so

$$\sum_{i=1}^{\infty}\dim M^i(w)_{w\cdot\lambda-\nu} = p_{w\cdot\lambda_0-\nu}\Big(R_{\bar{1}}\sum_{i=1}^{\infty}\operatorname{ch}\breve{M}^i\Big),$$

a contradiction.

We conclude that all submodules of $M(w)$ are generated by singular vectors (and lie in the image of Ψ). Combining the inclusions $\Psi(\breve{M}^i(w)) \subset M^i(w)$ and (37) we obtain $M^i(w) = \Psi(\breve{M}^i(w))$. This gives

$$M^i(w) = \Psi(\breve{M}^i(w)) = \sum_{\{z\geq_k w | l(z^{-1}w)\leq i\}} M(z).$$

9.2.4. Denote by $\breve{L}(\nu)$ (resp., $\breve{M}(\nu)$) the irreducible (resp., Verma) $\tilde{\mathfrak{g}}$-module with the highest weight ν. One has

$$\operatorname{ch} \breve{L}(w \cdot \lambda_0) = \sum_{y \in C} \operatorname{sgn}(yw^{-1}) \operatorname{ch} \breve{M}(y \cdot \lambda_0),$$

where $C := \{y \in W(\lambda_0)/Stab_{W(\lambda_0)}(\lambda_0 + \rho_{\overline{0}}) \mid y \cdot \lambda_0 \leq w \cdot \lambda_0\}$.

This implies

$$\operatorname{ch} L(w \cdot \lambda) = \sum_{y \in C} \operatorname{sgn}(yw^{-1}) \operatorname{ch} M(y \cdot \lambda).$$

From the construction of λ_0, one readily sees that

$$C = \{y \in W(\lambda)/Stab_{W(\lambda)}(\lambda + \rho) \mid y \cdot \lambda \leq w \cdot \lambda\}.$$

9.2.5. **Corollary.** *Let $\overline{\mathfrak{g}}^{\lambda}$ be one of the affine Lie superalgebra with the set of even roots which is the union of affine or finite root systems of rank at most two and let $\overline{\lambda}$ be a non-critical extremal typical weight. For each $w \in W(L)$ the Jantzen filtration $M^i(w \cdot \lambda)$ is given by*

$$M^i(w \cdot \lambda) = \sum_{\{z \in W(\lambda) | z \geq_K w,\ l(z^{-1}w) \leq i\}} M(z)$$

and

$$\operatorname{ch} L(w \cdot \lambda) = \sum_{y \in C} \operatorname{sgn}(yw^{-1}) \operatorname{ch} M(y \cdot \lambda),$$

where $C = \{y \in W(\lambda)/Stab_{W(\lambda)}(\lambda + \rho) \mid y \cdot \lambda \leq w \cdot \lambda\}$.

10. π-Relatively integrable modules

This section is continuation of §8. Throughout the section, L is an irreducible highest weight $\mathfrak{g}$-module.

10.1. Definition of π-relative integrability

We retain notations of §3.1.3. Recall the definition of $\Pi_0(L)$ from §7.5.

10.1.1. **Definition.** *For a subset $\pi \subset \Pi_0(L)$ we call L π-*relatively integrable *if $F(L)$ is π-integrable. We call L* relatively integrable *if $F(L)$ is integrable.*

10.1.2. In the light of Lemma 7.7.1, the above notion does not depend on the choice of Π (if $L(\lambda, \Pi) \cong L(\lambda', \Pi')$, then $L(\lambda, \Pi)$ is π-relatively integrable if and only if $L(\lambda', \Pi')$ is π-relatively integrable). Moreover, by Corollary 8.2.3, the linkage $\sim$ preserves relative integrability.

For each $\pi \subset \Pi_0(L)$ we denote by $W(\pi)$ the subgroup of $W(L)$ generated by $r_\alpha, \alpha \in \pi$.

10.1.3. In the affine Lie algebra case, relative integrability of L implies admissibility in the sense of [KW2]. Any boundary admissible module in the sense of [KW4] is $\Pi_0(L)$-relatively integrable (these are the modules L, such that $\dim F(L) = 1$).

10.2. Properties

In this section we will prove several useful properties of the characters of relatively integrable modules.

Note that the term $R_{\overline{0}} \operatorname{ch} L$ does not depend on the choice of Π (since $\Delta^+_{\overline{0}}$ is fixed) and lies in $\mathscr{R}(\Pi)$ for each Π.

10.2.1. **Lemma.** *Take $\gamma \in \Pi_0(L)$.*

(i) There exists $L' \sim L$ such that the root corresponding to γ in $\Pi_0(L')$ in the sense of Corollary 8.2.3 lies in Π_0.
(ii) Assume that f_γ acts locally nilpotently on $F(L)$. Denote by $\Delta^\#$ the connected component of $\Delta_{\overline{0}}$ which contains γ and by $\rho^\#$ the corresponding Weyl vector (i.e., the Weyl vector for $\Delta^\# \cap \Delta^+_{\overline{0}}$). The element $e^{\rho^\#} R_{\overline{0}} \operatorname{ch} L$ is a $W(\gamma)$-skew-invariant element of $\mathscr{R}_{W(\gamma)}$ for $W(\gamma) = \{r_\gamma, Id\}$.

Proof. Let us show that (ii) holds for $\gamma \in \Pi_0$. Choose Π such that $\gamma \in \Pi$ or $\gamma/2 \in \Pi$ and denote by λ the highest weight of L: $L = L(\lambda, \Pi)$. By Lemma 7.7.1 one has $F(L) = \overline{L}(\overline{\lambda}, \Pi(L))$. Since f_γ acts locally nilpotently on $F(L)$ one has

$$(\lambda + \rho_\Pi, \gamma^\vee) = (\overline{\lambda} + \rho_{L,\Pi}, \gamma^\vee) \in \mathbb{Z}_{>0}.$$

This means that f_γ acts locally nilpotently on L, so L can be decomposed as a direct sum of $\mathfrak{sl}_2(\gamma)$-modules. Therefore $(1 - e^{-\gamma})e^{\gamma/2} \operatorname{ch} L$ is a $W(\gamma)$-skew-invariant element of $\mathscr{R}_{W(\gamma)}$. Since $\rho^\# - \gamma/2$ and $\Delta_{\overline{0}} \setminus \{\gamma\}$ are r_γ-invariant, $e^{\rho^\#} R_{\overline{0}} \operatorname{ch} L$ is also a $W(\gamma)$-skew-invariant element of $\mathscr{R}_{W(\gamma)}$.

Now we prove (i) and (ii) by induction on γ. If $\gamma \not\in \Pi_0$, there exists $\alpha \in \Pi_0$ such that $r_\alpha \gamma < \gamma$. Since $\gamma \in \Pi_0(L)$, one has $r_{\gamma'}\gamma \geq \gamma$ for each $\gamma' \in \Pi_0(L)$, so $\alpha \not\in \Pi_0(L)$ and thus $\alpha \not\in \Delta(L)$ (because $\alpha \in \Pi_0$). Using the Enright functor

$T_\alpha(a)$ we obtain $L' = T_\alpha(a)(L) \sim L$ and $r_\alpha\gamma < \gamma$ is the root corresponding to γ in $\Pi_0(L')$. This proves (i).

For (ii) choose Π which contains α or $\alpha/2$. Recall that, by Corollary 8.2.3, $F(T_\alpha(a)(L)) \cong F(L')$ under the identification of Kac–Moody superalgebras with the sets of simple roots $\Pi(L)$ and $r_\alpha\Pi(L)$; under this identification $\mathbb{C} f_\gamma$ is identified with $\mathbb{C} f_{r_\alpha\gamma}$, so $f_{r_\alpha\gamma}$ acts locally nilpotently on $F(L')$.

Since $T_\alpha(a)$ provides the equivalence of categories, $Re^\rho \operatorname{ch} L' \in \mathscr{R}_{W(\alpha)}$ and $Re^\rho \operatorname{ch} L = r_\alpha(Re^\rho \operatorname{ch} L')$. Therefore

$$e^{\rho^\#} R_{\bar{0}} \operatorname{ch} L = R_{\bar{1}} e^{\rho^\# - \rho} Re^\rho \operatorname{ch} L = R_{\bar{1}} e^{\rho^\# - \rho} r_\alpha(Re^\rho \operatorname{ch} L').$$

Note that $R_{\bar{1}} e^{\rho^\# - \rho}$ is r_α-invariant element of $\mathscr{R}_{W(\alpha)}$ (if $\alpha \in \Pi$, then $\Delta_{\bar{1}}$ and $\rho^\# - \rho$ are r_α-invariant; if $\alpha/2 \in \Pi$, then $\Delta_{\bar{1}} \setminus \{\alpha/2\}$ and $e^{\rho^\# - \rho}(1 + e^{-\alpha/2})$ is r_α-invariant). Hence $e^{\rho^\#} R_{\bar{0}} \operatorname{ch} L' \in \mathscr{R}_{W(\alpha)}$ and

$$e^{\rho^\#} R_{\bar{0}} \operatorname{ch} L = r_\alpha(e^{\rho^\#} R_{\bar{0}} \operatorname{ch} L').$$

By induction hypothesis, $e^{\rho^\#} R_{\bar{0}} \operatorname{ch} L'$ is a $W(r_\alpha\gamma)$-skew-invariant element of $\mathscr{R}_{W(r_\alpha\gamma)}$, so $e^{\rho^\#} R_{\bar{0}} \operatorname{ch} L$ is a $W(\gamma)$-skew-invariant element of $\mathscr{R}_{W(\gamma)}$. □

10.2.2. **Corollary.** *Let L be π-relatively integrable ($\pi \subset \Pi_0(L)$). Assume that π lies in a connected component $\Delta^\#$ of $\Delta_{\bar{0}}$; let $\rho^\#$ be the corresponding Weyl vector.*

Then the element $R_{\bar{0}} e^{\rho^\#} \operatorname{ch} L$ is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}$ and the element $Re^\rho \operatorname{ch} L$ is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}[\mathscr{Y}^{-1}]$ (see §2.2.3 for notation).

Proof. From Lemma 10.2.1 it follows that $R_{\bar{0}} e^{\rho^\#} \operatorname{ch} L$ is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}$. Recall that $R_{\bar{1}} \in \mathscr{Y}$, so $Re^\rho \operatorname{ch} L \in \mathscr{R}_{W(\pi)}[\mathscr{Y}^{-1}]$. It remains to verify that $Re^\rho \operatorname{ch} L$ is $W(\pi)$-skew-invariant. Since $R_{\bar{0}} e^{\rho^\#}$ is $W(\Delta^\#)$-skew-invariant and Re^ρ is W-invariant, their ratio $R_{\bar{1}} e^{\rho^\# - \rho}$ is $W(\Delta^\#)$-skew-invariant, and, in particular, is $W(\pi)$-skew-invariant. Hence $Re^\rho \operatorname{ch} L = R_{\bar{1}} e^{\rho^\# - \rho} \cdot R_{\bar{0}} e^{\rho^\#} \operatorname{ch} L$ is $W(\pi)$-skew-invariant. □

10.2.3. Denote by $R_{L,\bar{0}}$, $R_{L,\bar{1}}(\Pi)$ the following analogues of $R_{\bar{0}}$, $R_{\bar{1}}(\Pi)$:

$$R_{L,\bar{0}} := \prod_{\alpha \in \Delta^+(L)_{\bar{0}}} (1 - e^{-\alpha}), \quad R_{L,\bar{1}}(\Pi) := \prod_{\alpha \in \Delta^+(L)_{\bar{1}}} (1 + e^{-\alpha}).$$

Note that these elements lie in $\mathscr{Y}$ (and $\mathscr{Y} \subset \mathscr{R}(\Pi')$ for each Π') and $R_L(\Pi) = R^{-1}_{L,\bar{1}}(\Pi) R_{L,\bar{0}}$.

Recall (see §2.2.6) that the elements $R(\Pi)e^{\rho_\Pi} \in \mathscr{R}(\Pi)$ are equivalent for all Π: the expansion of $R(\Pi)e^{\rho_\Pi}$ in $\mathscr{R}(\Pi)$ does not belong to $\mathscr{R}(\Pi')$, however, the expansion of $R(\Pi)e^{\rho_\Pi}$ in $\mathscr{R}(\Pi')$ coincides with the expansion of $R(\Pi')e^{\rho_{\Pi'}}$ in $\mathscr{R}(\Pi')$. Similarly, $R_L(\Pi)e^{\rho_{L,\Pi}}$ are equivalent for all Π. Hence the elements $R_{\bar{1}}(\Pi)e^{-\rho_\Pi}$, $R_{\bar{1},L}(\Pi)e^{-\rho_{L,\Pi}}$ are equivalent for all Π; since these elements lie in $\mathscr{Y}$, the expansion of $R_{\bar{1}}(\Pi)e^{-\rho_\Pi}$ (resp., of $R_{\bar{1},L}(\Pi)e^{-\rho_{L,\Pi}}$) in $\mathscr{R}(\Pi')$ is equal to the expansion of $R_{\bar{1}}(\Pi'')e^{-\rho_{\Pi''}}$ (resp., of $R_{\bar{1},L}(\Pi'')e^{-\rho_{L,\Pi''}}$) in $\mathscr{R}(\Pi''')$ for any sets of simple roots Π, Π', Π'', Π'''. This allows us to use the notation $R_{\bar{1},L}e^{-\rho_L}$ for $R_{\bar{1},L}(\Pi)e^{-\rho_{L,\Pi}}$.

10.2.4. **Lemma.** *For a set of simple roots* Π *let* $X(\Pi)$ *be the expansion of* $R_{\bar{1},L}Re^{\rho-\rho_L}\operatorname{ch} L$ *in* $\mathscr{R}(\Pi)$. *For any* Π *and* Π' *one has* $X(\Pi) = X(\Pi')$ *(in particular,* $X(\Pi) \in \mathscr{R}(\Pi')$*).*

Proof. By above, the elements $R_{\bar{1},L}Re^{\rho-\rho_L}\operatorname{ch} L$ are equivalent for all Π, so $X(\Pi)$ is equivalent to $X(\Pi')$. It remains to show that $X(\Pi) \in \mathscr{R}(\Pi')$.

Since any two sets of simple roots are connected by a chain of odd reflections, it is enough to consider the case $\Pi' = r_\beta\Pi$, where $\beta \in \Pi$ is an odd isotropic root. We denote by $R_{\bar{1},L}, R, \rho, \rho_L$ the corresponding elements for Π and set $\mathscr{R} := \mathscr{R}(\Pi), \mathscr{R}' := \mathscr{R}(r_\beta\Pi)$. Observe that $\operatorname{ch} L \in \mathscr{R} \cap \mathscr{R}'$, since L is an irreducible highest weight module.

If $\beta \in \Delta(L)$, then the element $R_{\bar{1},L}Re^{\rho-\rho_L}$ has the same expansion in $\mathscr{R}$ and in $\mathscr{R}'$, since

$$R_{\bar{1},L}Re^{\rho-\rho_{\bar{1},L}} = \prod_{\alpha\in\Delta_{\bar{0}}^+}(1-e^{-\alpha}) \cdot \prod_{\alpha\in\Delta_{\bar{1}}^+(\Pi)\setminus\Delta_{\bar{1}}(L)}(1+e^{-\alpha})^{-1}e^{\rho-\rho_{\bar{1},L}}$$

and $\Delta_{\bar{1}}^+(\Pi) \setminus \Delta_{\bar{1}}(L)$ lies in $\Delta^+(r_\beta\Pi) \cap \Delta^+(\Pi)$.

Consider the case $\beta \notin \Delta(L)$. Let us show that the expansion of $Re^\rho \operatorname{ch} L$ in $\mathscr{R}$ lies in $\mathscr{R}'$. Indeed, denote this expansion by Y. Since $\operatorname{ch} L \in \mathscr{R} \cap \mathscr{R}'$, Y has a pole of order ≤ 1 at β. By Lemma 2.2.9, in order to show that Y does not have a pole, i.e., that $Y \in \mathscr{R}'$, it is enough to verify that for each $\mu \in \mathfrak{h}^*$ the set $\operatorname{supp} Y \cap \{\mu + \mathbb{Z}\beta\}$ is finite. In fact this set contains at most one element, since the action of the Casimir element gives $\|\mu\|^2 = \|\mu + r\beta\|^2$ if $\mu, \mu + r\beta \in \operatorname{supp}(Y)$, so $(\mu, r\beta) = 0$. However, for $\mu \in \operatorname{supp} Y$ one has $\Delta(\mu) = \Delta(L)$, so $\beta \notin \Delta(\mu)$, that is $r = 0$. Hence $Y \in \mathscr{R}'$. Since the product $R_{\bar{1},L}e^{-\rho_L}$ lies in $\mathscr{R}$ and in $\mathscr{R}'$, we conclude that $X(\Pi) = R_{\bar{1},L}e^{-\rho_L}Y \in \mathscr{R}(\Pi')$. The assertion follows. □

10.2.5. By above, $X(\Pi)$ does not depend on Π (for fixed L). The next lemma shows that the equivalence relation $\sim$ preserves $X(\Pi)$.

Lemma. *Let $X(L)$ be the expansion of $R_{\bar{1},L}Re^{\rho-\rho_L}\operatorname{ch} L$ in $\mathscr{R}(\Pi)$. Recall that for $L\sim L'$ one has $F(L)\cong F(L')$ via the natural identification $\Delta(L)$ and $\Delta(L')$. Under this identification $X(L)=X(L')$.*

Proof. It is enough to verify the assertion for $L'=T_\alpha(a)(L)$, where $\alpha\in\Pi_0\setminus\Delta(L)$. Choose Π which contains α or $\frac{\alpha}{2}$. One has

$$Re^\rho\operatorname{ch} L(\lambda,\Pi')=\sum a_\nu e^\nu,\quad Re^\rho\operatorname{ch} L'=r_\alpha(Re^\rho\operatorname{ch} L).$$

By §8.2.2, $\Delta(L')=r_\alpha\Delta(L)$ and $\Pi(L')=r_\alpha\Pi(L)$. Thus

$$r_\alpha(R_{\bar{1},L})=R_{\bar{1},L'},\quad r_\alpha(\rho_L)=\rho_{L'}.$$

Hence $r_\alpha X(\Pi)$ is the expansion $R_{\bar{1},L'}Re^{\rho-\rho_{L'}}\operatorname{ch} L'$ in $\mathscr{R}(\Pi)$. □

10.2.6. Let L be π-relatively integrable ($\pi\subset\Pi_0(L)$). Assume that π admits a Weyl vector ρ_π ($(\rho_\pi,\alpha)=(\alpha,\alpha)/2$ for each $\alpha\in\pi$). By above, the element $R_{\bar{1},L}Re^{\rho+\rho_\pi-\rho_L}\operatorname{ch} L$ does not depend on Π (and have the same expansions in all algebras $\mathscr{R}(\Pi')$).

Proposition. *The element $R_{\bar{1},L}Re^{\rho+\rho_\pi-\rho_L}\operatorname{ch} L$ is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}$.*

Proof. Denote by $X(L)$ the expansion of $R_{\bar{1},L}Re^{\rho+\rho_\pi-\rho_L}\operatorname{ch} L$ in $\mathscr{R}(\Pi)$ (this does not depend on Π by Lemma 10.2.4). It is enough to verify that $X(L)e^{\rho_\pi}$ is $W(\gamma)$-skew-invariant element of $\mathscr{R}_{W(\gamma)}$ for each γ. We prove this by induction on $\gamma\in\Delta_{\bar{0}}^+$.

Assume first that $\gamma\in\Pi_0$. Take Π which contains γ or $\frac{\gamma}{2}$ and set $\mathscr{R}:=\mathscr{R}(\Pi)$. By Corollary 10.2.2, $R_{\bar{0}}e^{\rho^\#}\operatorname{ch} L$ is a W'-skew-invariant element of $\mathscr{R}_{W'}$. Clearly, $\Delta_{\bar{1}}^+\setminus\Delta_{\bar{1}}(L)^+$ is r_γ-invariant, so $R_{\bar{1},L}R_{\bar{1}}^{-1}$ is a $W(\gamma)$-invariant element of $\mathscr{R}_{W(\gamma)}$. Since γ or $\frac{\gamma}{2}$ lies in Π, $\rho-\rho_L$ is r_γ-invariant; since $\gamma\in\Pi_0$, $\rho^\#-\rho_\pi$ is r_γ-invariant. Hence $e^{\rho+\rho_\pi-\rho_L-\rho^\#}$ is a $W(\gamma)$-invariant element of $\mathscr{R}_{W(\gamma)}$, so $X(L)e^{\rho_\pi}$ is a $W(\gamma)$-skew-invariant element of $\mathscr{R}_{W(\gamma)}$ as well.

Now take $\gamma\notin\Pi_0$ and $\alpha\in\Pi_0$ such that $\gamma':=r_\alpha\gamma<\gamma$. Then $\alpha\notin\Delta(L)$ (see the proof of Lemma 10.2.1). Let $L':=T_\alpha(a)(L)$. By §10.1.2, we conclude that L' is $r_\alpha\pi$-relatively integrable. By induction $X(L')e^{\rho_{r_\alpha\pi}}$ is a $W(\gamma')$-skew-invariant element of $\mathscr{R}_{W(\gamma')}$. Clearly, $\rho_{r_\alpha\pi}$ can be chosen equal to $r_\alpha\rho_\pi$. Moreover, by Lemma 10.2.5, $X(L')=r_\alpha(X(L))$. Then $X(L)e^{\rho_\pi}=r_\alpha(X(L')e^{\rho_{r_\alpha\pi}})$ and thus $X(L)e^{\rho_\pi}$ is a $W(\gamma)$-skew-invariant element of $\mathscr{R}_{W(\gamma)}$, as required. □

11. Character formulas for some typical and relatively integrable modules

In this section we prove formula (35) from §7.7.2 for some cases.

11.1.

Recall that a module $L(\lambda, \Pi)$ is typical if $(\lambda + \rho, \beta) \neq 0$ for all isotropic $\beta \in \Delta_{\bar{1}}$.

Recall (see §9) that we call $\lambda \in \mathfrak{h}^*$ *extremal* if for each connected component π of $\Pi_0(\lambda)$, λ is maximal or minimal in its $W(\pi)$-orbit. We say that λ is regular if $Stab_W \lambda = \{Id\}$.

11.1.1. **Theorem.** *If $L = L(\lambda, \Pi)$ is typical and $\lambda + \rho$ is a regular extremal weight, then*

$$Re^{\rho} \operatorname{ch} L = \sum_{\{w \in W(L) | w(\lambda+\rho) \leq \lambda+\rho\}} \operatorname{sgn}(w) e^{w(\lambda+\rho)}$$

and (35) holds.

11.1.2. Recall that $\Pi_0(L)$ is the set of simple roots of $\Delta(L) \cap \Delta_0^+$.

Corollary. *If $L = L(\lambda, \Pi)$ is a typical module and $F(L)$ is $\Pi_0(L)$-integrable, then*

$$Re^{\rho} \operatorname{ch} L = \sum_{w \in W(L)} \operatorname{sgn}(w) e^{w(\lambda+\rho)}$$

and (35) holds.

The above theorem admits the following generalization.

11.1.3. **Theorem.** *Let $L = L(\lambda, \Pi)$ is a typical module. Set*

$$\pi := \{\alpha \in \Pi_0(L) \mid (\lambda + \rho, \alpha^{\vee}) > 0\}$$

(that is $\pi \subset \Pi_0(L)$ is maximal such that $F(L)$ is π-integrable). Write $\pi = \pi_f \coprod \pi_{aff}$, where π_f (resp., π_{aff}) is the union of connected finite (resp., affine) type diagrams in π. Assume that for each $\alpha \in \Pi_0(L) \setminus \pi$ one has

$$(\lambda + \rho, w_0 \alpha^{\vee}) < 0,$$

where w_0 is the product of the longest elements in $W(\pi_f)$. Then

$$Re^{\rho} \operatorname{ch} L = \sum_{w \in W(\pi)} \operatorname{sgn}(w) e^{w(\lambda+\rho)}.$$

11.1.4. Remark. We do not expect Theorem 11.1.3 to hold in general. Namely, the coefficients of the character formula may involve non-trivial Kazhdan–Lusztig polynomials.

11.2. Proof of Theorems 11.1.1, 11.1.3 and Corollary 11.1.2

First note that Theorem 11.1.1 is a particular case of Theorem 11.1.3: if λ is extremal, then π is a union of connected components of $\Pi_0(L)$ and each root $\alpha \in \Pi_0(L) \setminus \pi$ lies in a connected component $\tilde{\pi} \subset \Pi_0(L)$ such that λ is a minimal element in its $W(\tilde{\pi})$-orbit. One has $(\lambda + \rho, w\alpha^\vee) = (\lambda + \rho, \alpha^\vee)$ for each $w \in W(\pi)$; if λ is regular, then $(\lambda + \rho, \alpha^\vee) < 0$. Thus a regular typical extremal weight satisfies the assumptions of Theorem 11.1.3.

Now let us deduce Corollary 11.1.2 from Theorem 11.1.1. First, consider the case when $L' = L(\lambda', \Pi)$ is a typical module and $\mathfrak{g}_{\pm\alpha}$ acts nilpotently on L' for some $\alpha \in \Pi_0$. We claim that $(\lambda' + \rho, \alpha^\vee) \in \mathbb{Z}_{>0}$. Indeed, choosing Π' which contains α or $\alpha/2$ we obtain $L' = L(\lambda' + \rho - \rho', \Pi')$ and $(\lambda' + \rho - \rho', \alpha^\vee) \geq 0$; since $(\rho', \alpha^\vee) > 0$, we get $(\lambda' + \rho, \alpha^\vee) > 0$. Since $\mathfrak{g}_{\pm\alpha}$ acts locally nilpotently on L', $\alpha \in \Delta(L)$, so $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}_{>0}$, as required.

Next let $L = L(\lambda, \Pi)$ be a typical module such that $F(L)$ is $\Pi_0(L)$-integrable. By above, $(\lambda + \rho, \alpha^\vee) = (\overline{\lambda} + \rho_L, \alpha^\vee) \in \mathbb{Z}_{>0}$ for each $\alpha \in \Pi_0(L)$. Hence $\lambda + \rho$ is regular and extremal. Hence Corollary 11.1.2 follows from Theorem 11.1.1.

11.2.1. Proof of Theorem 11.1.3. We claim that

$$\operatorname{supp}(Re^\rho \operatorname{ch} L) \subset W(\pi)(\lambda + \rho).$$

Indeed, by Proposition 7.3.1 it is enough to verify that for each $w \in W(\pi)$ if $(w \cdot \lambda + \rho, \gamma^\vee) > 0$ for some non-isotropic even positive root γ, then $\gamma \in \Delta(\pi)$. One has $(w \cdot \lambda + \rho, \gamma^\vee) = (w_0\lambda + \rho, (w_0 w^{-1}\gamma)^\vee)$. Since $\gamma \notin \Delta(\pi)$, the root $w_0 w^{-1}\gamma$ lies in $\Delta^+ \setminus \Delta(\pi)$. By the assumptions, $(w_0 \cdot \lambda + \rho, \alpha^\vee) < 0$ for $\alpha \in \Pi_0(L) \setminus \pi$ and for $\alpha \in \pi_f$; therefore this inequality holds for $\alpha \in \Pi_0(L) \setminus \pi_{aff}$ and thus for a positive non-isotropic even root α which does not lie in $\Delta(\pi_{aff})$. Hence $(w \cdot \lambda + \rho, \gamma^\vee) < 0$, as required.

By Proposition 3.12 in [K3], $\lambda + \rho$ is a unique π-maximal element in its $W(\pi)$-orbit (by definition of π).

We claim that $F(L)$ is π-integrable. Indeed, take $\alpha \in \pi$ and let Π' be a subset of simple roots for $\Delta(L)$ such that α or $\alpha/2$ lies in Π'. Since $F(L)$ is typical, $F(L) = L(\lambda', \Pi')$, where $\lambda + \rho = \lambda' + \rho'$, so $(\lambda' + \rho', \alpha^\vee) > 0$. Since $\alpha \in \Delta(L)$, we conclude that $\mathfrak{g}_{\lambda,\pm\alpha}$ acts locally nilpotently on $F(L)$. Hence $F(L)$ is π-integrable.

Using Corollary 10.2.2, we conclude that $Re^\rho \operatorname{ch} L$ is $W(\pi)$-skew-invariant. Therefore $Re^\rho \operatorname{ch} L \in \mathscr{R}_{W(\pi)}$ and so

$$Re^\rho \operatorname{ch} L = c\mathscr{F}_{W(L)} e^{(\lambda+\rho)}.$$

Since the coefficient of $e^{\lambda+\rho}$ in $Re^\rho \operatorname{ch} L$ is 1, $c = 1$, as required. □

11.2.2. **Theorem.** *Let L be a non-critical module such that $(F(L), \Pi(L))$ satisfies the conditions of §4 and $S \subset \Pi$. Then (35) holds.*

11.2.3. **Theorem.** *Let L be a non-critical vacuum module such that $\Delta(L)$ is affine and $F(L)$ is integrable. Then (35) holds.*

11.2.4. **Corollary.** *If $\Delta(L)_{\overline{0}}$ is connected (i.e., $\Delta(L) = A(0,m), B(0,n), C(n)$, their untwisted affinizations, or $A(0,2n-1)^{(2)}, C(n+1)^{(2)}, A(0,2n)^{(4)}$) and $F(L)$ is $\Pi_0(L)$-integrable, then (35) holds.*

In particular, by the results of §4, we obtain the character formula for all admissible modules over $A(0,n)^{(1)}, C(n)^{(1)}$.

11.2.5. **Theorem.** *Let L be a non-critical $\mathfrak{g}$-module such that for each connected component Δ^i of $\Delta(L)$ one has either $(F(L), \Delta^i) = 0$ or $F(L)$ is $\Delta^i_{\overline{0}}$-integrable and Δ^i-typical (i.e., $(F(L), \beta) \neq 0$ for each $\beta \in \Delta^i$). Then (35) holds.*

11.2.6. **Corollary.** *Let L be a non-critical $\mathfrak{g}$-module such that* $\dim F(L) = 1$. *Then (35) holds.*

11.3. The case when $F(L)$ is a vacuum module

Consider the case when $F(L)$ is a vacuum module (i.e., $F(L) = L(\overline{\lambda}, \Pi(L))$, where $(\overline{\lambda}, \dot{\Pi}(L)) = 0$ for some finite part $\dot{\Pi}(L)$ of $\Pi(L)$), which is integrable (i.e., is π-integrable for a connected component π of $\Pi_0(L)$, see §3.1.3). We denote by α_0 the affine root in $\Pi(L)$, i.e.,

$$\Pi(L) = \dot{\Pi}(L) \cup \{\alpha_0\}.$$

Normalize the bilinear form in such a way that $\|\alpha\|^2 \in \mathbb{Q}_{>0}$ for $\alpha \in \pi$.

The following theorems improve the result of Theorem 11.2.2 in the case when $F(L)$ is an integrable vacuum module with $\Delta(L) \neq A(n,n)^{(1)}$.

11.3.1. **Theorem.** *Let $F(L)$ be an integrable vacuum module such that the dual Coxeter number of $\Delta(L)$ is non-zero.*

Assume that $\Pi(L)$ is such that $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi(L)$; if $\Pi(L) = C(m)$ or $\Pi(L) = A(m,n)^{(1)}, m > n$, assume, in addition, that the affine root in $\Pi(L)$ is not isotropic. Then (35) holds.

11.3.2. **Theorem.** *Let $F(L)$ be a non-critical integrable vacuum module.*

If $\Delta(L) = D(2,1,a)^{(1)}$ and $a \neq -\frac{1}{2}, -2$, then (35) holds for any $\Pi(L)$; for $\Delta(L) = D(2,1,-\frac{1}{2})^{(1)} = D(2,1,-2)^{(1)}$, (35) holds if $\|\alpha_0\|^2 \geq 0$.

If $\Delta(L) = A(2n-1,2n-1)^{(2)}, D(n+1,n)^{(1)}$ and $\Pi(L)$ is as in §6.4.1, then (35) holds.

If $\Delta(L) = A(2n,2n)^{(4)}, D(n+1,n)^{(2)}$ with $\Pi(L)$ is as in §6.4.2 and the level of $F(L)$ is not 1, then (35) holds.

If $\Delta(L) = A(2n,2n)^{(4)}, D(n+1,n)^{(2)}$ with $\Pi(L)$ is as in §6.4.2 and the level of $F(L)$ is 1, then (35) holds if $\dot{\Pi}(L) \subset \Pi$.

11.3.3. For each $\alpha \in \dot{\pi}$ we write $\alpha = \sum_{\beta \in \Pi(L)} x_{\alpha,\beta}\beta$, and let $\text{supp}(\alpha) := \{\beta \in \Pi \mid x_{\alpha,\beta} \neq 0\}$. Consider the following conditions on a set of simple roots $\Pi(L)$:

(A) $\|\alpha_0\|^2 \geq 0$;

(B) for each $\alpha \in \pi$ there exists $\beta \in \text{supp}(\alpha)$ such that $\beta \notin \text{supp}(\alpha')$ for each $\alpha' \in \pi$; this β is denoted by $b(\alpha)$;

(C) $\rho_L \in X_1 - X_2$, where

$$X_1 := \{\mu \in \mathfrak{h}^* \mid (\mu, \alpha) \in \mathbb{Q}_{\geq 0} \text{ for all } \alpha \in \Pi(L)\},$$
$$X_2 := \sum_{\alpha \in \dot{\Pi}(L)_0} \mathbb{Q}_{\geq 0}\alpha^\vee.$$

It is easy to see that (B) holds for each Π if $\Delta \neq F(4)^{(1)}$, see §13.5. Note that (C) holds if $\rho_L = 0$ and if $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi(L)$ (in this case $\rho_L \in X_1$). We give examples of sets of simple roots $\Pi(L)$ satisfying (A), (B), (C) in §13.5.

11.3.4. **Theorem.** *Let $F(L)$ be a non-critical integrable vacuum module such that $\Delta(L) \neq A(m,n)^{(1)}, C(n)^{(1)}$. If $\Pi(L)$ satisfies the conditions (A)–(C), then (35) holds.*

11.4. Proofs

Step 1. Set

$$Z := Re^{\rho}\,\text{ch}\,L - R_L e^{\rho_L}\,\text{ch}\,F(L) = 0, \quad Z' := R_{\overline{1},L} e^{-\rho_L} Z.$$

We have to prove that $Z' = 0$. Suppose that $Z' \neq 0$.

Denote by λ_Π (resp., $\overline{\lambda}_\Pi$) the highest weight of L (resp., of $\overline{L}$); recall that $\lambda_\Pi + \rho_\Pi = \overline{\lambda}_\Pi + \rho_{L,\Pi}$. By Corollary 7.3.4, $\text{supp}(Z) \subset \overline{\lambda}_\Pi + \rho_{L,\Pi} - \mathbb{Z}_{\geq 0}\Delta^+(L)$, so for each Π

$$\text{supp}(Z') \subset \overline{\lambda}_\Pi - \mathbb{Z}_{\geq 0}\Pi(L). \tag{38}$$

Clearly, $\lambda_\Pi + \rho_\Pi \not\in \mathrm{supp}(Z)$, so $\overline{\lambda}_\Pi \not\in \mathrm{supp}(Z')$.

Let $F(L)$ be π-integrable, where $\pi \subset \Pi_0(L)$ is connected. Since π is connected, it admits a Weyl vector $\rho_\pi \in \mathfrak{h}^*$ ($(\rho_\pi, \alpha^\vee) = 1$ for each $\alpha \in \pi$). By Proposition 10.2.6, for each $\pi \subset \Pi_0(L)$ the element $e^{\rho_\pi} Z'$ is a $W(\pi)$-skew-invariant element of $\mathscr{R}_{W(\pi)}$. Therefore $\mathrm{supp}(e^{\rho_\pi} Z')$ is a union of regular $W(\pi)$-orbits (the regularity means that the stabilizer of each element is trivial). From (38) it follows that each orbit has a π-maximal element (i.e., maximal with respect to the following order: $\nu' \geq_\pi \nu''$ if $\nu' - \nu'' \in \mathbb{Z}_{\geq 0}\pi$). Let $\mu = \overline{\lambda}_\Pi - \nu$ be a π-maximal element in its orbit. Using the regularity of the orbit we obtain $(\overline{\lambda}_\Pi - \nu, \alpha^\vee) \geq 0$ for each $\alpha \in \pi$.

Combining maximality of ν and regularity of the orbit we obtain

$$(\overline{\lambda}_\Pi - \nu, \alpha^\vee) \geq 0 \quad \text{for each } \alpha \in \pi. \tag{39}$$

Now let $\overline{\lambda}_\Pi + \rho_L - \nu$ be a maximal element in $\mathrm{supp}(Z)$ with respect to the order $\nu' \geq \nu''$ if $\nu' - \nu'' \in \mathbb{Z}_{\geq 0}\Delta^+$. Clearly, $\nu \neq 0$. Then $\overline{\lambda}_\Pi - \nu$ is a maximal element in $\mathrm{supp}(Z')$ with respect to the same order and so this is a $\Pi_0(L)$-maximal element in $\mathrm{supp}(Z')$. Since $\overline{\lambda}_\Pi + \rho_L - \nu \in \mathrm{supp}(Z)$ we have $2(\overline{\lambda}_\Pi + \rho_L, \nu) = (\nu, \nu)$. Combining with (38) we get

$$2(\overline{\lambda}_\Pi + \rho_L, \nu) = (\nu, \nu), \quad \nu \in \mathbb{Z}_{\geq 0}\Pi(L), \quad \nu \neq 0. \tag{40}$$

We will show that (39) contradicts (40).

11.5. Proofs of Theorems 11.2.2, 11.2.3 and Corollary 11.2.4

11.5.1. Proof of Theorem 11.2.2. Arguing as in §4.3.2 we deduce from (39) and (40) that $\nu \in \mathbb{Z}_{\geq 0}S$. Write $\nu = \sum_{\beta \in S} x_\beta \beta$, $x_\beta \geq 0$. Let β be such that $x_\beta \neq 0$. By Lemma 10.2.4, Z' does not depend on the choice of Π, so $\mathrm{supp}(Z') \subset \overline{\lambda}_{\Pi'} - \mathbb{Z}_{\geq 0}\Pi'(L)$ for any Π'. In particular, $\overline{\lambda}_\Pi - \nu \in \overline{\lambda}_{\Pi'} - \mathbb{Z}_{\geq 0}\Pi'(L)$ for any Π'. For $\Pi' = r_\beta \Pi$ we have $\overline{\lambda}_{\Pi'} = \overline{\lambda}_\Pi$, so $\nu \in \mathbb{Z}_{\geq 0}\Pi'(L)$. However, $-\beta \in \Pi'(L)$ and $S \setminus \{\beta\} \in \Pi'(L)$, a contradiction. □

11.5.2. Proof of Theorem 11.2.3. Recall that L is a vacuum module means that $L = L(\lambda, \Pi)$, where $(\lambda, \dot{\Delta}) = 0$ for some finite part $\dot{\Delta}$ of Δ. In particular, $\dot{\Delta} \subset \Delta(L)$ and the inclusion is strict, since $\Delta(L)$ is affine. Note that the elements of $\dot{\Pi}$ are indecomposable in $\Delta(L)^+ = \Delta^+ \cap \Delta(L)$, so $\dot{\Pi} \subset \Pi(L)$. Hence $\dot{\Pi}$ (resp., $\dot{\Delta}$) is a finite part of $\Pi(L)$ (resp., of $\Delta(L)$).

Consider $\Delta' := \Delta(L)$ with a set of simple roots $\Pi' := \Pi(L)$ and a finite part $\dot{\Delta}$.

If the dual Coxeter number of Δ' is non-zero or $\Delta' = A(n,n)^{(1)}$, then, by §13.2, there exists a chain of odd reflections with respect to the roots in $\dot{\Delta}$ which transform Π' to a set of simple roots Π'' with the following property: for each $\alpha \in \Pi''$ one has $\|\alpha\|^2 \geq 0$ (where $(-,-)$ is normalized in such a way that $(\rho', \delta) \geq 0$). Acting by the same chain of odd reflections of Π, we obtain a set of simple roots $\tilde{\Pi}$ for Δ such that $\tilde{\Pi}(L) = \Pi''$. Clearly, $L = L(\tilde{\lambda}, \tilde{\Pi})$, where $(\tilde{\lambda}, \dot{\Delta}) = 0$. Hence $(F(L), \tilde{\Pi}(L))$ satisfies the conditions of §4 and $S \subset \Pi$. By Theorem 11.2.2, (35) holds.

Similarly, if the dual Coxeter number of Δ' is zero and $\Delta' \neq A(n,n)^{(1)}$, then, by §13.2, there exists a chain of odd reflections with respect to the roots in $\dot{\Delta}$ which transform Π' to a set of simple roots Π'' given in §6.1, §6.4.1, §6.4.2, respectively. Acting by the same chain of odd reflections of Π, we obtain a set of simple roots $\tilde{\Pi}$ for Δ such that $\tilde{\Pi}(L) = \Pi''$. One has $L = L(\tilde{\lambda}, \tilde{\Pi})$, where $(\tilde{\lambda}, \dot{\Delta}) = 0$ and $F(L)$ is integrable. In this case the statement follows from Theorem 11.3.2. □

11.5.3. Proof of Corollary 11.2.4. If L is typical, the assertion follows from Corollary 11.1.2. Assume that L is not typical.

In the light of Theorem 11.2.2 it is enough to verify that there exists $L' \sim L$ and Π such that $(\lambda'_\Pi + \rho_\Pi, \beta) = 0$ for some $\beta \in \Pi$, where $L' = L(\lambda'_\Pi, \Pi)$. Assume that this is not the case. Then for any L' one has $\lambda'_\Pi + \rho_\Pi = \lambda'_{\Pi'} + \rho_{\Pi'}$ for all Π, Π'. Fix any Π and let $\beta \in \Delta(L)$ be such that $(\lambda_\Pi + \rho_\Pi, \beta) = 0$.

First, consider the case when $\beta \in \Pi'$ for some Π'. Then $\lambda_\Pi + \rho_\Pi = \lambda_{\Pi'} + \rho_{\Pi'}$ implies that $(\lambda_{\Pi'} + \rho_{\Pi'}, \beta) = 0$, so the assertion holds for $L' = L$ and $\beta \in \Pi'$.

Assume that $\beta \notin \Pi'$ for any Π'. Then $\Delta \neq A(m,n), C(n), A(m,n)^{(1)}$, $C(n)^{(1)}$. In particular, Π_0 is not connected. Let π be a connected component of Π_0 such that $\Delta(\pi) \cap \Delta(L) = \emptyset$. There exists $w \in W(\pi)$ and Π' such that $w\beta \in \Pi'$ (by Lemma 13.4 the stronger assertion holds). Write w as a product of simple reflections and let T be the product of the corresponding Enright functors. Then $w\beta \in \Delta(T(L))$ and $T(L) = L(w(\lambda_\Pi + \rho_\Pi) - \rho_\Pi, \Pi)$, so the assertion holds for $L' = T(L)$ and $w\beta \in \Pi'$. □

11.6. Proofs for vacuum cases

Since $F(L)$ is a vacuum module, $F(L)$ is not only π-integrable, but also $\dot{\Pi}_0(L)$-integrable, so $(\overline{\lambda}_\Pi - \nu, \alpha^\vee) \geq 0$ for each $\alpha \in \dot{\Pi}_0(L)$. For $\alpha \in \dot{\Pi}(L)$ one has $(\overline{\lambda}_\Pi, \alpha^\vee) = 0$, so we obtain

(41) $$(\nu, \alpha^\vee) \leq 0 \quad \text{for each } \alpha \in \dot{\Pi}_0(L).$$

We will frequently use the following statement, which is a part of Theorem 4.3 in [K3]:

(Fin) If A is a Cartan matrix of a semisimple Lie algebra and v is a vector with rational coordinates, then $Av \geq 0$ implies $v > 0$ or $v = 0$;

(Aff) If A is a Cartan matrix of an affine Lie algebra and v is a vector with rational coordinates, then $Av \geq 0$ implies $v \in \mathbb{Q}\delta$.

11.6.1. Proof of Theorem 11.3.1. Arguing as in §11.5.1 we obtain that for any $S \subset \dot{\Pi}(L)$ satisfying the conditions in §3.3 one has $\nu \in \mathbb{Z}_{\geq 0}S$, $\nu \neq 0$. In particular, $\nu \in \mathbb{Z}_{\geq 0}\dot{\Pi}$.

Consider the case $\dot{\Delta}(L) \neq A(m,n), C(n)$, i.e., $\mathbb{Q}\dot{\Delta}_{\overline{0}} = \mathbb{Q}\dot{\Delta}$. Combining (41) and (Fin) we obtain $\nu \in -\mathbb{Q}_{\geq 0}\dot{\Delta}^{+}_{\overline{0}}$. Since $\nu \in \mathbb{Z}_{\geq 0}\Delta^{+}$ we get $\nu = 0$, a contradiction.

If $\Pi(L) = C(m)$ or $\Pi(L) = A(m,n)^{(1)}, m > n$, then using the condition that affine root is not isotropic, we obtain that the set of isotropic roots $Iso \subset \Pi(L)$ lies in $\dot{\Pi}(L)$. It is easy to see that the condition $\|\alpha\|^2 \geq 0$ implies that each connected component of *Iso* is of type $A(n+1,n)$ (and not of type $A(n,n)$). By above, it is enough to verify that for some S,

$$\mathbb{Z}_{\geq 0}S \cap \{\mu \mid \forall\alpha \in \dot{\Pi}_0(L)\ (\mu,\alpha^\vee) \leq 0\} = \{0\}.$$

Clearly, it is enough to check the assertion for each connected component of $Iso \subset \dot{\Pi}(L)$. Retain notation of §3.3.1 and choose $S = \{\varepsilon_i - \delta_i\}$ in each component of *Iso*. Write $\nu = \sum k_i(\varepsilon_i - \delta_i)$. Taking $\alpha = \varepsilon_i - \varepsilon_{i+1}, \delta_i - \delta_{i+1} \in \Pi_0$, we obtain $k_i = k_{i+1}$ for each i, that is $\nu = k\sum_{i=1}^{n}(\varepsilon_i - \delta_i)$ for some $k \geq 0$. Then $(\nu, (\varepsilon_n - \varepsilon_{n+1})^\vee) = k$, so $k \leq 0$, that is $\nu = 0$, a contradiction. □

11.6.2. Proof of Theorem 11.3.2. Combining (40) and (41), we obtain that ν satisfies the formulas (27). Then arguing as in §6.4.1 (resp., §6.4.2, §6.1) we get the assertion for $A(2n-1,2n-1)^{(2)}$, $D(n+1,n)^{(1)}$ (resp., $A(2n,2n)^{(4)}$, $D(n+1,n)^{(2)}$ and $D(2,1,a)^{(1)}$); the restriction that $\dot{\Pi}(L) \subset \Pi$ for $\Delta(L) = A(2n,2n)^{(4)}, D(n+1,n)^{(2)}$ with level 1, comes from the use of odd reflections in the proof of this case.

Now consider the remaining case $\Delta(L) = \mathfrak{g} = D(2,1,a)^{(1)}$. Recall that $a \neq 0, -1$ and $D(2,1,a) \cong D(2,1,a^{-1}) \cong D(2,1,-1-a)$; if $a \in \mathbb{Q}$, we assume (without loss of generality) that $-1 < a < 0$.

One has $D(2,1,a)_{\overline{0}} = A_1 \times A_1 \times A_1$; if we denote the root in ith copy of A_1 by $2\varepsilon_i$, then $\|2\varepsilon_1\|^2 : \|2\varepsilon_2\|^2 : \|2\varepsilon_3\|^2 = 1 : a : (-a-1)$.

Let $L(\lambda)$ be a π-integrable vacuum module of level k for some $\pi \subset \Pi_0$. If $\pi \setminus \dot{\Pi}_0$ contains one root, then $\pi = \{\delta - 2\varepsilon_r, \varepsilon_r\}$ and $L(\lambda)$ is π-integrable if and only if $2k/\|2\varepsilon_r\|^2 \in \mathbb{Z}_{\geq 0}$. If $\pi \setminus \dot{\Pi}_0$ contains two roots, then $\pi = \{\delta - 2\varepsilon_r, \delta - 2\varepsilon_q, 2\varepsilon_r, 2\varepsilon_q\}$, and, by above, $L(\lambda)$ is π-integrable if and only if $2k/\|2\varepsilon_r\|^2$, $2k/\|2\varepsilon_q\|^2 \in \mathbb{Z}_{\geq 0}$; in particular, if $k \neq 0$, then $\|2\varepsilon_r\|^2/\|2\varepsilon_q\|^2 \in \mathbb{Q}_{>0}$, so $a \in \mathbb{Q}$ and, since $-1 < a < 0$, $r, q = 2, 3$. If $\pi \setminus \dot{\Pi}_0$ contains three roots, then $\pi = \Pi_0$ and $k = 0$.

We see that $L(\lambda)$ with $k \neq 0$ can be $A_1^{(1)}$-integrable for any copy $A_1^{(1)}$ in Π_0, but it is $A_1^{(1)} \times A_1^{(1)}$-integrable only if $a \in \mathbb{Q}$ and the roots of π have positive integral square length for some normalization of $(-,-)$.

Recall that $\pi = \{\alpha \in \Pi_0 \mid \|\alpha\|^2 \in \mathbb{Q}_{>0}\}$ for some normalization of $(-,-)$. If $a \notin \mathbb{Q}$,then π can be any copy of $A_1^{(1)}$. If $a \in \mathbb{Q}$, then either $\pi = A_1^{(1)}$, which corresponds to the longest root (the absolute value of $\|2\varepsilon_i\|^2$ is maximal; this is $2\varepsilon_1$ if $-1 < a < 0$), or $\pi = A_1^{(1)} \times A_1^{(1)}$ (and then $\dot{\pi} = \{2\varepsilon_2, 2\varepsilon_3\}$, by above).

Let us show that (40) contradicts (39).

Recall that there are 4 sets of simple roots:

$$\Pi_1 = \{\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3, -\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2 + \varepsilon_3\},$$

and $\Pi_2 := \{\delta - 2\varepsilon_1, \varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2, 2\varepsilon_3\}$, with similar Π_3, Π_4 ($\delta - 2\varepsilon_i \in \Pi_{i+1}$).

Take Π_s. Write $\mu = j\delta - \sum_{i=1}^{3} e_i\varepsilon_i$. By (39), $e_1, e_2, e_3 \geq 0$. By (40), $\mu \in \mathbb{Z}_{\geq 0}\Pi_i$; in all 4 cases, using $e_1, e_2, e_3 \geq 0$, we get $e_1, e_2, e_3 \leq 2j$. Hence

$$(42) \qquad 0 \leq e_i \leq 2j \quad \text{for } i = 1, 2, 3.$$

In particular, $\mu = 0$ if $j = 0$. Since $\mu = 0$ contradicts (40), we assume that $j > 0$.

Consider the case $\pi = \{2\varepsilon_1, \delta - 2\varepsilon_1\}$. Normalize $(-,-)$ by $\|2\varepsilon_1\|^2 = 2$ (then $\|2\varepsilon_2\|^2 = 2a,\ \|2\varepsilon_3\|^2 = -2(a+1)$). Recall that $k \in \mathbb{Z}_{>0}$. By (39) one has $e_1 \leq k$ and $\|\mu\|^2 = 2(\lambda + \rho, \mu)$, that is $e_1^2 + ae_2^2 - (a+1)e_3^2 = 4jk$. If $a \notin \mathbb{Q}$, then we get $e_2 = e_3$ and $e_1^2 - e_3^2 = 4jk$; if $a \in \mathbb{Q}$, then, by our assumption, $-1 < a < 0$ and we obtain $e_1^2 \geq 4jk$. Combining with $e_1 \leq k$ with (42), we get $j = 0$, a contradiction.

Consider the case $\pi = \{\delta - 2\varepsilon_2, \delta - 2\varepsilon_3, 2\varepsilon_2, 2\varepsilon_3\}$ (with $-1 < a < 0$). Normalize $(-,-)$ by $\|2\varepsilon_2\|^2 = 2$ (then $\|2\varepsilon_1\|^2 = 2/a,\ \|2\varepsilon_3\|^2 = -2(a+1)/a$). Recall that $k, -ka/(a+1) \in \mathbb{Z}_{>0}$. By (39) one has $e_2 \leq k, e_3 \leq -ka/(a+1)$ and $\|\mu\|^2 = 2(\lambda + \rho, \mu)$, that is

$$e_1^2/a + e_2^2 - \frac{a+1}{a}e_3^2 = 4jk.$$

Recall that $j > 0$. By (42), $e_2, e_3 \leq 2j$, so we obtain $e_1 = 0, e_2 = 2j = k, e_3 = 2j = -ka/(a+1)$. If $a \neq -\frac{1}{2}$ this is impossible. For $a = -\frac{1}{2}$ we get $\mu = j(\delta - 2\varepsilon_2 - 2\varepsilon_3)$, which does not lie in $\mathbb{Z}_{\geq 0}\Pi_s$ for $s = 1, 3, 4$. Hence (40) contradicts (39) for Π_i with $i = 1, 3, 4$. □

11.6.3. Proof of Theorem 11.3.4. By (40), $2(\overline{\lambda}_\Pi + \rho_L, \nu) = \|\nu\|^2$. The condition (C) implies $\rho_L = \xi_1 - \xi_2$, where $(\xi_1, \nu) \geq 0$ (since $\nu \in \mathbb{Z}_{\geq 0}\Pi(L)$) and $(\xi_2, \nu) \leq 0$ (by (41)). Hence $(\rho_L, \nu) \geq 0$, that is

$$2(\overline{\lambda}_\Pi, \nu) \leq \|\nu\|^2.$$

Let $s\delta$ be the minimal imaginary root in $\Delta(L)$ (δ is the minimal imaginary root in Δ). Write $\dot{\Pi}(L)_0 = \dot{\pi} \cup \pi''$ and recall that $\|\alpha\|^2 \not\in \mathbb{Q}_{>0}$ for $\alpha \in \pi''$.

Write $\nu = j(s\delta) - \nu' - \nu''$, where $\nu' \in \mathbb{Q}\dot{\pi},\ \nu'' \in \mathbb{Q}\pi''$. From (Fin) (see §11.6) we deduce from the condition (41) that

$$\nu' \in \mathbb{Q}_{\geq 0}\dot{\pi}, \quad \nu'' \in \mathbb{Q}_{\geq 0}\pi''.$$

Note that $j(s\delta)-\nu' = \nu+\nu'' \in \mathbb{Z}_{\geq 0}\Pi(L)$ because $\pi'' \subset \mathbb{Z}_{\geq 0}\Pi(L)$. Therefore $j(s\delta) - \nu' \in \mathbb{Q}\pi \cap \mathbb{Z}_{\geq 0}\Pi(L)$.

We claim that $j(s\delta) - \nu' \in \mathbb{Q}_{\geq 0}\pi$. Indeed, write $\pi = \dot{\pi} \cup \{\alpha_0^\sharp\}$. One has

$$j(s\delta) - \nu' = j\alpha_0^\sharp + \sum_{\alpha\in\dot{\pi}} y_\alpha \alpha = j \sum_{\beta\in\text{supp}(\alpha_0^\sharp)} x_{\alpha_0^\sharp,\beta} + \sum_{\alpha\in\dot{\pi}} \sum_{\beta\in\text{supp}(\alpha)} y_\alpha x_{\alpha,\beta}\beta.$$

In the light of assumption (B), the coefficient of the root $b(\alpha)$ is equal to $y_\alpha x_{\alpha,b(\alpha)}$; thus $j(s\delta) - \nu \in \mathbb{Z}_{\geq 0}\Pi(L)$ gives $y_\alpha x_{b(\alpha)} \geq 0$, so $y_\alpha \geq 0$. Hence $j(s\delta) - \nu' \in \mathbb{Q}_{\geq 0}\pi$, as required.

Set $k := (\overline{\lambda}, s\delta)$. Then $k = (\overline{\lambda}, \alpha_0^\sharp)$, so $(\nu, \alpha_0^\sharp) \leq k$ by (39). Using (41) we get

$$\begin{aligned}
&\|\nu\|^2 \\
=&\|j(s\delta) - \nu'\|^2 + \|\nu''\|^2 \leq \|j(s\delta) - \nu'\|^2 \\
=&j(j(s\delta) - \nu', \alpha_0^\sharp) + \sum_{\alpha\in\dot{\pi}} y_\alpha (j(s\delta) - \nu', \alpha) \\
=&j(\nu, \alpha_0^\sharp) + \sum_{\alpha\in\dot{\pi}} \sum y_\alpha (\nu, \alpha) \leq jk.
\end{aligned}$$

Since $2(\overline{\lambda}, \nu) = 2jk$ we obtain $j = 0$, so $\nu = -\nu' - \nu''$. Since $\nu \in \mathbb{Z}_{\geq 0}\Pi(L)$ with $\nu' \in \mathbb{Q}_{\geq 0}\dot{\pi}^\sharp,\ \nu'' \in \mathbb{Q}_{\geq 0}\dot{\pi}''$ we get $\nu = 0$, a contradiction. □

11.7. Proof of Theorem 11.2.5

Let L be a non-critical $\mathfrak{g}$-module such that for each connected component Δ^i of $\Delta(L)$ one has either $(F(L), \Delta^i) = 0$ or $F(L)$ is $\Delta^i_{\overline{0}}$-integrable and Δ^i-typical (i.e., $(F(L), \beta) \neq 0$ for each $\beta \in \Delta^i$).

We want to prove that (35) holds.

By Lemma 10.2.4, Z' does not depend on the choice of Π and is preserved by $\sim$ ($Z'(L) = Z'(L')$ if $L' \sim L$).

Decompose $\Delta(L)$ in the union of irreducible components $\Delta(L) = \coprod \Delta^j$. By (38), we can decompose $\overline{\lambda}_\Pi - \mu = \sum \mu_j$ with $\mu_j \in \mathbb{Z}\Delta^j$. This decomposition might be not unique (even for fixed Π): μ_s is uniquely defined if Δ^s

is finite, but $\mu_s - \mu'_s \in \mathbb{Z}\delta$ for different decompositions of μ, if Δ^s is affine. By (38), there exists a decomposition, where $\mu_j \in \mathbb{Z}(\Delta^j \cap \Delta^+)$.

Note that $\overline{\lambda}_{r_\beta \Pi} = \overline{\lambda}$ if $\beta \not\in \Delta(L)$ or $(\overline{\lambda}_\Pi, \beta) = 0$, and $\overline{\lambda}_{r_\beta \Pi} = \overline{\lambda} - \beta$ otherwise. In particular, if $(\overline{\lambda}_\Pi, \Delta^j) = 0$ for some Π, then this holds for each Π.

Assume that $(\overline{\lambda}_\Pi, \Delta^j) = 0$. We claim that for any two decompositions $\overline{\lambda}_\Pi - \mu = \sum \mu_s$ and $\overline{\lambda}_{\Pi'} - \mu = \sum \mu'_s$ one has $\mu_j - \mu'_j \in \mathbb{Z}\delta$ (in particular, $\mu_j = \mu'_j$ if Δ^j is finite). Indeed, it is enough to consider the case $\Pi' = r_\beta \Pi$ and $\overline{\lambda}_\Pi \neq \overline{\lambda}_{\Pi'}$. Then $\overline{\lambda}_{\Pi'} = \overline{\lambda} - \beta$ and $(\overline{\lambda}_\Pi, \beta) \neq 0$, that is $\beta \not\in \Delta^j$. Therefore there exists a decomposition $\overline{\lambda}_{\Pi'} - \mu = \sum \mu''_s$ with $\mu''_j = \mu_j$; since $\mu''_j - \mu_j \in \mathbb{Z}\delta$ the claim follows.

Let us show that

$$(43)\quad (\overline{\lambda}_\Pi, \Delta^j) = 0 \implies \begin{cases} \mu_j \in \mathbb{Z}\delta, & \text{if } \Delta^j \neq A(m,n)^{(1)}, C(m)^{(1)}, \\ \mu_j \in \mathbb{Z}\delta + \mathbb{Z}\xi_j, & \text{if } \Delta^j = A(m,n)^{(1)}, C(m)^{(1)}, \end{cases}$$

where for $\Delta^j = A(m,n)^{(1)}, C(m)^{(1)}$ the element $\xi_j \in \mathbb{Z}\Delta^j$ is such that $(\xi_j, \Delta^j_{\overline{0}}) = 0$.

Take j such that $(\overline{\lambda}_\Pi, \Delta^j) = 0$. Since $(\Delta^j, \Delta^s) = 0$ for $s \neq j$, (39) gives

$$(44)\qquad (\mu_j, \alpha^\vee) \leq 0 \quad \text{for each } \alpha \in \Pi_0(L) \cap \Delta^s.$$

11.7.1. Assume that Δ^j is finite and $\mathbb{Q}\Delta^j = \mathbb{Q}\Delta^j_{\overline{0}}$. Then, from (Fin) (see §11.6), $\mu_j \in -(\mathbb{Q}\Delta^j_{\overline{0}} \cap \Delta^+)$; since $\mu_j \in \mathbb{Z}(\Delta^j \cap \Delta^+)$, we get $\mu_j = 0$, as required.

Assume that Δ^j is affine and $\mathbb{Q}\Delta^j = \mathbb{Q}\Delta^j_{\overline{0}}$. Then, from (Aff) (see §11.6), $\mu_j \in \mathbb{Z}\delta$, as required.

Assume that Δ^j is $A(m,n)^{(1)}, C(n)^{(1)}$. Since L is non-critical, $(\rho_L, \delta) \neq 0$, so the case $A(n,n)^{(1)}$ is excluded. Let Δ^j be of type $A(m,n)^{(1)}$ with $m \neq n$, or $C(n)^{(1)}$. In this case $\mathbb{Q}\Delta^j$ lies in $\mathbb{Q}\Delta^j_{\overline{0}} + \mathbb{Z}\xi_j$, where $\xi_j \in \mathfrak{h}^*$ is orthogonal $\Delta^j_{\overline{0}}$. Combining (Aff) and (44) we obtain $\mu_j \in \mathbb{Z}\delta + \mathbb{Z}\xi_j$, as required.

11.7.2. Now consider the remaining case $\Delta^j = A(m,n), C(m)$. By above, μ_j is uniquely defined, and, in particular, does not depend on Π.

One has $\mathbb{Q}\Delta^j = \mathbb{Q}\Delta^j_{\overline{0}} + \mathbb{Z}\xi$, where $\xi \in \mathfrak{h}^*$ is orthogonal $\Delta^j_{\overline{0}}$. Combining (Fin) in §11.6 and (44) we obtain $\mu_j = x\xi - \mu'$, where $\mu' \in \mathbb{Q}^+\Pi^j_{\overline{0}}$. Since $\mu_j \in \mathbb{Z}_{\geq 0}\Pi^j$ for each Π, we obtain $x\xi \in \mathbb{Z}_{\geq 0}\Pi^j$ for each Π. Thus for each set of simple roots Π the corresponding set of simple roots Π^j (the set of simple roots for $\Delta^j \cap \Delta^+(\Pi)$) is such that $x\xi \in \mathbb{Z}_{\geq 0}\Pi^j$.

Consider the root $\varepsilon_1-\delta_1$ in $A(m,n)$ or $C(m)$. Assume that this root is simple in Δ, i.e., lies in a set of simple roots Π. If Π^j is the set of simple roots in $\Delta^j\cap\Delta^+(\Pi)$, then $r_{\varepsilon_1-\delta_1}(\Pi^j)$ is the set of simple roots in $\Delta^j\cap\Delta^+(r_{\varepsilon_1-\delta_1}\Pi)$. Write $\xi=n\sum_{i=1}^m\varepsilon_i-m\sum_{i=1}^n\delta_i$ for $A(m,n)$ and $\xi=-\delta_1$ for $C(m)$. Since $x\xi\in\mathbb{Z}_{\geq 0}\Pi^j$ and $\varepsilon_1-\delta_1\in\Pi^j$, one has $x\geq 0$; similarly, $x\xi\in\mathbb{Z}_{\geq 0}r_{\varepsilon_1-\delta_1}(\Pi^j)$ gives $x\leq 0$. Hence $x=0$. Since $x=0$, we have $\mu_j=-\mu'\in-\mathbb{Q}^+\Pi_0^j$. Combining with $\mu_j\in\mathbb{Z}_{\geq 0}\Pi^j$, we get $\mu_j=0$.

It remains to show that $\varepsilon_1-\delta_1$ lies in some set of simple roots for Δ. If Δ is of type $A(m',n')$, $C(n')$ or $A(m',n')^{(1)}$, $C(n')^{(1)}$, then any odd root lies in a set of simple roots, so this holds.

Let us show that for other root systems Δ this can be achieved for some $L'\sim L$. Denote by ι_L the embedding $A(m,n)\to\Delta$ with the image Δ^j. Since $m,n>1$, Δ is not exceptional or affinization of exceptional. In the light of Lemma 13.4 the root $\iota_L(\varepsilon_1-\delta_1)$ or $\iota_L(\delta_1-\varepsilon_1)\in X$ (see Lemma 13.4 for notations). We may (and will) assume that $\iota_L(\varepsilon_1-\delta_1)\in X$. Denote by Δ' the connected component of $\Delta_{\bar{0}}$ containing $\delta_1-\delta_2$ and by π' its set of simple roots ($\pi'\subset\Pi_0$). Take $\alpha\in\pi'$. If $\alpha\in\Delta(L)$ and $(\alpha,\beta)\neq 0$, then $\alpha\in\Delta^j$, and so $\alpha\in\Pi_0^j$ (since $\alpha\in\Pi_0$), which implies $(\beta,\alpha^\vee)=-1$. As a result, if $\alpha\in\pi'$ is such that $(\beta,\alpha^\vee)>0$, then $\alpha\not\in\Delta(L)$, and we can apply the Enright functor $T_\alpha(a)$ to L. Set $L':=T_\alpha(a)(L)$. Clearly, $\Delta(L')=r_\alpha(\Delta(L))$ and $\iota_{L'}=r_\alpha\iota_L$. In particular, $r_\alpha\Delta^j=A(m,n)$ and the $\iota_{L'}(\varepsilon_1-\delta_1)=r_\alpha\beta<\beta$. By Lemma 13.4, repeating this procedure we obtain $L''\sim L$, where $\beta'':=\iota_{L''}(\varepsilon_1-\delta_1)$ is an essentially simple isotropic root (see §13.1), that is $\beta''\in\Pi$ for some Π (and $\iota_{L''}(\delta_1-\varepsilon_1)\in\Pi'$ for $\Pi'=r_{\beta''}(\Pi)$).

11.7.3. Now fix Π and let $\overline{\lambda}_\Pi+\rho_L-\nu\in\operatorname{supp}(Z)$. From Corollary 7.3.4 it follows that ν can be decomposed as a sum $\nu=\sum\nu_j$ with

$$2(\overline{\lambda}_\Pi+\rho_L,\nu_j)=\|\nu_j\|^2,\quad \nu_j\in\mathbb{Z}_{\geq 0}(\Delta^j\cap\Delta^+) \tag{45}$$

and, moreover, that $\overline{\lambda}_\Pi+\rho_L-\nu_j\in W(\Delta^j)(\overline{\lambda}_\Pi+\rho_L)$ if the restriction of $\overline{\lambda}_\Pi$ to Δ^j is typical, i.e., $(\overline{\lambda}_\Pi+\rho_{L,\Pi},\beta)\neq 0$ for each $\beta\in\Delta^j$.

Consider the case when the restriction of $\overline{\lambda}_\Pi$ to Δ^j is typical, i.e., $(\overline{\lambda}_\Pi+\rho_{L,\Pi},\beta)\neq 0$ for each $\beta\in\Delta^j$. By above, $\overline{\lambda}_\Pi+\rho_L-\nu_j\in W(\Delta^j)(\overline{\lambda}_\Pi+\rho_L)$ and $\overline{\lambda}_\Pi+\rho_L$ is $\Pi_0(L)$-maximal in its $W(\Delta^j)$-orbit by Theorem 11.1.1. Since $(\nu-\nu_j,\Delta^j)=0$, $\overline{\lambda}_\Pi+\rho_L-\nu\in W(\Delta^j)(\overline{\lambda}_\Pi+\rho_L-(\nu-\nu_j))$ and $\overline{\lambda}_\Pi+\rho_L-(\nu-\nu_j)$ is $\Pi_0(L)$-maximal in its $W(\Delta^j)$-orbit. Therefore $Z\in\mathscr{R}_{W(\Delta^j)}$. Since Z is $W(\Delta^j)$-skew-invariant (see Corollary 10.2.2), we conclude that $\overline{\lambda}_\Pi+\rho_L-(\nu-\nu_j)\in\operatorname{supp}(Z)$. Note that $\overline{\lambda}_\Pi+\rho_L-(\nu-\nu_j)>_\Pi\overline{\lambda}_\Pi+\rho_L-\nu$.

Now let $\overline{\lambda}_\Pi+\rho_L-\nu$ be a maximal element in $\operatorname{supp}(Z)$ with respect to the order $\nu'\geq\nu''$ if $\nu'-\nu''\in\mathbb{Z}_{\geq 0}\Delta^+$. Then $\overline{\lambda}_\Pi-\nu$ is a maximal element in

$\operatorname{supp}(Z')$ with respect to the same order and so is a $\Pi_0(L)$-maximal element in its $W(L)$-orbit. By above, $\nu_j = 0$ for each j such that the restriction of $\overline{\lambda}_\Pi$ to Δ^j is typical.

Let us show that $\nu_j = 0$ for each j. By the assumption $(\overline{\lambda}_\Pi, \Delta^j) = 0$. Then, by (43), $\nu_j = 0$ if Δ^j is finite and $\nu_j = k_j\delta$ if $\Delta^j \neq A(m,n)^{(1)}, C(m)^{(1)}$ is affine; in the latter case, since Δ^j is not critical, $(\rho_L, \delta) \neq 0$, so (45) forces $k_j = 0$. If $\Delta^j = A(m,n)^{(1)}$ or $C(m)^{(1)}$, then $\nu_j = x_j\xi_j + k_j\delta$. In the light of Lemma 13.3, (45) forces $\nu_j = 0$ for $\Delta^j = A(m,n)^{(1)}$. For $\Delta^j = C(n)^{(1)}$ we can (and will) normalize the form in such a way that $\|\alpha\|^2 \geq 0$ for $\alpha \in \Delta^j$; then $\|\xi_j\|^2 < 0$ and $(\rho_L, \nu_j) \geq 0$ since $\nu_j \in \mathbb{Z}_{\geq 0}(\Delta^j \cap \Delta^+)$. Now (45) forces $x_j = 0$, that is $2(\rho_L, k_j\delta) = 0$; hence $\nu_j = 0$, as required.

We conclude that $\nu = 0$, a contradiction. □

12. Examples: $\mathfrak{g}^\lambda$ of type $A(1,1)^{(1)}$ and $B(1,1)^{(1)}$

In this section we establish the KW-formula in two more cases: $\overline{\mathfrak{g}}^\lambda$ is of types $A(1,1)^{(1)}$ or $B(1,1)^{(1)}$.

12.1. Case $\mathfrak{g} = \mathfrak{gl}(2,2)^{(1)}$, $\mathfrak{sl}(2,2)^{(1)}$, $\mathfrak{psl}(2,2)^{(1)}$

Consider $\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$, where $\|\alpha_i\|^2 = 0$ and $(\alpha_i, \alpha_{i+1}) \neq 0$ (where $\alpha_4 = \alpha_0$). Let $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3$ be the corresponding fundamental weights, i.e., $(\Lambda_i, \alpha_j) = \delta_{ij}$. Let $L = L(\lambda, \Pi)$ be a non-critical module and $\Delta(L) \cong \Delta$. We show that (35) holds if L is a non-critical module such that $F(L) \cong L(\lambda', \Pi)$, where $\lambda' = k_0\Lambda_0 + k_2\Lambda_2$ or $\lambda = k_0\Lambda_0 + k_1\Lambda_1$, $k_0, k_1, k_2 \in \mathbb{Z}_{\geq 0}$.

Note that we do not assume that $\overline{\lambda}$ is of above form (or that $\Pi(L) \cong \Pi$).

12.2. Marked diagrams

Fix a irreducible highest weight module L. For each Π take λ such that $L = L(\lambda, \Pi)$. Consider the Dynkin diagram of $\Pi(L)$ and assign to each edge $\alpha - \alpha'$ the scalar product (α, α') and to each node α the number $x_\alpha := (\lambda + \rho_\Pi, \alpha)$ (which is integral).

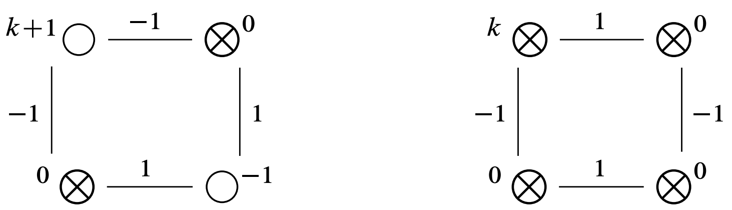
We call the diagram $\Pi(L)$ endowed by these numbers a *marked diagram* $D(L, \Pi)$ corresponding to (L, Π).

12.2.1. If β is an odd node of a marked diagram $\Pi(L)$, we define the action of r_β on the marked diagram $D(L, \Pi)$ in such a way that $r_\beta * D(L, \Pi) = D(L, r_\beta\Pi)$ in the case when $\Pi(L) = \Pi$. This means that

- the nodes connected to β change their parity and other nodes (including β) preserve the parity;
- the scalar products between the node corresponding to β and other nodes change to the opposite; other scalar products do not change;
- the mark x_β of β is changed to $-x_\beta$; if the mark $x_\beta \neq 0$, the new mark of the node β' connected to β is $x_\beta + x_{\beta'}$, and if $x_\beta = 0$, the new marks of the node β' connected to β is $x_{\beta'} + (\beta, \beta')$; other marks do not change.

12.2.2.

Example 1. The second diagram is obtained from the first one by the reflection with respect to the upper-right node:



12.2.3. By §7.5.4, $D(L, r_\beta \Pi) = r_\beta * D(L, \Pi)$ if $\beta \in \Pi$ and $D(L, r_\beta \Pi) = D(L, \Pi)$ otherwise.

We say that two marked diagrams D, D' are *connected by an odd reflection* if $D' = r_\beta D$ for some odd node $\beta \in D$.

Denote by $DM(L)$ the set of marked diagrams $D(L', \Pi')$ for all Π' (compatible with Π_0) and all L' such that $L \sim L'$. One readily sees that any two diagrams in $DM(L)$ are connected by a chain of odd reflections $r_\beta *$. In Corollary 12.2.5 below we show that if $D \in DM(L)$ and $v \in D$ is an odd node, then $s_v * D \in DM(L)$. This implies that $DM(L)$ is the set of diagrams obtained from D by the action of chains of odd reflections; in particular, $DM(L) = DM(L')$ if $DM(L) \cap DM(L')$ is non-empty. Take a pair (L, Π) and let L' be such that $D(L, \Pi) = D(L', \Pi)$ with $\Delta(L') = \Delta$. Then L' is partially integrable and $DM(L) = DM(L')$.

12.2.4. **Lemma.** *For each odd node of the marked diagram $D \in DM(L)$ there exists a pair (L', Π') such that $L \sim L'$, $D(L', \Pi') = D$ and the root in $\Pi'(L')$ which corresponds to this node is simple, i.e., lies in Π'.*

Proof. Let $D = D(L, \Pi)$ and let $\beta \in \Pi(L)$ be the root corresponding to the odd node in D. We prove the assertion by induction on $\mathrm{ht}_\Pi(\beta)$.

If $\mathrm{ht}_\Pi \beta = 1$, then $\beta \in \Pi$, as required.

If β is of the form $\beta = j\delta + \beta'$ for $\beta' \in \Pi$, $j \in \mathbb{Z}_{>0}$, then $\beta' \notin \Pi(L)$ and for $\Pi' := r_{\beta'}\Pi$ one has $\Pi'(L) = \Pi(L)$. Moreover, $\mathrm{ht}_\beta = 4j + 1$ and

$\mathrm{ht}_{\Pi'}\beta = 4j-1$, because $\beta' \in \Pi'$, so $\mathrm{ht}_{\Pi'}(\delta+\beta') = 3$. Hence $\mathrm{ht}_{\Pi'}\beta < \mathrm{ht}_{\Pi}\beta$ and the assertion follows by induction.

Assume that $\beta \neq j\delta + \beta'$ for $\beta' \in \Pi, j \in \mathbb{Z}_{\geq 0}$. Take $\alpha \in \Pi_0$ such that $\|\alpha\|^2 = 2$ and $r_\alpha\beta < \beta$ (thus $(\alpha,\beta) = 1$). Note that $\alpha \notin \Pi(L)$, because for $\alpha,\beta \in \Pi(L)$ one has $r_\alpha\beta \geq \beta$. Since $\alpha \in \Pi_0$, one has $1 \leq \mathrm{ht}_\Pi\alpha \leq 3$.

If $\mathrm{ht}_\Pi\alpha = 1$, i.e., $\alpha \in \Pi$, then $\alpha \notin \Delta(L)$ (because $\alpha \notin \Pi(L)$). Applying the Enright functor T_α (see §8) we have

$$\Pi(T_\alpha(L)) = r_\alpha\Pi(L), \quad D(T_\alpha(L),\Pi) = D(L,\Pi)$$

and the node corresponding to β is $r_\alpha\beta = \beta - t\alpha, t > 0$. Thus $\mathrm{ht}_\Pi r_\alpha\beta = \mathrm{ht}_\Pi\beta - t < \mathrm{ht}_\Pi\beta$ and the assertion follows by induction.

Assume that $\mathrm{ht}_\Pi\alpha = 2$. Then $\alpha = \alpha_1+\alpha_2$, where $\alpha_1,\alpha_2 \in \Pi$ are odd roots and $(\alpha_1,\alpha_2) = 1$. If $\alpha_1,\alpha_2 \in \Pi(L)$, then $\Pi(L)$ contains three odd roots β,α_1,α_2 and since $(\alpha_1,\alpha_2) = 1$ one has $\{(\beta,\alpha_1),(\beta,\alpha_2)\} = \{0,-1\}$ that is $(\beta,\alpha_1+\alpha_2) = -1$, a contradiction. Thus at least one of the roots α_1,α_2, say α_1, is not in $\Pi(L)$. Since π contains two non-orthogonal odd roots, it contains only odd roots, so $\beta = j\delta \pm \beta'$ for some $\beta' \in \Pi, j \in \mathbb{Z}_{>0}$. From the above assumption we obtain $\beta = j\delta - \beta'$. Since $\beta \in \Pi(L)$ one has $\beta' \notin \Pi(L)$. Moreover, $(\beta,\alpha_1+\alpha_2) = (\alpha_1,\alpha_2)$ forces $\beta' \notin \{\alpha_1,\alpha_2\}$. For $\Pi' := r_{\alpha_1}\Pi$ one has $\alpha \in \Pi'$, $\Pi'(L) = \Pi(L)$. Since $\alpha_1 \neq \beta'$, one has $\mathrm{ht}_{\Pi'}(\delta-\beta') \leq \mathrm{ht}_\Pi(\delta-\beta')$ so $\mathrm{ht}_{\Pi'}\beta \leq \mathrm{ht}_\Pi\beta$. Since $\alpha \in \Pi'$, the assertion follows by induction from the above.

Now assume that $\mathrm{ht}_\Pi\alpha = 3$, i.e., $\alpha = \alpha_1+\alpha_2+\alpha_3$, where $\alpha_i \in \Pi$ is odd for $i = 1,3$ and even for $i = 2$. One readily sees that $\|\alpha\|^2 = 2$ forces $\|\alpha_2\|^2 = -2$. Since $(\beta,\alpha) = 1$, β is of the form $j\delta+\alpha_i$ or $j\delta+\alpha_i+\alpha_2$ for $i \in \{1,3\}$ and $j \in \mathbb{Z}_{\geq 0}$. From the above assumption we get $\beta = j\delta+\alpha_i+\alpha_2$. Since $\beta \in \Pi(L)$, one has $\alpha_i \notin \Pi(L)$. Set $\Pi' := r_{\alpha_i}\Pi$. One has $\Pi'(L) = \Pi(L)$ and $\mathrm{ht}_{\Pi'}\beta < \mathrm{ht}_\Pi\beta$ since $1 = \mathrm{ht}_{\Pi'}(\alpha_i+\alpha_2) < \mathrm{ht}_\Pi(\alpha-i+\alpha_2) = 2$. The assertion follows by induction. □

12.2.5. **Corollary.** *If $D \in DM(L)$ and v is an odd node of D, then $s_v * D \in DM(L)$.*

In the light of Corollary 12.2.5 it is enough to verify formula (35) for one set of simple roots $\Pi(L)$ (if the formula holds for L and $F(L) \cong F(L')$, then the formula holds for L'). We check the formula for the cases $\overline{\lambda} = k\Lambda_0 + j\Lambda_2, \overline{\lambda} = k\Lambda_0 + j\Lambda_1$, $k,j \in \mathbb{Z}_{\geq 0}, k+j \neq 0$, where $\Pi(L)$ consists of odd roots (in particular, $\overline{\rho} = 0$).

12.2.6. Case $\overline{\lambda} = k_0\Lambda_0 + k_2\Lambda_2, k_0,k_2 \in \mathbb{Z}_{\geq 0}, k_0+k_2 \neq 0$. In this case, $F(L)$ is integrable and $(\overline{\lambda}+\rho,\alpha_i) = 0$ for $i = 1,3$. Set $S := \{\alpha_1,\alpha_3\}$ and

$Z(\Pi)$ is the expansion of

$$(46) \qquad Z(\Pi) := Re^{\rho}\,\mathrm{ch}\,L - \mathscr{F}_{W(\pi)}\Big(\frac{e^{(\lambda+\rho)}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)$$

in $\mathscr{R}(\Pi)$. Arguing as in §4, we obtain that the Π-maximal element in $\mathrm{supp}\,Z(\Pi)$ is $\overline{\lambda}-\mu$, where $\mu\in\mathbb{Z}S$, that is $\mu=a_1\alpha_1+a_3\alpha_3$. where $a_1,a_3\geq 0, a_1+a_3\neq 0$. By Lemma 12.2.4, we can assume that α_1 is a simple root. Then $L(\lambda,\Pi)=L(\lambda,r_{\alpha_1}\Pi)$. Since $\lambda+\rho-(a_1-1)\alpha_1-a_3\alpha_3\not\in\mathrm{supp}\,Z(\Pi)$ and $\lambda+\rho-a_1\alpha_1-a_3\alpha_3\not\in\mathrm{supp}\,Z(\Pi)$, Lemma 2.2.9 gives $\lambda+\rho-(a_1-1)\alpha_1-a_3\alpha_3\in\mathrm{supp}\,Z(\Pi')$ or $\lambda+\rho-a_1\alpha_1-a_3\alpha_3\in\mathrm{supp}\,Z(\Pi')$, where $\Pi':=r_{\alpha_1}\Pi$. It is easy to see that $\mathrm{supp}\,Z(\Pi')\subset\lambda+\rho'-\mathbb{Z}_{\geq 0}\Pi'$, so $-(a_1-1)\alpha_1-a_3\alpha_3$ or $-a_1\alpha_1-a_3\alpha_3$ lie in $\rho'-\mathbb{Z}_{\geq 0}\Pi'=\alpha_1-\mathbb{Z}_{\geq 0}\Pi'$; since $-\alpha_1,\alpha_3\in\Pi'$, we obtain $a_1\leq 0$, that is $a_1=0$. Similarly, $a_3=0$, a contradiction. Hence $\mathrm{supp}\,Z$ is empty, that is $Z=0$ and (35) holds.

12.2.7. Case $\overline{\lambda}=k\Lambda_0+j\Lambda_1$, $k,j\in\mathbb{Z}_{\geq 1}$. In this case $F(L)$ is $\pi\cup\{\alpha_0+\alpha_1\}$-integrable, where $\pi=\{\alpha_0+\alpha_3,\alpha_1+\alpha_2\}\cong A_1^{(1)}$. Set $S:=\{\alpha_2\}$ and define $Z:=Z(\Pi)$ as in (46). Arguing as in §4, we obtain that the Π-maximal element in $\mathrm{supp}\,Z$ is of the form $\overline{\lambda}-\mu$, where $\mu\in\mathbb{Z}_{\geq 0}\Pi(L)$, $\mu\neq 0$, $\|\mu\|^2=2(\overline{\lambda},\mu)$ and $(\overline{\lambda}-\mu,\alpha^{\vee})\geq 0$ for $\alpha\in\pi\cup\{\alpha_0+\alpha_1\}$.

Set

$$\alpha_1=\varepsilon_1-\delta_1,\quad \alpha_2=\delta_1-\varepsilon_2,\quad \alpha_3=\varepsilon_2-\delta_2,\quad \alpha_0=s\delta-\varepsilon_1+\delta_2$$

(δ is the minimal imaginary root in Δ, $s\delta$ is the minimal imaginary root in $\Delta(L)$). Note that $\xi:=\alpha_1+\alpha_3$ is orthogonal to $\Delta(L)$. Write

$$\begin{aligned}\mu&=a(s\delta)+b\xi+(d_1+d_2)\varepsilon_2-d_1\delta_1-d_2\delta_2\\&=a\alpha_0+(a+b)\alpha_1+(a-d_1)\alpha_2+(a+b+d_2)\alpha_3.\end{aligned}$$

The condition $\mu\in\mathbb{Z}_{\geq 0}\Pi(L)$ gives

$$a,b,d_1,d_2\in\mathbb{Z},\quad a,a+b,a-d_1,a+b+d_2\geq 0.$$

The above conditions $(\overline{\lambda}-\mu,\alpha^{\vee})\geq 0$ give $d_1\geq d_2, -j\leq d_1+d_2\leq k$. The condition $\|\mu\|^2=2(\overline{\lambda},\mu)$ is equivalent to

$$d_1d_2=ka+j(a+b).$$

Since $a,a+b\geq 0$ one has $d_1d_2\geq 0$. If $d_1+d_2\geq 0$, then the above inequalities imply $0\leq d_1\leq a, d_1+d_2\leq k$, so $d_1d_2+d_1^2\leq ka$ and thus $d_1=d_2=0$. If $d_1+d_2<0$, then $d_1,d_2<0$ and the above inequalities imply $-d_2\leq a+b, -d_1-d_2\leq j$, so $d_1d_2+d_2^2\leq j(a+b)$ and thus again $d_1=d_2=0$. Therefore $d_1=d_2=0$ and thus $a=a+b=0$ (since $j,k>0$). Hence $\mu=0$, a contradiction, so (35) holds.

12.3. Case $\mathfrak{g}_L = B(1,1)^{(1)}$

Let $\mathfrak{g}$ be an affine Lie superalgebra and L be a $\mathfrak{g}$-module such that $\Delta(L) = B(1,1)^{(1)}$ and $F(L)$ satisfies the KW-condition. The case of typical $F(L)$ was considered in §9. Below we establish the KW-character formula for atypical case when the KW-condition holds and $F(L)$ is π-integrable for "sufficiently large" π (in the standard notation $\pi = \{2\delta_1, \delta - 2\delta_1\}$ or $\pi = \{\varepsilon_1, \delta - \varepsilon_1, 2\delta_1\}$).

12.3.1. Consider the embedding $\iota_L : B(1,1)^{(1)} \to \Delta$ given by the identification $\Delta(L) \cong B(1,1)^{(1)}$.

Recall (see §13.1) that a root is called essentially simple if it lies in some set of simple roots. Let $\beta \in B(1,1)^{(1)}$ be an isotropic essentially simple root; let us show that for some $L' \sim L$ the root $\iota_{L'}(\beta)$ is essentially simple.

Indeed, the non-isotropic roots of $B(1,1)^{(1)}$ are $A_1^{(1)} \times B(0,1)^{(1)}$; each essentially simple root of $B(1,1)^{(1)}$ is of the form $\pm(\alpha_1 - \alpha_2)$, were α_1 (resp., α_2) is a simple root of $A_1^{(1)}$ (resp., of $B(0,1)^{(1)}$). Assume that $\iota_L(\alpha_1) \not\in \Pi_0$. Then there exists $\gamma \in \Pi_0$ such that $r_\gamma \iota(\alpha_1) < \alpha_1$ (see §2.1.3 for $<$); note that $\gamma \not\in \iota(\Delta(L))$ (since $r_{\alpha'}\alpha \geq \alpha$ for $\alpha' \in \Pi_0(L) \setminus \{\alpha\}$) and that $r_\gamma \alpha_2 = \alpha_2$. For $L' := T_\gamma(L)$ we have $\iota_{L'}(\alpha_1) < \iota_L(\alpha_1)$ and $\iota_{L'}(\alpha_2) = \iota_{L'}(\alpha_2)$. A similar reasoning works if $\iota_L(\alpha_2) \not\in \Pi_0$. Hence there exists L' such that $\iota_{L'}(\alpha_1), \iota_{L'}(\alpha_2) \in \Pi_0$. By §13.1.2, $\iota_{L'}(\alpha_1) - \iota_{L'}(\alpha_2)$ is an essentially simple root, as required.

12.3.2. Now assume that $L = L(\lambda, \Pi)$ is such that $F(L)$ satisfies KW-condition. Let us show that for some set of simple roots Π' of Δ and some $L' \sim L$ one has $L' = L(\lambda', \Pi')$, where $(\lambda' + \rho, \beta') = 0$ for some $\beta' \in \Pi'$.

Since $F(L)$ satisfies KW-condition and $\mathfrak{g}_L = B(1,1)^{(1)}$, one has $(\lambda + \rho, \iota_L(\beta)) = 0$, where $\beta \in B(1,1)^{(1)}$ is essentially simple.

Indeed, if this does not hold, then for all $L' \sim L$ the value $\lambda' + \rho'$ does not depend on Π' (i.e., if $L' = L(\lambda', \Pi') = L(\lambda'', \Pi'')$, then $\lambda' + \rho' = \lambda'' + \rho''$). Then taking $L' = T_{\gamma_1} \cdots T_{\gamma_s}(L)$ as above, we obtain $L' = L(\lambda', \Pi')$, where $\lambda' + \rho' = w(\lambda + \rho)$, and $\iota_{L'}(\beta) = w\iota_L(\beta)$, where $w := r_{\gamma_1} \cdots r_{\gamma_s}$. Hence $(\lambda' + \rho', \iota_{L'}(\beta)) = 0$. Since $\beta' := \iota_{L'}(\beta)$, we can choose Π' containing β' and L', Π' satisfies our requirements.

12.3.3. By above, we can (and will) assume that $L = L(\lambda, \Pi)$ is such that $(\lambda + \rho, \beta) = 0$ for some $\beta \in \Pi$.

For $B(1,1)^{(1)}$ there are three sets of simple roots:

$$\Pi_1 := \{\delta - \delta_1 - \varepsilon_1, \varepsilon_1 - \delta_1, \delta_1\}, \quad \Pi_2 := \{\delta - 2\delta_1, \delta_1 - \varepsilon_1, \varepsilon_1\},$$
$$\Pi_3 := \{\delta_1 + \varepsilon_1 - \delta, \delta - 2\delta_1, \delta - \varepsilon_1\}$$

and the set Π_1 can be obtained from Π_2 (resp., from Π_3) by an odd reflection with respect to the unique isotropic root in Π_2 (resp., in Π_3). Hence we can (and will) assume that $\Pi(L) \cong \Pi_1$; we identify $\Pi(L)$ and Π_1.

If $F(L)$ is integrable, (35) follows from Theorem 11.2.2 (since $S = \{\beta\} \subset \Pi$ and Π_1 satisfies the conditions of §4).

12.3.4. Now consider the case when $F(L)$ is non-critical π-integrable, where $\pi = \{\delta - \varepsilon_1, \varepsilon_1, 2\delta_1\}$ (subprincipal case in [KW4]). The formula for $\overline{\lambda} = 0$ is proved in Theorem 11.2.5, so we assume that $\overline{\lambda} \neq 0$. We will show that if $\overline{\lambda} \neq 0$ and KW-condition holds, i.e., $(\lambda + \rho, \beta) = 0$ for some $\beta \in \Pi(L)$, then

$$Re^{\rho} \operatorname{ch} L = \frac{1}{2} \mathscr{F}_{W(\pi)} \Big(\frac{e^{\lambda+\rho}}{1+e^{-\beta}} \Big). \tag{47}$$

This implies (35).

12.3.5. Set $y_0 := (\overline{\lambda}, \delta - \delta_1 - \varepsilon_1)$, $y_1 := (\overline{\lambda}, \varepsilon_1 - \delta_1)$, $y_2 := (\overline{\lambda}, \delta_1)$. The module $L(\overline{\lambda}, \Pi_1)$ is π-integrable if and only if either $\overline{\lambda} = 0$ or it is one of the following cases, *cf.* [KW4]:

$$\begin{aligned} &y_1 = y_2 = 0, \quad -2(y_0+1) \in \mathbb{Z}_{\geq 0};\\ &y_0 = y_2 = 0, \quad -2(y_1+1) \in \mathbb{Z}_{\geq 0};\\ &y_0, y_1 \neq 0, \quad 2y_2, -2(y_0+y_2+1), -2(y_1+y_2+1) \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

KW-condition holds for first two cases and does not hold for atypical modules in the third case (since $y_0, y_1 \neq 0$). Note that $B(1,1)^{(1)}$ admits an automorphism given by $\varepsilon_1 \mapsto \delta - \varepsilon_1$, $\delta_1 \mapsto \delta_1$, which interchanges the isotropic roots of Π_1; this automorphisms interchanges the first and the second cases. For the third case KW-condition does not hold.

Therefore we may (and will) consider the first case (when $y_1 = y_2 = 0$, i.e., $F(L)$ is a vacuum module). In this case $\beta = \varepsilon_1 - \delta_1$. Since $\overline{\lambda}$ is non-critical, $y_0 \neq -1$. Hence

$$y_1 = y_2 = 0, \quad 2y_0 \in \mathbb{Z}, \quad y_0 < -1.$$

Using the denominator identity for $B(1,1)$ (see (18)) we rewrite (47) in the form

$$Re^{\rho} \operatorname{ch} L = \mathscr{F}_{W(\pi')} \Big(\frac{e^{\lambda+\rho}}{1+e^{-\beta}} \Big),$$

where $\pi' := \{\delta - \varepsilon_1, \varepsilon_1\} \cong A_1^{(1)}$. Set $\alpha_1 = \varepsilon_1, \alpha_0 := \delta - \varepsilon_1$ and let $r_1, r_0 \in W(\pi')$ be the corresponding reflections.

First, let us show that the support of the right-hand side is in $\lambda+\rho-\mathbb{Z}_{\geq 0}\Pi_1$ and that the coefficient of $e^{\lambda+\rho}$ is equal to 1. For $w\in W(\pi')$ let

$$Y_w := \frac{e^{w(\lambda+\rho+\beta)}}{1+e^{w\beta}} \in \mathscr{R}(\Pi_1)$$

(i.e., Y_w is the expansion in $\mathscr{R}(\Pi_1)$ of the fraction in the right-hand side). It is enough to verify that for each $w\in W(\pi')$,

$$\text{supp}\, Y_w \subset \lambda+\rho-\mathbb{Z}_{\geq 0}\Pi_1 \tag{48}$$

and for $w\neq Id$,

$$\lambda+\rho\not\in \text{supp}\, Y_w. \tag{49}$$

Our reasoning is based on the formula (15). One has $(\lambda+\rho,\alpha_1^\vee)=-1,\ (\lambda+\rho,\alpha_0^\vee)=-2y_0-1$ and so $(r_1(\lambda+\rho),\alpha_1^\vee)=1, (r_1(\lambda+\rho),\alpha_0^\vee)=-2y_0-3\geq 0$. Therefore $r_1(\lambda+\rho)$ is maximal in its $W(\pi')$-orbit; this establishes (48) for $w\neq Id, r_1, r_0r_1$ and (49) for the same w if $y_0<-3/2$; for $y_0=-3/2$ this gives (49) for $w\neq r_1, r_0r_1, r_1r_0r_1$. For $w=r_1, r_0r_1, r_1r_0r_1$ one has $w(-\beta)\in\Delta^+(\Pi_1)$ and, moreover, $w(-\beta)\geq_1 \delta_1+\varepsilon_1$, where $\geq_1$ stands for Π_1-partial order. Thus for such w one has

$$\begin{aligned}\text{supp}\, Y_w &\subset w(\lambda+\rho)+w\beta \leq_1 r_1(\lambda+\rho)+w\beta\\ &\leq_1 \lambda+\rho+\varepsilon_1-(\delta_1+\varepsilon_1)=\lambda+\rho-\delta_1.\end{aligned}$$

This establishes (48), (49).

Now for $Z := Re^\rho \,\text{ch}\, L-\frac{1}{2}\mathscr{F}_{W(\pi)}(\frac{e^{\lambda+\rho+\beta}}{1+e^{\beta}})$ we have $\text{supp}\, Z\subset \lambda+\rho-\mathbb{Z}_{\geq 0}\Pi_1$ and $\lambda+\rho\not\in\text{supp}\, Z$.

Since $F(L)$ is π-integrable, Z is $W(\pi)$-skew-invariant. Arguing as in §11.4 we conclude that it is enough to verify that if $\mu\in\mathbb{Z}_{\geq 0}\Pi_1$ satisfies $2(\lambda+\rho,\mu)=\|\mu\|^2$ and $(\lambda-\mu,\alpha^\vee)\geq 0$ for $\alpha\in\pi$, then $\mu=0$. Write

$$\mu=s_0(\delta-\delta_1-\varepsilon_1)+s_1(\varepsilon_1-\delta_1)+s_2\delta_1,\quad s_0,s_1,s_2\geq 0.$$

Then $(\lambda-\mu,\alpha^\vee)\geq 0$ for $\alpha\in\pi$ gives

$$s_0-s_1, s_0+s_1-s_2\geq 0,\quad s_1-s_0\geq y_0$$

and $2(\lambda+\rho,\mu)=\|\mu\|^2$ gives

$$(s_0+s_1-s_2)^2-(s_0-s_1)^2=2y_0s_0+s_2.$$

The last formula can be rewritten as

$$(s_0+s_1-s_2)^2=(s_0-s_1)^2+y_0(2s_0-s_2)+(y_0+1)s_2.$$

By above, $y_0<-1, 0\leq s_0-s_1\leq -y_0, 2s_0-s_2$, so the right-hand side is at most $(y_0+1)s_2<-s_2$. Hence $s_2=s_0+s_1-s_2=0$ that is $\mu=0$, as required. This completes the proof of (47).

13. Appendix

Let Δ be a root system of a finite-dimensional basic Lie superalgebra or an associated (untwisted or twisted) affine Lie superalgebra. As before, we fix $\Delta^+_{\bar{0}}$ and consider sets of simple roots Π such that $\Delta(\Pi)^+$ contains $\Delta^+_{\bar{0}}$.

13.1. Essentially simple roots

Let us call a root β *essentially simple* if there exists Π which contains β.

The set of non-isotropic essentially simple roots coincides with the set of simple roots for the non-isotropic part of Δ, see Proposition 2.1.1 (b). For $\Delta = A(m,n), C(n), A(m,n)^{(1)}$ and $C(n)^{(1)}$ any odd isotropic root is essentially simple.

13.1.1. We say that Δ' is a root subsystem of Δ if Δ (resp., Δ') has a subset of simple roots Π (resp., Π') such that $\Pi' \subset \Pi$.

Note that if Δ' is a root subsystem of Δ, then the essentially simple roots of Δ' are essentially simple for Δ (if β is essential for Δ', then β lies in a subset of simple roots for Δ' which is obtained from Π' by a chain of odd reflections; therefore β lies in a subset of simple roots for Δ which is obtained from Π by the same chain of odd reflections).

13.1.2. The following fact is useful. If α_1, α_2 are non-isotropic essentially simple roots and $\alpha_1 - \alpha_2$ is an isotropic root, then $\alpha_1 - \alpha_2$ is essentially simple (for instance, $\Delta = B(m,n)$ and $\alpha_1 = \varepsilon_m, \alpha_2 = \delta_n$).

Indeed, let Π_i be a set of simple roots containing α_i. Since $\alpha_1 - \alpha_2 \in \Delta$, $\Pi_1 \neq \Pi_2$. Since $\alpha_1, \alpha_2 \in \Delta^+(\Pi_i)$ for $i = 1, 2$ we have $\alpha_1 - \alpha_2 \in \Delta^+(\Pi_2), \alpha_2 - \alpha_1 \in \Delta^+(\Pi_1)$. Recall that Π_1 can be obtained from Π_2 by a chain of odd reflections. Since for an odd reflection r_γ we have

$$\Delta^+(r_\gamma \Pi) = (\Delta^+(\Pi) \setminus \{\gamma\}) \cup \{-\gamma\},$$

$r_{\alpha_1-\alpha_2}$ is one of the reflections in this chain. Hence $\alpha_1 - \alpha_2$ is essentially simple.

13.2. Finite parts

Let X be an affine Dynkin diagram. We call its connected subdiagram $\dot{X}$ its *finite part*, if $X \setminus \dot{X}$ contains exactly one root. By Lemma 3.1.1, $\dot{X}$ is of finite type. We call a root subsystem $\dot{\Delta}$ a finite part of the affine root system Δ if $\dot{\Delta}$ has a subset of simple roots $\dot{\Pi}$ which is a finite part of a subset of simple roots for Δ.

In this section we will describe the finite parts of affine root systems.

Let $\Delta \neq F(4)^{(1)}, G(3)^{(1)}$ and $\dot{\Delta}$ be a finite part of Δ. We will show that each set of simple roots $\dot{\Pi}$ such that $\dot{\Delta} = \Delta(\dot{\Pi})$ is a finite part of some Π (where $\Delta = \Delta(\Pi)$). Moreover, we will show that each $\dot{\Pi}$ can be uniquely extended to Π: if $\dot{\Delta} = \Delta(\dot{\Pi})$, then there exists a unique subset of simple roots Π for Δ containing $\dot{\Pi}$; moreover, $\dot{\Pi} \cong \dot{\Pi}'$ forces $\Pi \cong \Pi'$: if there exists a $(-,-)$-preserving map $\iota : \dot{\Delta} \to \dot{\Delta}'$ such that $\iota(\dot{\Pi}) = \dot{\Pi}'$, then ι can be extended to a $(-,-)$-preserving map $\tilde{\iota} : \Delta \to \Delta$ such that $\iota(\Pi) = \Pi'$.

Now let X be a disjoint union of Dynkin diagrams of affine Lie algebras. We call $\dot{X} \subset X$ a finite part of X if for each connected component X^j of X, $\dot{X} \cap X^j$ is a finite part of X^j. From the description of finite parts given below it follows that for $\Delta \neq F(4)^{(1)}, G(3)^{(1)}$ the following holds: if $\dot{\Delta}$ is a finite part of Δ, then $\dot{\Delta}_{\overline{0}}$ is a finite part of $\Delta_{\overline{0}}$.

13.2.1. The finite part of $X_l^{(1)}$ is X_l if $X = A, C, D$. The finite parts of other relevant to this paper affine Lie algebras are given by the following tables:

$B_k^{(1)}$	$A_{2k}^{(2)}$	$A_{2k-1}^{(2)}$	$D_{k+1}^{(2)}$	$G_2^{(1)}$
B_k, D_k	B_k, C_k	C_k, D_k	B_k	G_2

We claim that the finite parts of affine root systems for classical affine Lie superalgebras, which are not Lie algebras, are given by the following tables:

$A(k,l)^{(1)}$	$B(0,l)^{(1)}$	$B(k,l)^{(1)}, k>1$	$C(k)^{(1)}$	$D(k,l)^{(1)}$
$A(k,l)$	$B(0,l), C(l)$	$B(k,l), D(k,l)$	$C(k)$	$D(k,l)$

$A(2k,2l-1)^{(2)}$	$A(2k-1,2l-1)^{(2)}$	$A(2k,2l)^{(4)}$	$D(k+1,l)^{(2)}$	$C(l+1)^{(2)}$
$B(k,l), D(l,k)$	$D(k,l), D(l,k)$	$B(k,l), B(l,k)$	$B(k,l)$	$B(0,l)$

where we take $D(1,n) := C(n+1)$, $D(1,1) := A(1,0)$.

The finite parts of $G(3)^{(1)}$ are $G(3)$, $D(2,1,-\frac{3}{4})$, $A(2,0)$; the finite parts of $F(4)^{(1)}$ are $F(4)$, $A(3,0)$; the finite part of $D(2,1,a)^{(1)}$ is $D(2,1,a)$.

13.2.2. Let $\dot{\Delta}$ be a finite part of Δ. By definition, $\dot{\Delta}$ has a subset of simple roots $\dot{\Pi}$ which is a finite part of a subset of simple roots for Δ, which we denote by Π. Since any other subset of simple roots for $\dot{\Delta}$ can be obtained via a chain of odd reflections, any subset of simple roots $\dot{\Pi}'$ of $\dot{\Delta}$ is a finite part of some Π'. Let us show that Π' is unique and that $\dot{\Pi} \cong \dot{\Pi}'$ implies $\Pi \cong \Pi'$: i.e., if there exists a $(-,-)$-preserving map $\iota : \dot{\Delta} \to \dot{\Delta}'$ such that $\iota(\dot{\Pi}) = \dot{\Pi}'$, then ι can be extended to a $(-,-)$-preserving map $\tilde{\iota} : \Delta \to \Delta$ such that $\iota(\Pi) = \Pi'$.

For $\Delta = A(m,n)^{(1)}$ all Dynkin diagrams are cycles and the sum of all simple roots is δ; hence $\dot{\Pi}$ determines Π.

Since any set of simple roots for $\dot{\Delta}$ are connected by a chain of odd reflections, it is enough to verify the assertion for one choice of $\dot{\Pi}$. Let $\dot{\Pi}$ be a finite part of Π. Denote the unique root in $\Pi \setminus \dot{\Pi}$ by α_0.

Consider the case $\Delta \neq A(m,n)^{(1)}, F(4)^{(1)}, G(3)^{(1)}$. For each Π any proper connected subdiagram of Π is of finite type which is not $F(4), G(3)$. Hence any proper connected subdiagram has at most one branching node β, this node has three branches, two of these branches have length one (excluding the "branching" point) and consist of the nodes γ_1, γ_2 respectively with $\|\gamma_1\|^2 = \|\gamma_2\|^2$, $(\gamma_1, \beta) = (\gamma_2, \beta)$, which are connected if and only if $\|\gamma_1\|^2 = 0$.

It is not hard to show that $\dot{\Delta} \neq A(k,l)$ for $(k,l) \neq (1,1)$. Therefore $\dot{\Delta}$ is $B(k,l)$ or $D(k,l)$. Take $\dot{\Pi} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_k - \delta_1, \ldots, a\delta_l\}$ ($a = 1$ for $B(k,l)$, $a = 2$ for $D(k,l)$) and write $\alpha_1 := \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{k+l} := a\delta_l$. If in Π the node α_0 is connected to a node which is not α_1, then, by above, $\Pi \setminus \{\alpha_1\}$ is of type A, B, C, D and contains $\alpha_{k+l} = a\delta_l$; then $(\alpha_0, \alpha_i) = 0$ for $i > 2$ and α_2 is a "branching" node; since $\Pi \setminus \{\alpha_l\}$ is of finite type, $\|\alpha_0\|^2 = \|\alpha_1\|^2$, $(\alpha_0, \alpha_2) = (\alpha_1, \alpha_2)$ and $(\alpha_1, \alpha_2) = 0$; hence Π is uniquely defined (and it is of the type $B(k,l)^{(1)}$, $D(k,l)^{(1)}$ respectively). Consider the remaining case when α_0 is connected only to α_1; then the subdiagram $\alpha_0 - \alpha_1$ can be one of the following:

$$\otimes - \bigcirc; \quad \bigcirc - \bigcirc; \quad \bigcirc \Longrightarrow \bigcirc; \quad \bigcirc \Longleftarrow \bigcirc; \quad \bullet - \bigcirc.$$

For $\dot{\Pi} = B(k,l)$ the above subdiagrams correspond to Π of the types $B(k,l+1)$, $B(k+1,l)$, $A(2k,2l-1)^{(2)}$, $D(k+1,l)^{(2)}$ and $A(2k,2l)^{(4)}$ respectively. For $\dot{\Pi} = D(k,l)$ the above subdiagrams corresponds to Π of the types $D(k,l+1)$, $D(k+1,l)$, $A(2k-1,2l-1)^{(2)}$, $B(k,l)^{(1)}$ and $A(2l,2k-1)^{(2)}$ respectively. We conclude that in each type of Δ the set of simple roots Π containing $\dot{\Pi}$ is uniquely defined (up to isomorphism).

13.2.3. Root systems $F(4), F(4)^{(1)}$. Recall that the non-isotropic roots of $F(4)$ are $B_3 \times A_1$.

The root system $F(4)$ has 6 sets of simple roots, which are pairwise non-isomorphic; the root system $F(4)^{(1)}$ has 7 sets of simple roots, among them 4 non-isomorphic.

The finite parts of $F(4)^{(1)}$ are $F(4)$ and $A(3,0)$. Each Π for $F(4)^{(1)}$ contains $\dot{\Pi}$ of type $F(4)$ and each $\dot{\Pi}$ of type $F(4)$ can be uniquely, up to isomorphism, extended to some Π for $F(4)^{(1)}$. Only 4 (out of 7) sets of simple roots for $F(4)^{(1)}$ contain $\dot{\Pi}$ of the type $A(3,0)$; three of these sets of simple roots are non-isomorphic and each $\dot{\Pi}$ of the type $A(3,0)$ can be uniquely, up to isomorphism, extended to some Π.

There are 4 subsets $\dot{\Pi}_1, \ldots, \dot{\Pi}_4$ of simple roots for $F(4)$ satisfying $\|\alpha\|^2 \geq 0$ for each $\alpha \in \dot{\Pi}$; there are three subsets Π_1, Π_2, Π_3 of simple roots for $F(4)^{(1)}$ satisfying $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi$ ($\dot{\Pi}_i \subset \Pi_i$); one has $\Pi_1 \cong \Pi_3$. If we consider the affine root $\alpha_0(i) := \Pi_i \setminus \dot{\Pi}_i$ ($i = 1,2,3$), then $\alpha_0(1), \alpha_0(2)$ are long roots in B_3 and $\alpha_0(3)$ is isotropic.

Each of the sets Π_1, Π_2, Π_3 contains a finite part $A(3,0)$: if we denote by $\alpha_0(i)'$ the corresponding affine root, then this root is isotropic for Π_1, Π_3 and is a short root for Π_2.

13.2.4. Root systems $G(3), G(3)^{(1)}$. The non-isotropic roots of $G(3)$ are $G_2 \times B(0,1)$. We write G_2 in terms $\varepsilon_1, \varepsilon_2, \varepsilon_3$ subject to the relations $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, $\|\varepsilon_1\|^2 = \|\varepsilon_2\|^2 = \|\varepsilon_3\|^2$; we choose the set of simple roots $\varepsilon_2 - \varepsilon_3, \varepsilon_3$. For $B(0,1)$ we take the simple root δ_1 with $\|\delta_1\|^2 = -\|\varepsilon_i\|^2$. Then the isotropic roots are $\pm\delta_1 \pm \varepsilon_i, i = 1,2,3$.

The sets of simple roots for $G(3)$ are the following:

$$\begin{aligned}
&\delta_1 + \varepsilon_1, \varepsilon_2 - \varepsilon_3, \varepsilon_3;\\
&-\delta_1 - \varepsilon_1, \varepsilon_2 - \varepsilon_3, \delta_1 - \varepsilon_2;\\
&\varepsilon_3, \delta_1 - \varepsilon_3, -\delta_1 + \varepsilon_2;\\
&\delta_1, -\delta_1 + \varepsilon_3, \varepsilon_2 - \varepsilon_3.
\end{aligned}$$

Recall that we call an isotropic root essentially simple if it lies in some set of simple roots. The set of essentially simple roots for $G(3)$ is $\{\pm(\delta_1+\varepsilon_1); \pm(\delta_1 - \varepsilon_2); \pm(\delta_1 - \varepsilon_3)\}$.

The non-isotropic roots of $G(3)^{(1)}$ are $G_2^{(1)} \times B(0,1)^{(1)}$. Using the above notations, we write the set of simple roots for $G_3^{(1)}$ (resp., for $B(0,1)^{(1)}$) as $\delta + \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3$ (resp., $\delta - 2\delta_1, \delta_1$). Here is the sets of simple roots for $G(3)^{(1)}$ and their finite parts:

$$\begin{aligned}
&\delta - 2\delta_1, \delta_1 + \varepsilon_1, \varepsilon_2 - \varepsilon_3, \varepsilon_3; && G(3), D\Big(2,1,-\frac{3}{4}\Big),\\
&\delta - \delta_1 + \varepsilon_1, -\delta_1 - \varepsilon_1, \varepsilon_2 - \varepsilon_3, \delta_1 - \varepsilon_2; && G(3), D\Big(2,1,-\frac{3}{4}\Big), A(2,0),\\
&\delta + \varepsilon_1 - \varepsilon_2, \varepsilon_3, \delta_1 - \varepsilon_3, -\delta_1 + \varepsilon_2; && G(3), D\Big(2,1,-\frac{3}{4}\Big), A(2,0),\\
&\delta + \varepsilon_1 - \varepsilon_2, \delta_1, -\delta_1 + \varepsilon_3, \varepsilon_2 - \varepsilon_3; && G(3), A(2,0),\\
&-\delta + \delta_1 - \varepsilon_1, \delta - 2\delta_1, \varepsilon_2 - \varepsilon_3, \delta + \varepsilon_1 - \varepsilon_2; && D\Big(2,1,-\frac{3}{4}\Big), A(2,0).
\end{aligned}$$

The set of essentially simple roots is the union of the corresponding set for $G(3)$ with $\{\pm(\delta - \delta_1 + \varepsilon_1)\}$.

Note that $G(3)$ and $D(2,1,-\frac{3}{4})$ have 4 sets of simple roots; each set occurs exactly once as a finite part of a set of simple roots for $G(3)^{(1)}$ (in other words, $\dot{\Pi}$ can be uniquely extended to Π).

13.3.

In this subsection we will prove the following statement.

Lemma. *Let* $\Delta = A(m,n)^{(1)}, m \neq n$. *If* $\nu \in \mathbb{Z}_{\geq 0}(\Pi)$ *is such that* $(\nu, \Delta_{\overline{0}}) = 0$ *and* $2(\rho,\nu) = (\nu,\nu)$, *then* $\nu = 0$.

13.3.1. We start with the following problem. Let $X = (x_1, \ldots, x_{m+n})$ be a sequence of $m + n$ numbers, where m numbers are equal to 1 and n numbers to -1. Let $f(X)$ be the "total number of disorders":

$$f(X) := \sum_{i<j} \frac{1}{2}(x_i - x_j).$$

Clearly, $|f(X)| \le mn$. Let $X_1, \ldots, X_{m+n}$ be the sequences obtained from X by cyclic permutations. We claim that there exist i, j such that $0 \le f(X_i) < 2m$ and $-2n < f(X_j) \le 0$.

Indeed, let $\sigma(X)$ be the sequence obtained from X by moving last element to the first place: $\sigma(x)_i := x_{i-1}$ if $1 < i \le m + n$, $\sigma(x)_1 := x_{m+n}$. Set $f_k(X) := f(\sigma^k(X))$. Since $\sigma^{m+n} = Id$, f_k has period $m + n$.

We claim that $\sum_{k=0}^{m+n-1} f_k(X) = 0$ for each X. Indeed, let X be any sequence and $s \in \{1, 2, \ldots, m + n - 1\}$ be such that $x_s = 1$ and $x_{s+1} = -1$; let X' be the sequence obtained from X by switching x_s and x_{s+1} ($x'_i = x_i$ if $i \neq s$, $x'_s = -1, x'_{s+1} = 1$). Then $f(X') = f(X) - 2$. Note that $\sigma^j(X')$ is obtained from $\sigma^j(X)$ by the same operation (for different index s) if $j + s \not\equiv 0$ modulo $m + n$; if $j + s \not\equiv 0$ modulo $m + n$, then for $Y := \sigma^j(X)$ we have $y_1 = -1, y_{m+n} = 1$ and $Y' := \sigma^j(X')$ is obtained from Y by switching y_1 and y_{m+n}. Clearly, $f(Y') = f(Y) + 2(m + n - 1)$; by above, if $j + s \not\equiv 0$ modulo $m + n$, then $f(\sigma^j(X')) = f(\sigma^j(X')) - 2$. Hence $\sum_{k=0}^{m+n-1} f_k(X) = \sum_{k=0}^{m+n-1} f_k(X')$. Since any sequence can be obtained from the sequence $X_0 = (1, \ldots, 1, -1, \ldots, -1)$ by a chain of above operations, we obtain $\sum_{k=0}^{m+n-1} f_k(X) = \sum_{k=0}^{m+n-1} f_k(X_0)$. One has readily sees that $\sum_{k=0}^{m+n-1} f_k(X_0) = 0$, as required.

Note that $f(\sigma(X)) = f(X) + 2n$ if $x_{m+n} = 1$ and $f(\sigma(X)) = f(X) - 2m$ if $x_{m+n} = -1$, so $f_{i+1} - f_i$ is $2n$ or $-2m$. Since $\sum_{k=0}^{m+n-1} f_k(X) = 0$, $f_k(X)$ contains positive and negative elements. If i is such that $f_{i+1}(X) < 0 \le f_i(X)$, then $f_i(X) < 2m$, and if j is such that $f_{j+1}(X) > 0 \ge f_j(X)$, then $f_j(X) > -2n$. The claim follows.

13.3.2. Recall that a set of simple roots for $A(m, n)$ can be naturally encoded as a sequence of m dots and n crosses, see §5.6): for instance, $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3\}$ is encoded by the sequence $\cdot\ \cdot\ \times\ \cdot$; putting 1 instead of dots and -1 instead of crosses, we obtain a sequence considered in §13.3.1. Similarly, a set of simple roots for $A(m, n)^{(1)}$ can be encoded by the same sequence viewed as a cycle. The inverse procedure can be described as follows: to a sequence X as in §13.3.1 we assign the Dynkin diagram of $A(m, n)$-type with $\|\alpha_i\|^2 = x_{i+1} + x_i$ for $i = 1, \ldots, m + n - 1$ and the Dynkin diagram of $A(m, n)^{(1)}$-type with $\|\alpha_i\|^2 = x_{i+1} + x_i$ for $i = 1, \ldots, m + n - 1$ and $\|\alpha_0\|^2 = x_1 + x_{m+n}$. Clearly, X and $\sigma(X)$ give the same Dynkin diagram of $A(m, n)^{(1)}$-type.

13.3.3. Let Π be a set of simple roots for $A(m,n)^{(1)}$ and $\tilde{X}$ be the corresponding cyclic sequence (which we view as a set containing $m+n$ usual sequences). Take a sequence $X \in \tilde{X}$ such that $-2n < f(X) \leq 0$. Let $\dot{\Pi}$ be the corresponding Dynkin diagram of $A(m,n)$-type. Since $m \neq n$, we may (and will) assume that $m > n$ and use the standard notations for $A(m,n)$. We set

$$E := \sum_{i=1}^{m} \varepsilon_i, \quad D := \sum_{i=1}^{n} \delta_i.$$

We take the standard form $(-,-)$ (i.e., $\|\varepsilon_i\|^2 = -\|\delta_j\|^2 = 1$). Then

$$2(\rho, E) = -\Big(\sum_{\alpha \in \dot{\Delta}_{\bar{1}}^+} \alpha, E \Big) = -f(X).$$

The condition $(\nu, \Delta_{\bar{0}}) = 0$ is equivalent to $\nu = j\delta + u(nE - mD)$; since $\nu \in \mathbb{Z}_{\geq 0}(\Pi)$, we have $j \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{Z}\frac{1}{d}$, where $d := GCD(m,n)$. One has $(\rho, \delta) = m - n$. Since $(\Delta, E - D) = 0$, we have $(\rho, \nu) = j(m-n) - (m-n)u(\rho, E)$, so $2(\rho, \nu) = (\nu, \nu)$ gives

$$2j(m-n) + (m-n)uf(X) = -u^2 mn(m-n),$$

that is $u^2 mn + uf(X) + 2j = 0$. Writing $u = s/d$ with $s \in \mathbb{Z}$, we obtain

$$s^2 mn + 2jd^2 = -df(X)s.$$

Since $0 \leq -f(X) < 2n$, we get $s = 0$ or $s > 0$ and $s^2 mn + 2jd^2 < 2nds$. One has $2d \leq m$ (because $n < m$), so the only solution is $s = j = 0$. Hence $\nu = 0$, as required.

13.4.

Recall that an odd isotropic root β is essentially simple if it belongs to a set of simple roots.

Let π be a connected component of Π_0. For $\nu \in \mathfrak{h}^*$ write $\nu \succ_\pi r_\alpha \nu$ if $\alpha \in \pi$ and $\nu - r_\alpha \nu \in \mathbb{Z}_{>0}\alpha$, and consider the order $\succ_\pi$ on $\mathfrak{h}^*$ generated by this ($\nu \succ_\pi \mu$ if $\mu = r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_s} \nu$ with $(r_{\alpha_i} r_{\alpha_i+1} \cdots r_{\alpha_s} \nu, \alpha_{i-1}^\vee) \in \mathbb{Z}$); write $\nu \succeq_\pi \mu$ if $\nu \succ_\pi \mu$ or $\nu = \mu$.

Lemma. *Let Δ be a finite or affine root system and Iso be the set of odd isotropic roots. Let $X \subset$ Iso be the set of odd isotropic roots β with the following property: for each connected component π of Π_0 there exists an essentially simple root β' such that $\beta' \succeq_\pi \beta$. Then Iso $\subset (X \cup (-X))$.*

Proof. Let π be one of the root systems B_m, C_m, D_m with the standard notations. Clearly, $\varepsilon_i \succeq_\pi \varepsilon_m \succeq_\pi -\varepsilon_{m-1}$ and $\pm\varepsilon_i \succeq_\pi -\varepsilon_1$ for each $i = 1, \dots, m$. In particular, ε_m is the minimal element in $\{\varepsilon_i\}_{i=1}^m$ with respect to the order $\succeq_\pi$.

Now let π be one of the root systems $B_m^{(1)}, C_m^{(1)}, D_m^{(1)}, A_{2m-1}^{(2)}$ with the standard notations, or $\pi = A_{2m}^2 = \{\delta - 2\varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_m\}$ and $Y := \{\varepsilon_i, s\delta \pm \varepsilon_i : s \in \mathbb{Z}_{>0}\}_{i=1}^m$. We claim that ε_m is again the minimal element in Y with respect to the order $\succeq_\pi$. Indeed, the finite part of π is B_m, C_m or D_m, so $s\delta \pm \varepsilon_i \succeq_\pi s\delta - \varepsilon_1$. For $B_m^{(1)}, D_m^{(1)}, A_{2m-1}^{(2)}$ the affine root is $\delta - \varepsilon_1 - \varepsilon_2$, so $s\delta - \varepsilon_1 \succeq_\pi (s-1)\delta + \varepsilon_2$; for $C_m^{(1)}, A_{2m}^{(2)}$ the affine root is $\delta - 2\varepsilon_1$, so $s\delta - \varepsilon_1 \succeq_\pi (s-1)\delta + \varepsilon_1$. Hence, for $s > 0$ one has $s\delta \pm \varepsilon_i \succeq_\pi \varepsilon_m$, as required.

Now consider $\pi = D_m^{(2)} = \{\delta - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_m\}$ with $Y := \{\varepsilon_i, 2s\delta \pm \varepsilon_i, s \in \mathbb{Z}_{>0}\}_{i=1}^m$. Since $2s\delta - \varepsilon_1 \succeq_\pi 2(s-1)\delta + \varepsilon_1$, ε_m is again the minimal element in Y with respect to the order $\succeq_\pi$.

Recall that for $\Delta = A(m,n), C(n), A(m,n)^{(1)}, C(n)^{(1)}$ any odd root is essentially simple, so $X = \Delta_{\bar{1}}$. Take $\Delta \neq A(m,n), C(n), A(m,n)^{(1)}, C(n)^{(1)}$.

If Δ is $B(m,n)$ or $D(m,n)$, then $Iso = \{\pm\varepsilon_i \pm \delta_j\}$. The roots $\pm(\varepsilon_i - \delta_j)$ are essentially simple, so X contains the roots $\pm(\varepsilon_i - \delta_j), \varepsilon_i + \delta_j$. Hence $Iso \subset (X \cup (-X))$.

For $D(2,1,a)$, one has $Iso = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}$ and all roots are essentially simple except for $\pm(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$. Thus $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \in X$, so $Iso \subset (X \cup (-X))$.

For $F(4)$ recall that $\Pi_0 = A_1 \times B_3$ and choose $\Pi_0 = \{\delta_1; \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3\}$. In this case $Iso = \{\pm\frac{1}{2}(\delta_1 \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)\}$. Take $\beta = \frac{1}{2}(\delta_1 \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)$; if at least two signs $\pm$ are $-$, then β is essentially simple, so $\beta \in X$. If β is not essentially simple, then $-(\beta - \delta_1)$ is essentially simple, so $\beta - \delta_1$ is essentially simple. For $\pi = B_3$ we have

$$\begin{aligned} \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \succeq_\pi \varepsilon_1 + \varepsilon_2 - \varepsilon_3 \succeq_\pi \varepsilon_1 - \varepsilon_2 + \varepsilon_3 \succeq_\pi -\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ \succeq_\pi -\varepsilon_1 + \varepsilon_2 - \varepsilon_3, \end{aligned}$$

so $\beta \succeq_\pi \frac{1}{2}(\delta_1 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3)$; the last root is essentially simple. Hence $\beta \in X$. Thus X contains the roots of the form $\frac{1}{2}(\delta_1 \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)$, so $Iso \subset (X \cup (-X))$.

For $G(3)$ one has $Iso = \{\pm\delta_1 \pm \varepsilon_i\}_{i=1}^3$ and the essentially simple roots are $\pm(\delta_1 + \varepsilon_1), \pm(\delta_1 - \varepsilon_i), i = 2, 3$. It is easy to see that $X = Iso \setminus \{-\delta_1 - \varepsilon_i\}_{i=1}^3$.

In the remaining cases Δ is affine. Let $\dot{\Delta}$ be a finite part of Δ. Clearly, $\dot{Iso} = Iso \cap \dot{\Delta}$ is the set of isotropic odd roots in $\dot{\Delta}$; let $\dot{X} \subset \dot{Iso}$ be the corresponding set for $\dot{\Delta}$. Recall that any essentially simple root for $\dot{\Delta}$ is essentially simple for Δ, so $\dot{X} \subset X$.

Note that $Iso = \dot{Iso} + \mathbb{Z}\delta$, except for $C(m)^{(2)}, D(m,n)^{(2)}$ with $Iso = \dot{Iso} + \mathbb{Z}\delta$. Let us show that X contains $\dot{Iso} + j\delta$ for $j \in \mathbb{Z}_{>0}$, where j is even for $C(m)^{(2)}, D(m,n)^{(2)}$.

First, consider the case when $\dot{\Delta} \neq F(4), G(3)$. Then $\dot{\Delta}$ is $B(m,n)$ or $D(m,n)$ and π is one of the root systems $A_1^{(1)}, B_m^{(1)}, C_m^{(1)}, D_m^{(1)}, A_{2m-1}^{(2)}, A_{2m}^2,$

$D_m^{(2)}$. If π lies in the span of ε_is, then, by above, for $s > 0$ (and s even for $C(m)^{(2)}, D(m,n)^{(2)}$) one has $s\delta \pm \varepsilon_i \succeq_\pi \varepsilon_m$, so $s\delta \pm \varepsilon_i \pm \delta_j \succeq_\pi \varepsilon_m \pm \delta_j$; $\varepsilon_m - \delta_j$ is essentially simple and $\varepsilon_m + \delta_j \succeq_\pi -\varepsilon_1 \pm \delta_j$, where $-\varepsilon_1 \pm \delta_j$ is also essentially simple. Hence X contains the roots $s\delta \pm \varepsilon_i \pm \delta_j$ for $s > 0$ (and s even for $C(m)^{(2)}, D(m,n)^{(2)}$), as required. The similar reasoning shows that X contains $\dot{Iso} + \mathbb{Z}_{>0}\delta$ for $F(4)^{(1)}$ and $G(3)^{(1)}$.

Combining $Iso = \dot{Iso} + \mathbb{Z}\delta$ (resp., $Iso = \dot{Iso} + \mathbb{Z}\delta$ for $C(m)^{(2)}, D(m,n)^{(2)}$), the fact that X contains $\dot{X}$ and $\dot{Iso} + \mathbb{Z}_{>0}\delta$ (resp., $\dot{Iso} + 2\mathbb{Z}_{>0}\delta$ for $C(m)^{(2)}$, $D(m,n)^{(2)}$), and the inclusion $\dot{Iso} \subset (\dot{X} \cup (-\dot{X}))$, we obtain $Iso \subset (X \cup (-X))$ as required. □

13.5.

Let $\Pi \neq A(m,n)^{(1)}, C(n)^{(1)}$ be a set of simple roots of affine type, $\dot{\Pi}$ be its finite part and $\pi \subset \Pi_0$ be as in §3.1.3: $\pi := \{\alpha \in \Pi_0 \mid (\alpha,\alpha) \in \mathbb{Q}_{>0}\}$ if the form $(-,-)$ is such that $(\rho,\delta) \in \mathbb{Q}_{\geq 0}$. Recall the conditions (A)–(C) from Theorem 11.3.4:

(A) $\|\alpha_0\|^2 \geq 0$;
(B) for each $\alpha \in \pi$ there exists $\beta \in \operatorname{supp}(\alpha)$ such that $\beta \notin \operatorname{supp}(\alpha')$ for each $\alpha' \in \pi$; this β is denoted by $b(\alpha)$;
(C) $\rho \in X_1 - X_2$, where

$$X_1 := \{\mu \in \mathfrak{h}^* \mid (\mu,\alpha) \in \mathbb{Q}_{\geq 0} \text{ for all } \alpha \in \Pi\}, \quad X_2 := \sum_{\alpha \in \dot{\Pi}_0} \mathbb{Q}_{\geq 0}\alpha^\vee.$$

Let us give some examples when these conditions hold.

If $\Delta \neq F(4)$ is finite and π is a connected component of Π_0, then $\operatorname{supp}(\alpha) \cap \operatorname{supp}(\alpha') = \emptyset$ for each $\alpha, \alpha' \in \pi$, except for the pair $\alpha, \alpha' = \varepsilon_{m-1} \pm \varepsilon_m$, for $\pi = D_m$. Since for affine Δ, $\dot{\Delta}$ is finite, this implies that (B) holds for all affine sets of simple roots Π which are not of type $F(4)^{(1)}$.

13.5.1. Case $B(m,n)^{(1)}, D(m,n)^{(1)}$. By above, the condition (B) always hold. Condition (C) holds if $\|\alpha\|^2 \geq 0$ for each $\alpha \in \Pi(L)$ (since $\rho \in X_1$). Let us describe other cases when (C) holds.

We write $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_{m+n}\}$ with a standard enumeration: this means $(\alpha_i, \alpha_{i+1}) \neq 0$ for each i, except for the case $D(m,n)^{(1)}$, where sometimes $(\alpha_{m+n-1}, \alpha_{m+n}) = 0$, but $(\alpha_{m+n-2}, \alpha_{m+n}) \neq 0$ (i.e., α_{m+n-2} is a "branching" point of the Dynkin diagram). Note that this convention determines the enumeration, except for $D(m,n)^{(1)}$ with the "branching" point α_{m+n-2}, where we can interchange α_{m+n-1} and α_{m+n}; note that in this case $\|\alpha_{m+n-1}\|^2 = \|\alpha_{m+n}\|^2$.

Introduce the numbers $d_{m+n}, \ldots, d_1$ by the following rule:

$$d_{m+n} := \max(-\|\alpha_{m+n}\|^2, 0), \quad d_i := \max(d_{i+1} - \|\alpha_i\|^2, 0)$$
$$\text{for } i = 1, \ldots, m+n-1,$$

if α_{m-n-2} is not a "branching" point and

$$d_{m+n} = d_{m+n-1} := \max(-\|\alpha_{m+n}\|^2, 0), \quad d_i := \max(d_{i+1} - \|\alpha_i\|^2, 0)$$
$$\text{for } i = 1, \ldots, m+n-2,$$

if α_{m-n-2} is a "branching" point (in this case $\|\alpha_{m+n}\|^2 = \|\alpha_{m+n-1}\|^2$).

The property (C) holds if and only if the sum of d_i with $(\alpha_0, \alpha_i) \neq 0$ is not greater than 2. If $m > n$, then α_1 is a "branching" point (i.e., $(\alpha_0, \alpha_2) \neq 0$) and (C) is equivalent to $d_1 + d_2 \leq 2$; if $m \leq n$, (C) is equivalent to $d_1 \leq 2$. For instance, for $B(m,n)^{(1)}$ with $m \leq n$, if Π has j roots of negative square length and there are j roots of positive square length which precede (counting from α_0) the roots of negative square length, then Π satisfies (C).

13.5.2. Conditions (B) and (C) for exceptional Lie superalgebras. For $D(2,1,a)^{(1)}$ we have two sets of simple roots satisfying (A): one consists of isotropic roots and another one with $\|\alpha_0\|^2 > 0$ (for $a \in \mathbb{Q},\ 0 < a < 1$, it takes the form $\Pi = \{\delta - 2\varepsilon_1, \varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2, 2\varepsilon_3\}$). Both of them satisfy conditions (B) and (C).

For $G(3)^{(1)}$ there are three sets of simple roots, which satisfy (A), all of them satisfy (B) and (C); these are the second, the third and the fourth sets in §13.2.4.

For $F(4)^{(1)}$ there are two sets of simple roots which satisfy (A)–(C); they correspond to the third and the fourth sets of simple roots in [K1], 2.5.4.

13.5.3. For $D(n+1,n)^{(1)}, A(2n-1,2n-1)^{(2)}$ for Π as in §6.4.1 we have $\|\alpha_0\|^2 = 0$ (so (A) holds) and $\rho = 0$ (so (C) holds); it is easy to verify that (B) holds for both choices of π, so (A)–(C) hold for this Π (for both choices of π).

For $D(n+1,n)^{(2)}, A(2n,2n)^{(4)}$ with fixed π, we choose a presentation of Δ via ε_i, δ_j, where π lies in the span of ε_is (as it was done in §6.4.2). The set of simple root $\{\delta - \delta_1, \delta_1 - \varepsilon_1, \ldots, \varepsilon_n\}$ chosen in §6.4.2 does not satisfy (A). However, $\Pi = \{\delta - \varepsilon_1, \varepsilon_1 - \delta_1, \ldots, \varepsilon_n - \delta_n, \delta_n\}$ satisfies (A)–(C). Indeed, we can fix $(-,-)$ with $\|\varepsilon_i\|^2 = 1$. Then $\|\delta - \varepsilon_1\|^2 = 1$, so (A) holds. One has $\pi = \{a'(\delta - \varepsilon_1), \varepsilon_1 - \varepsilon_2, \ldots, a\varepsilon_n\}$ with $a, a' \in \{1,2\}$; thus $\operatorname{supp}(\alpha)\operatorname{supp}(\alpha') = \emptyset$ for $\alpha \neq \alpha' \in \pi$ and (B) holds. Note that ε_i or $2\varepsilon_i$ (resp., δ_i or $2\delta_i$) lies in $\dot{\Delta}_{\bar{0}}^+$, so $\varepsilon_i, -\delta_i \in X_2$. One has $2\rho = \sum_{i=1}^n (\delta_i - \varepsilon_i)$, so $\rho \in -X_2$; hence (C) holds.

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