MIT Libraries | DSpace@MIT

MIT Open Access Articles

Anisotropic Growth of Random Surfaces in 2 + 1 Dimensions

Massachusetts Institute of Technology

Anisotropic Growth of Random Surfaces in 2 + 1 Dimensions

Alexei Borodin[1], **Patrik L. Ferrari**[2]

[1] Massachusetts Institute of Technology, Department of Mathematics, 77 Massachusetts Avenue, Cambridge,
MA 02139-4307, USA. E-mail: borodin@math.mit.edu
[2] Bonn University, Institute for Applied Mathematics, Endenicher Allee 60, 53115 Bonn, Germany.
E-mail: ferrari@uni-bonn.de

Abstract: We construct a family of stochastic growth models in 2 + 1 dimensions, that belong to the anisotropic KPZ class. Appropriate projections of these models yield 1 + 1 dimensional growth models in the KPZ class and random tiling models. We show that correlation functions associated to our models have determinantal structure, and we study large time asymptotics for one of the models.

The main asymptotic results are: (1) The growing surface has a limit shape that consists of facets interpolated by a curved piece. (2) The one-point fluctuations of the height function in the curved part are asymptotically normal with variance of order $\ln(t)$ for time $t \gg 1$. (3) There is a map of the $(2 + 1)$-dimensional space-time to the upper half-plane $\mathbb{H}$ such that on space-like submanifolds the multi-point fluctuations of the height function are asymptotically equal to those of the pullback of the Gaussian free (massless) field on $\mathbb{H}$.

Contents

1. Introduction

In recent years there has been a lot of progress in understanding large time fluctuations of driven interacting particle systems on the one-dimensional lattice, see e.g. [2, 3, 5, 13–17,

43,47,48,58,68–70,72]. Evolution of such systems is commonly interpreted as random growth of a one-dimensional interface, and if one views the time as an extra variable, the evolution produces a random surface (see e.g. Fig. 4.5 in [66] for a nice illustration). In a different direction, substantial progress has also been achieved in studying the asymptotics of random surfaces arising from dimers on planar bipartite graphs, see the review [50] and references therein.

Although random surfaces of these two kinds were shown to share certain asymptotic properties (also common to random matrix models), no direct connection between them was known. One goal of this paper is to establish such a connection.

We construct a class of two-dimensional random growth models (that is, the principal object is a randomly growing surface, embedded in the four-dimensional space-time). In two different projections these models yield random surfaces of the two kinds mentioned above (one reduces the spatial dimension by one, the second projection is fixing time). We partially compute the correlation functions of an associated (three-dimensional) random point process and show that they have determinantal form that is typical for determinantal point processes.

For one specific growth model we compute the correlation kernel explicitly, and use it to establish Gaussian fluctuations of the growing random surface. We then determine the covariance structure.

Let us describe our results in more detail.

1.1. A two-dimensional growth model. Consider a continuous time Markov chain on the state space of interlacing variables

$$S^{(n)} = \left\{ \{x_k^m\}_{\substack{k=1,\ldots,m \\ m=1,\ldots,n}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_{k-1}^m < x_{k-1}^{m-1} \le x_k^m \right\}, \quad n = 1, 2, \ldots. \tag{1.1}$$

x_k^m can be interpreted as the position of particle with label (k, m), but we will also refer to a given particle as x_k^m. As initial condition, we consider the fully-packed one, namely at time moment $t = 0$ we have $x_k^m(0) = k - m - 1$ for all k, m, see Fig. 1.

The particles evolve according to the following dynamics. Each of the particles x_k^m has an independent exponential clock of rate one, and when the x_k^m-clock rings the particle attempts to jump to the right by one. If at that moment $x_k^m = x_k^{m-1} - 1$ then the jump is blocked. If that is not the case, we find the largest $c \ge 1$ such that
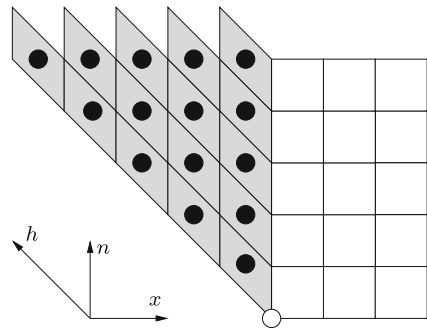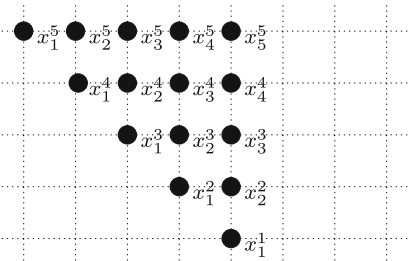

Fig. 1. Illustration of the initial conditions for the particles system and the corresponding lozenge tilings. In the height function picture, the white circle has coordinates $(x, n, h) = (-1/2, 0, 0)$
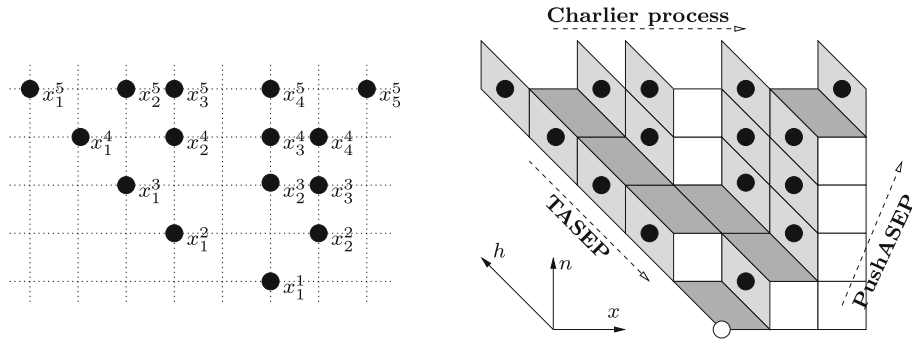
Fig. 2. From particle configurations (*left*) to 3d visualization via lozenge tilings (*right*). The corner with the white circle has coordinates $(x, n, h) = (-1/2, 0, 0)$

$x_k^m = x_{k+1}^{m+1} = \cdots = x_{k+c-1}^{m+c-1}$, and all c particles in this string jump to the right by one. For any $t \geq 0$ denote by $\mathcal{M}^{(n)}(t)$ the resulting measure on $\mathcal{S}^{(n)}$ at time moment t.

Informally speaking, the particles with smaller upper indices are heavier than those with larger upper indices, so that the heavier particles block and push the lighter ones in order for the interlacing conditions to be preserved. This anisotropy is essential, see more details in Sect. 1.4.

Let us illustrate the dynamics using Fig. 2, which shows a possible configuration of particles obtained from our initial condition. If in this state of the system the x_1^3-clock rings, then particle x_1^3 does not move, because it is blocked by particle x_1^2. If it is the x_2^2-clock that rings, then particle x_2^2 moves to the right by one unit, but to keep the interlacing property satisfied, also particles x_3^3 and x_4^4 move by one unit at the same time. This aspect of the dynamics is called "pushing".

Observe that $\mathcal{S}^{(n_1)} \subset \mathcal{S}^{(n_2)}$ for $n_1 \leq n_2$, and the definition of the evolution implies that $\mathcal{M}^{(n_1)}(t)$ is a marginal of $\mathcal{M}^{(n_2)}(t)$ for any $t \geq 0$. Thus, we can think of $\mathcal{M}^{(n)}$'s as marginals of the measure $\mathcal{M} = \lim \mathcal{M}^{(n)}$ on $\mathcal{S} = \lim \mathcal{S}^{(n)}$. In other words, $\mathcal{M}(t)$ are measures on the space $\mathcal{S}$ of infinite point configurations $\{x_k^m\}_{k=1,\dots,m, \, m \geq 1}$.

Before stating the main results, it is interesting to notice that the Markov chain has different interpretations. Also, some projections of the Markov chain to subsets of $\mathcal{S}^{(n)}$ are still Markov chains.

1. The evolution of x_1^1 is the one-dimensional Poisson process of rate one.
2. The row $\{x_1^m\}_{m \geq 1}$ evolves as a Markov chain on $\mathbb{Z}$ known as the *Totally Asymmetric Simple Exclusion Process* (TASEP), and the initial condition $x_1^m(0) = -m$ is commonly referred to as the *step initial condition*. In this case, particle x_1^k jumps to its right with unit rate, provided the arrival site is empty (exclusion constraint).
3. The row $\{x_m^m\}_{m \geq 1}$ also evolves as a Markov chain on $\mathbb{Z}$ that is sometimes called "long range TASEP"; it was also called PushASEP in [13]. It is convenient to view $\{x_m^m + m\}_{m \geq 1}$ as particle locations in $\mathbb{Z}$. Then, when the x_k^k-clock rings, the particle $x_k^k + k$ jumps to its right and pushes by one unit the (maybe empty) block of particles sitting next to it. If one disregards the particle labeling, one can think of particles as independently jumping to the next free site on their right with unit rate.
4. For our initial condition, the evolution of each row $\{x_k^m\}_{k=1,\dots,m}, m = 1, 2, \dots$, is also a Markov chain. It was called the *Charlier process* in [55] because of its relation to the classical orthogonal Charlier polynomials. It can be defined as the Doob h-transform

for m independent rate one Poisson processes with the harmonic function h equal to the Vandermonde determinant.

5. Infinite point configurations $\{x_k^m\} \in \mathcal{S}$ can be viewed as *Gelfand-Tsetlin schemes*. Then $\mathcal{M}(t)$ is the "Fourier transform" of a suitable irreducible character of the infinite-dimensional unitary group $U(\infty)$, see [22]. Interestingly enough, increasing t corresponds to a deterministic flow on the space of irreducible characters of $U(\infty)$.

6. Elements of $\mathcal{S}$ can also be viewed as the lozenge tiling of a sector in the plane. To see that one surrounds each particle location by a rhombus of one type and draws edges through locations where there are no particles, see Fig. 2. Our initial condition corresponds to a perfectly regular tiling, see Fig. 1.

7. The random tiling defined by $\mathcal{M}(t)$ is the limit of the uniformly distributed lozenge tilings of hexagons with side lengths (a, b, c), when $a, b, c \to \infty$ so that $ab/c \to t$, and we observe the hexagon tiling at finite distances from the corner between sides of lengths a and b.

8. Finally, Fig. 2 has a clear three-dimensional connotation. Given the random configuration $\{x_k^n(t)\} \in \mathcal{S}$ at time moment t, define the random *height function*

$$
\begin{aligned}
&h : (\mathbb{Z} + \tfrac{1}{2}) \times \mathbb{Z}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}, \\
&h(x, n, t) = \#\{k \in \{1, \ldots, n\} \mid x_k^n(t) > x\}.
\end{aligned}
\tag{1.2}
$$

In terms of the tiling on Fig. 2, the height function is defined at the vertices of rhombi, and it counts the number of particles to the right from a given vertex. (This definition differs by a simple linear function of (x, n) from the standard definition of the height function for lozenge tilings, see e.g. [50,51].) The initial condition corresponds to starting with perfectly flat facets.

Thus, our Markov chain can be viewed as a random growth model of the surface given by the height function. In terms of the step surface of Fig. 2, the evolution consists of removing all columns of (x, n, h)-dimensions $(1, *, 1)$ that could be removed, independently with exponential waiting times of rate one. For example, if x_2^2 jumps to its right, then three consecutive cubes (associated to x_2^2, x_3^3, x_4^4) are removed. Clearly, in this dynamics the directions x and n do not play symmetric roles. Indeed, this model belongs to the $2 + 1$ anisotropic KPZ class of stochastic growth models, see Sect. 1.4.

1.2. Determinantal formula, limit shape and one-point fluctuations. The first result about the Markov chain $\mathcal{M}(t)$ that we prove is the (partial) determinantal structure of the correlation functions. Introduce the notation

$$
(n_1, t_1) \prec (n_2, t_2) \quad \text{iff} \quad n_1 \leq n_2, t_1 \geq t_2, \text{ and } (n_1, t_1) \neq (n_2, t_2). \tag{1.3}
$$

Theorem 1.1. *For any $N = 1, 2, \ldots$, pick N triples,*

$$
\varkappa_j = (x_j, n_j, t_j) \in \mathbb{Z} \times \mathbb{Z}_{>0} \times \mathbb{R}_{\geq 0},
$$

such that

$$
t_1 \leq t_2 \leq \cdots \leq t_N, \qquad n_1 \geq n_2 \geq \cdots \geq n_N. \tag{1.4}
$$

Then

$$
\begin{aligned}
\mathbb{P}\{&\textit{For each } j = 1, \ldots, N \textit{ there exists a } k_j, \\
&1 \leq k_j \leq n_j \textit{ such that } x_{k_j}^{n_j}(t_j) = x_j\} = \det\left[\mathcal{K}(\varkappa_i, \varkappa_j)\right]_{i,j=1}^N, \tag{1.5}
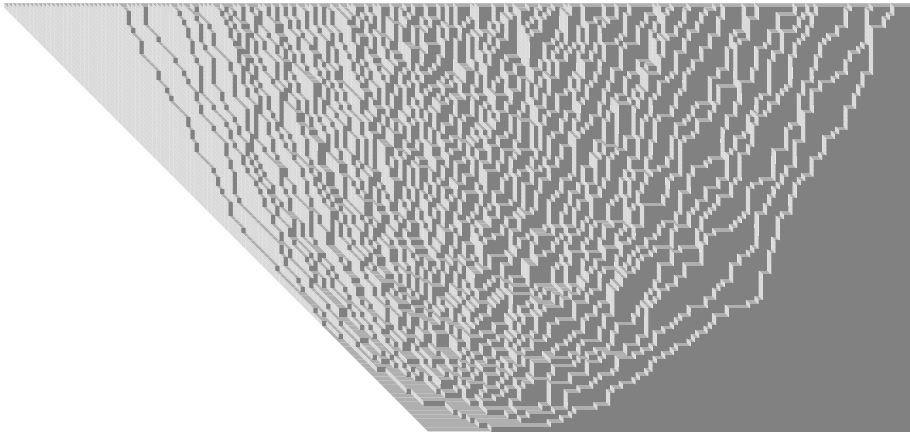\end{aligned}
$$

Fig. 3. A configuration of the model analyzed with $N = 100$ particles at time $t = 25$, using the same representation as in Fig. 2. In [38] there is a Java animation of the model

where

$$\mathcal{K}(x_1, n_1, t_1; x_2, n_2, t_2) = -\frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{w^{x_2-x_1+1}} \frac{e^{(t_1-t_2)/w}}{(1-w)^{n_2-n_1}} \mathbb{1}_{[(n_1,t_1) \prec (n_2,t_2)]}$$
$$+ \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz \frac{e^{t_1/w}}{e^{t_2/z}} \frac{(1-w)^{n_1}}{(1-z)^{n_2}} \frac{w^{x_1}}{z^{x_2+1}} \frac{1}{w-z}, \quad (1.6)$$

the contours Γ_0, Γ_1 *are simple positively oriented closed paths that include the poles* 0 *and* 1, *respectively, and no other poles (hence, they are disjoint).*

This result is proved at the end of Sect. 2.8. The above kernel has in fact already appeared in [13] in connection with PushASEP. The determinantal structure makes it possible to study the asymptotics. On a macroscopic scale (large time limit and hydrodynamic scaling) the model has a limit shape, which we now describe, see Fig. 3. Since we look at heights at different times, we cannot use time as a large parameter. Instead, we introduce a large parameter L and consider space and time coordinates that are comparable to L. The limit shape consists of three facets interpolated by a curved piece. To describe it, consider the set

$$\mathcal{D} = \{(\nu, \eta, \tau) \in \mathbb{R}_{>0}^3 \mid (\sqrt{\eta} - \sqrt{\tau})^2 < \nu < (\sqrt{\eta} + \sqrt{\tau})^2\}. \quad (1.7)$$

It is exactly the set of triples $(\nu, \eta, \tau) \in \mathbb{R}_{>0}^3$ for which there exists a nondegenerate triangle with side lengths $(\sqrt{\nu}, \sqrt{\eta}, \sqrt{\tau})$. Denote by $(\pi_\nu, \pi_\eta, \pi_\tau)$ the angles of this triangle that are opposite to the corresponding sides (see Fig. 4 too).

Our second result concerns the limit shape and the Gaussian fluctuations in the curved region, living on a $\sqrt{\ln L}$ scale.

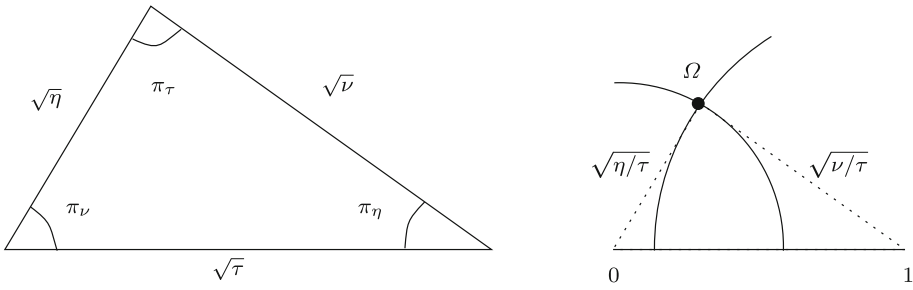**Theorem 1.2.** *For any* $(\nu, \eta, \tau) \in \mathcal{D}$ *we have the moment convergence of random variables*

Fig. 4. The triangle of (3.3) on the *left* and its scaled version defined by intersection of circles on the *right*

$$\lim_{L\to\infty} \frac{h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L) - \mathbb{E}h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)}{\sqrt{\kappa \ln L}} = \xi \sim \mathcal{N}(0,1),$$

$$(1.8)$$

with $\kappa = (2\pi^2)^{-1}$.

We also give an explicit formula for the limit shape:

$$\lim_{L\to\infty} \frac{\mathbb{E}h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)}{L} =: \boldsymbol{h}(\nu,\eta,\tau)$$

$$= \frac{1}{\pi}\left(-\nu\pi_\eta + \eta(\pi-\pi_\nu) + \tau\frac{\sin\pi_\nu \sin\pi_\eta}{\sin\pi_\tau}\right). \qquad (1.9)$$

Theorem 1.2 describes the limit shape $\boldsymbol{h}$ of our growing surface, and the domain $\mathcal{D}$ describes the points where this limit shape is *curved*. The logarithmic fluctuations is essentially a consequence of the local asymptotic behavior being governed by the discrete sine kernel (this local behavior occurs also in tiling models [42,49,63]). Using the connection with the Charlier ensembles, see above, the formula (1.9) for the limit shape can be read off the formulas of [7].

Using Theorem 1.1 it is not hard to verify (see Proposition 3.1 below) that near every point of the limit shape in the curved region, at any fixed time moment the random lozenge tiling approaches the unique translation invariant measure $M_{\pi_\nu,\pi_\eta,\pi_\tau}$ on lozenge tilings of the plane with prescribed slope (see [27,50,53] and references therein for discussions of these measures). The slope is exactly the slope of the tangent plane to the limit shape, given by

$$\frac{\partial\boldsymbol{h}}{\partial\nu} = -\frac{\pi_\eta}{\pi}, \qquad \frac{\partial\boldsymbol{h}}{\partial\eta} = 1 - \frac{\pi_\nu}{\pi}. \qquad (1.10)$$

This implies in particular, that $(\pi_\nu/\pi, \pi_\eta/\pi, \pi_\tau/\pi)$ are the asymptotic proportions of lozenges of three different types in the neighborhood of the point of the limit shape. One also computes the growth velocity (see (1.12) for the definition of Ω)

$$\frac{\partial\boldsymbol{h}}{\partial\tau} = \frac{1}{\pi}\frac{\sin\pi_\nu \sin\pi_\eta}{\sin\pi_\tau} = \frac{\text{Im}(\Omega(\nu,\eta,\tau))}{\pi}. \qquad (1.11)$$

Since the right-hand side depends only on the slope of the tangent plane, this suggests that it should be possible to extend the definition of our surface evolution to the random

surfaces distributed according to measures $M_{\pi_v, \pi_\eta, \pi_\tau}$; these measures have to remain invariant under evolution, and the speed of the height growth should be given by the right-hand side of (1.11). This is an interesting open problem that we do not address in this paper.

1.3. Complex structure and multipoint fluctuations. To describe the correlations of the interface, we first need to introduce a complex structure. Set $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ and define the map $\Omega : \mathcal{D} \to \mathbb{H}$ by

$$|\Omega(v, \eta, \tau)| = \sqrt{\eta/\tau}, \quad |1 - \Omega(v, \eta, \tau)| = \sqrt{v/\tau}. \tag{1.12}$$

Observe that $\arg \Omega = \pi_v$ and $\arg(1 - \Omega) = -\pi_\eta$. The preimage of any $\Omega \in \mathbb{H}$ is a ray in $\mathcal{D}$ that consists of triples (v, η, τ) with constant ratios $(v : \eta : \tau)$. Denote this ray by R_Ω. One sees that R_Ω's are also the level sets of the slope of the tangent plane to the limit shape. Since $h(\alpha v, \alpha \eta, \alpha \tau) = \alpha h(v, \eta, \tau)$ for any $\alpha > 0$, the height function grows linearly in time along each R_Ω. Note also that the map Ω satisfies

$$(1 - \Omega)\frac{\partial \Omega}{\partial v} = \Omega \frac{\partial \Omega}{\partial \eta} = -\frac{\partial \Omega}{\partial \tau}, \tag{1.13}$$

and the first of these relations is the complex Burgers equation, cf. [52].

From Theorem 1.2 one might think that to get non-trivial correlations we need to consider $(h - \mathbb{E}(h))/\sqrt{\ln L}$. However, this is not true and the division by $\sqrt{\ln L}$ is not needed. To state the precise result, denote by

$$\mathcal{G}(z, w) = -\frac{1}{2\pi} \ln \left| \frac{z - w}{z - \bar{w}} \right| \tag{1.14}$$

the Green function of the Laplace operator on $\mathbb{H}$ with Dirichlet boundary conditions.

Theorem 1.3. *For any $N = 1, 2, \ldots$, let $\varkappa_j = (v_j, \eta_j, \tau_j) \in \mathcal{D}$ be any distinct N triples such that*

$$\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N, \quad \eta_1 \geq \eta_2 \geq \cdots \geq \eta_N. \tag{1.15}$$

Denote

$$H_L(v, \eta, \tau) := \sqrt{\pi} \left(h([(v - \eta)L] + \tfrac{1}{2}, [\eta L], \tau L) - \mathbb{E}h([(v - \eta)L] + \tfrac{1}{2}, [\eta L], \tau L) \right), \tag{1.16}$$

and $\Omega_j = \Omega(v_j, \eta_j, \tau_j)$. Then

$$\lim_{L \to \infty} \mathbb{E}\left(H_L(\varkappa_1) \cdots H_L(\varkappa_N)\right) = \begin{cases} \sum_{\sigma \in \mathcal{F}_N} \prod_{j=1}^{N/2} \mathcal{G}(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & N \text{ is even,} \\ 0, & N \text{ is odd,} \end{cases} \tag{1.17}$$

where the summation is taken over all fixed point free involutions σ on $\{1, \ldots, N\}$.

The result of the theorem means that as $L \to \infty$, $H_L(\Omega^{-1}(z))$ is a Gaussian process with covariance given by $\mathcal{G}$, i.e., it has correlation of the Gaussian Free Field on $\mathbb{H}$. We can make this statement more precise. Indeed, in addition to Theorem 1.3, a simple consequence of Theorem 1.2 gives (see Lemma 5.4),

$$\mathbb{E}\left(H_L(\varkappa_1) \cdots H_L(\varkappa_N)\right) = O(L^\epsilon), \quad L \to \infty, \tag{1.18}$$

for any $\varkappa_j \in \mathcal{D}$ and any $\epsilon > 0$. This bounds the moments of $H_L(\varkappa_j)$ for infinitesimally close points $\varkappa_j$. A small extension of Theorem 1.3 together with this estimate immediately implies that on suitable surfaces in $\mathcal{D}$, the random function $H_L(\nu, \eta, \tau)$ converges to the Ω-pullback of the Gaussian free field on $\mathbb{H}$, see Theorem 5.6 and Theorem 5.8 in Sect. 5.5 for more details.

Conjecture 1.4. *The statement of Theorem 1.3 holds without the assumption (1.15), provided that Ω-images of all the triples are pairwise distinct.*

Theorem 1.3 and Conjecture 1.4 indicate that the fluctuations of the height function along the rays R_Ω vary slower than in any other space-time direction. This statement can be rephrased more generally: the height function has smaller fluctuations along the curves where the slope of the limit shape remains constant. We have been able to find evidence for such a claim in one-dimensional random growth models as well [30,39].

1.4. Universality class. In the terminology of physics literature, see e.g. [4], our Markov chain falls into the class of local growth models with relaxation and lateral growth, described by the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h = \Delta h + Q(\partial_x h, \partial_y h) + \text{white noise}, \tag{1.19}$$

where Q is a quadratic form. Relations (1.10) and (1.11) imply that for our growth model the determinant of the Hessian of $\partial_t h$, viewed as a function of the slope, is strictly negative, which means that the form Q in our case has signature $(-1, 1)$. In such a situation Eq. (1.19) is called an *anisotropic* KPZ or AKPZ equation.

An example of such system is growth of vicinal surfaces, which are naturally anisotropic because the tilt direction of the surface is special. Using non-rigorous renormalization group analysis based on one-loop expansion, Wolf [77] predicted that large time fluctuations (the roughness) of the growth models described by the AKPZ equation should be similar to those of linear models described by the Edwards-Wilkinson equation (heat equation with random term)

$$\partial_t h = \Delta h + \text{white noise}. \tag{1.20}$$

Our results can be viewed as the first rigorous analysis of a non-equilibrium growth model in the AKPZ class. (Some results, like logarithmic fluctuations, for an AKPZ model in a steady state were obtained in [67]. Some numerical results are described in [45,46,54]). Indeed, Wolf's prediction correctly identifies the logarithmic behavior of height fluctuations. However, it does not (at least explicitly) predict the appearance of the Gaussian free field, and in particular the complete structure (map Ω) of the fluctuations described in the previous section.

On the other hand, universality considerations imply that analogs of Theorems 1.2 and 1.3, as well as possibly Conjecture 1.4, should hold in any AKPZ growth model.

1.5. More general growth models. It turns out that the determinantal structure of the correlations functions stated in Theorem 1.1 holds for a much more general class of two-dimensional growth models. In the first part of the paper we develop an algebraic formalism needed to show that. At least three examples where this formalism applies, other than the Markov chain considered above, are worth mentioning.

1. In the Markov chain considered above one can make the particle jump rates depend on the upper index *m* in an arbitrary way. One can also allow the particles to jump both right and left, with ratio of left and right jump rates possibly changing in time [13].
2. The shuffling algorithm for domino tilings of Aztec diamonds introduced in [37] also fits into our formalism. The corresponding discrete time Markov chain is described in Sect. 2 below, and its equivalence to domino shuffling is established in the recent paper [59].
3. A shuffling algorithm for lozenge tilings of the hexagon (also known as *boxed plane partitions*) has been constructed in [19] using the formalism developed in this paper, see [19] for details.

   Our original Markov chain is a suitable degeneration of each of these examples.

   We expect our asymptotic methods to be applicable to many other two-dimensional growth models produced by the general formalism, and we plan to return to this discussion in a later publication.

1.6. Other connections. We have so far discussed the global asymptotic behavior of our growing surface, and its bulk properties (measures $M_{\pi_\nu, \pi_\eta, \pi_\tau}$), but have not discussed the edge asymptotics. As was mentioned above, rows $\{x_1^m\}_{m \geq 1}$ and $\{x_m^m\}_{m \geq 1}$ can be viewed as one-dimensional growth models on their own, and their asymptotic behavior was studied in [13] using essentially the same Theorem 1.1. This is exactly the edge behavior of our two-dimensional growth model.

Of course, the successive projections to $\{x_1^m\}_{m \geq 1}$ and then to a fixed (large) time commute. In the first ordering, this can be seen as the large time interface associated to the TASEP. In the second ordering, it corresponds to considering a tiling problem of a large region and focusing on the border of the facet.

Interestingly enough, an analog of Theorem 1.1 remains useful for the edge computations even in the cases when the measure on the space $\mathcal{S}$ is no longer positive (but its projections to $\{x_1^m\}_{m \geq 1}$ and $\{x_m^m\}_{m \geq 1}$ remain positive). These computations lead to the asymptotic results of [13–17,72] for one-dimensional growth models with more general types of initial conditions.

Another natural asymptotic question that was not discussed is the limiting behavior of $\mathcal{M}^{(n)}(t)$ when $t \to \infty$ but *n* remains fixed. After proper normalization, in the limit one obtains the Markov chain investigated in [76].

Two of the four one-dimensional growth models constructed in [35] (namely, "Bernoulli with blocking" and "Bernoulli with pushing") are projections to $\{x_1^m\}_{m \geq 1}$ and $\{x_m^m\}_{m \geq 1}$ of one of our two-dimensional growth models, see Sect. 2 below. It remains unclear however, how to interpret the other two models of [35] in a similar fashion.

Finally, let us mention that our proof of Theorem 1.1 is based on the argument of [31] and [74], the proof of Theorem 1.3 uses several ideas from [51], and the algebraic formalism for two-dimensional growth models employs a crucial idea of constructing bivariate Markov chains out of commuting univariate ones from [34].

Outline. The rest of the paper is organized as follows. It has essentially two main parts. The first part is Sect. 2. It contains the construction of the Markov chains, with the final

result being the determinantal structure and the associated kernel (Theorem 2.25). Its continuous time analogue is Corollary 2.26, whose further specialization to particle-independent jump rate leads to Theorem 1.1.

The second main part concerns the limit results for the continuous time model that we analyze. We start by collecting various geometric identities in Sect. 3. We also shortly discuss why our model is in the AKPZ class. In Sect. 4 we first give a shifted version of the kernel, whose asymptotic analysis is the content of Sect. 6. These results then allow us to prove Theorem 1.2 in Sect. 4 and Theorem 1.3 in Sect. 5.

Finally, we report in Appendix B certain developments that originated from the present work since the appearance of its preprint version on the arXiv.

2. Two Dimensional Dynamics

All the constructions below are based on the following basic idea. Consider two Markov operators P and P^* on state spaces $\mathcal{S}$ and $\mathcal{S}^*$, and a Markov link $\Lambda : \mathcal{S}^* \to \mathcal{S}$ that intertwines P and P^*, that is $\Lambda P = P^* \Lambda$. Then one can construct Markov chains on (subsets of) $\mathcal{S}^* \times \mathcal{S}$ that in some sense has both P and P^* as their projections. There is more than one way to realize this idea, and in this paper we discuss two variants.

In one of them the image (y^*, y) of $(x^*, x) \in \mathcal{S}^* \times \mathcal{S}$ under the Markov operator is determined by a *sequential update*: One first chooses y according to $P(x, y)$, and then one chooses y^* so that the needed projection properties are satisfied. A characteristic feature of the construction is that x and y^* are independent, given x^* and y. This bivariate Markov chain is denoted P_Λ; its construction is borrowed from [34].

In the second variant, the images y^* and y are independent, given (x, x^*), and we say that they are obtained by *parallel update*. The distribution of y is still $P(x, y)$, independently of what x^* is. This Markov chain is denoted P_Δ for the operator $\Delta = \Lambda P = P^* \Lambda$ that plays an important role.

By induction, one constructs multivariate Markov chains out of finitely many univariate ones and links that intertwine them. Again, we use two variants of the construction — with sequential and parallel updates.

The key property that makes these constructions useful is the following: If the chains P, P^*, and Λ, are h-Doob transforms of some (simpler) Markov chains, and the harmonic functions h used are consistent, then the transition probabilities of the multivariate Markov chains do not depend on h. Thus, participating multivariate Markov chains may be fairly complex, while the transition probabilities of the univariate Markov chains remain simple.

Below we first explain the abstract construction of P_Λ, P_Δ, and their multivariate extensions. Then we exhibit a class of examples that are of interest to us. Finally, we show how the knowledge of certain averages (correlation functions) for the univariate Markov chains allows one to compute similar averages for the multivariate chains.

2.1. Bivariate Markov chains. Let $\mathcal{S}$ and $\mathcal{S}^*$ be discrete sets, and let P and P^* be stochastic matrices on these sets:

$$\sum_{y \in \mathcal{S}} P(x, y) = 1, \quad x \in \mathcal{S}; \qquad \sum_{y^* \in \mathcal{S}^*} P^*(x^*, y^*) = 1, \quad x^* \in \mathcal{S}^*. \qquad (2.1)$$

Assume that there exists a third stochastic matrix $\Lambda = \|\Lambda(x^*, x)\|_{x^* \in \mathcal{S}^*, x \in \mathcal{S}}$ such that for any $x^* \in \mathcal{S}^*$ and $y \in \mathcal{S}$,

$$\sum_{x \in \mathcal{S}} \Lambda(x^*, x) P(x, y) = \sum_{y^* \in \mathcal{S}^*} P^*(x^*, y^*) \Lambda(y^*, y). \tag{2.2}$$

Let us denote the above quantity by $\Delta(x^*, y)$. In matrix notation

$$\Delta = \Lambda P = P^* \Lambda. \tag{2.3}$$

Set

$$\mathcal{S}_\Lambda = \{(x^*, x) \in \mathcal{S}^* \times \mathcal{S} \mid \Lambda(x^*, x) > 0\},$$
$$\mathcal{S}_\Delta = \{(x^*, x) \in \mathcal{S}^* \times \mathcal{S} \mid \Delta(x^*, x) > 0\}.$$

Define bivariate Markov chains on $\mathcal{S}_\Lambda$ and $\mathcal{S}_\Delta$ by their corresponding transition probabilities

$$P_\Lambda((x^*, x), (y^*, y)) = \begin{cases} \frac{P(x,y) P^*(x^*, y^*) \Lambda(y^*, y)}{\Delta(x^*, y)}, & \Delta(x^*, y) > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

$$P_\Delta((x^*, x), (y^*, y)) = \frac{P(x, y) P^*(x^*, y^*) \Lambda(y^*, x)}{\Delta(x^*, x)}. \tag{2.5}$$

It is immediately verified that both matrices P_Λ and P_Δ are stochastic.

The chain P_Λ was introduced by Diaconis-Fill in [34], and we are using the notation of that paper.

One could think of P_Λ and P_Δ as follows.

For P_Λ, starting from (x^*, x) we first choose y according to the transition matrix $P(x, y)$, and then choose y^* using $\frac{P^*(x^*, y^*) \Lambda(y^*, y)}{\Delta(x^*, y)}$, which is the conditional distribution of the middle point in the successive application of P^* and Λ provided that we start at x^* and finish at y.

For P_Δ, starting from (x^*, x) we independently choose y according to $P(x, y)$ and y^* according to $\frac{P^*(x^*, y^*) \Lambda(y^*, x)}{\Delta(x^*, x)}$, which is the conditional distribution of the middle point in the successive application of P^* and Λ provided that we start at x^* and finish at x.

Lemma 2.1. *For any $(x^*, x) \in \mathcal{S}_\Lambda$, $y \in \mathcal{S}$, we have*

$$\sum_{y^* \in \mathcal{S}^*: (y^*, y) \in \mathcal{S}_\Lambda} P_\Lambda((x^*, x), (y^*, y)) = P(x, y),$$
$$\sum_{y^* \in \mathcal{S}^*: (y^*, y) \in \mathcal{S}_\Delta} P_\Delta((x^*, x), (y^*, y)) = P(x, y), \tag{2.6}$$

and for any $x^* \in \mathcal{S}^*$, $(y^*, y) \in \mathcal{S}_\Lambda$,

$$\sum_{x \in \mathcal{S}: (x^*, x) \in \mathcal{S}_\Lambda} \Lambda(x^*, x) P_\Lambda((x^*, x), (y^*, y)) = P^*(x^*, y^*) \Lambda(y^*, y),$$
$$\sum_{x \in \mathcal{S}: (x^*, x) \in \mathcal{S}_\Delta} \Delta(x^*, x) P_\Delta((x^*, x), (y^*, y)) = P^*(x^*, y^*) \Delta(y^*, y). \tag{2.7}$$

Proof of Lemma 2.1. Straightforward computation using the relation $\Delta = \Lambda P = P^* \Lambda$.
□

Proposition 2.2. *Let $m^*(x^*)$ be a probability measure on $\mathcal{S}^*$. Consider the evolution of the measure $m(x^*)\Lambda(x^*, x)$ on $\mathcal{S}_\Lambda$ under the Markov chain P_Λ and denote by $(x^*(j), x(j))$ the result after $j = 0, 1, 2, \ldots$ steps. Then for any $k, l = 0, 1, \ldots$ the joint distribution of*

$$(x^*(0), x^*(1), \ldots, x^*(k), x(k), x(k+1), \ldots, x(k+l)) \tag{2.8}$$

coincides with the stochastic evolution of m^* under transition matrices

$$(\underbrace{P^*, \ldots, P^*}_{k}, \Lambda, \underbrace{P, \ldots, P}_{l}). \tag{2.9}$$

Exactly the same statement holds for the Markov chain P_Δ and the initial condition $m^*(x^*)\Delta(x^*, x)$ with Λ replaced by Δ in the above sequence of matrices.

Proof of Proposition 2.2. Successive application of the first relations of Lemma 2.1 to evaluate the sums over $x^*(k+l), \ldots, x^*(k+1)$, and of the second relations to evaluate the sums over $x(1), \ldots, x(k-1)$. $\quad\square$

Note that Proposition 2.2 also implies that the joint distribution of $x^*(k)$ and $x(k)$ has the form $m_k^*(x^*(k))\Lambda(x^*(k), x(k))$, where m_k^* is the result of k-fold application of P^* to m^*.

The above constructions can be generalized to the nonautonomous situation.

Assume that we have a time variable $t \in \mathbb{Z}$, and our state spaces as well as transition matrices depend on t, which we will indicate as follows:

$$\mathcal{S}(t), \quad \mathcal{S}^*(t), \quad P(x, y \,|\, t), \quad P^*(x^*, y^* \,|\, t), \quad \Lambda(x^*, x \,|\, t), \quad P(t), \quad P^*(t), \quad \Lambda(t). \tag{2.10}$$

The commutation relation (1.3) is replaced by $\Lambda(t)P(t) = P^*(t)\Lambda(t+1)$ or

$$\Delta(x^*, y \,|\, t) := \sum_{x \in \mathcal{S}(t)} \Lambda(x^*, x \,|\, t) P(x, y \,|\, t) = \sum_{y^* \in \mathcal{S}^*(t+1)} P^*(x^*, y^* \,|\, t)\, \Lambda(y^*, y \,|\, t+1). \tag{2.11}$$

Further, we set

$$\begin{aligned} \mathcal{S}_\Lambda(t) &= \{(x^*, x) \in \mathcal{S}^*(t) \times \mathcal{S}(t) \,|\, \Lambda(x^*, x \,|\, t) > 0\}, \\ \mathcal{S}_\Delta(t) &= \{(x^*, x) \in \mathcal{S}^*(t) \times \mathcal{S}(t+1) \,|\, \Delta(x^*, x \,|\, t) > 0\}, \end{aligned} \tag{2.12}$$

and

$$P_\Lambda((x^*, x), (y^*, y) \,|\, t) = \begin{cases} \dfrac{P(x,y \,|\, t)P^*(x^*,y^* \,|\, t)\Lambda(y^*,y \,|\, t+1)}{\Delta(x^*,y \,|\, t)} & \Delta(x^*, y \,|\, t) > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.13}$$

$$P_\Delta((x^*, x), (y^*, y) \,|\, t) = \frac{P(x, y \,|\, t+1)P^*(x^*, y^* \,|\, t)\Lambda(y^*, x \,|\, t+1)}{\Delta(x^*, x \,|\, t)}. \tag{2.14}$$

The nonautonomous generalization of Proposition 2.2 is proved in exactly the same way as Proposition 2.2. Let us state it.

Proposition 2.3. *Fix $t_0 \in \mathbb{Z}$, and let $m^*(x^*)$ be a probability measure on $\mathcal{S}^*(t_0)$. Consider the evolution of the measure $m(x^*) \Lambda(x^*, x \mid t_0)$ on $\mathcal{S}_\Lambda(t_0)$ under the Markov chain $P_\Lambda(t)$, and denote by $(x^*(t_0 + j), x(t_0 + j)) \in \mathcal{S}_\Lambda(t_0 + j)$ the result after $j = 0, 1, 2, \ldots$ steps. Then for any $k, l = 0, 1, \ldots$ the joint distribution of*

$$(x^*(t_0), x^*(t_0 + 1), \ldots, x^*(t_0 + k), x(t_0 + k), x(t_0 + k + 1), \ldots, x(t_0 + k + l)) \quad (2.15)$$

coincides with the stochastic evolution of m^* under transition matrices

$$P^*(t_0), \ldots, P^*(t_0 + k - 1), \Lambda(t_0 + k), P(t_0 + k), \ldots, P(t_0 + k + l - 1) \quad (2.16)$$

(for $k = l = 0$ only $\Lambda(t_0)$ remains in this string).

   A similar statement holds for the Markov chain $P_\Delta(t)$ and the initial condition $m^*(x^*) \Delta(x^*, x \mid t_0)$: For any $k, l = 0, 1, \ldots$ the joint distribution of

$$(x^*(t_0), x^*(t_0 + 1), \ldots, x^*(t_0 + k), x(t_0 + k + 1), x(t_0 + k + 2), \ldots, x(t_0 + k + l + 1))$$
$$(2.17)$$

coincides with the stochastic evolution of m^* under transition matrices

$$P^*(t_0), \ldots, P^*(t_0 + k - 1), \Delta(t_0 + k), P(t_0 + k + 1), \ldots, P(t_0 + k + l). \quad (2.18)$$

Remark 2.4. Observe that there is a difference in the sequences of times used in (2.8) and (2.17). The reason is that for nonautonomous P_Δ, the state space at time t is a subset of $\mathcal{S}^*(t) \times \mathcal{S}(t + 1)$, and we denote its elements as $(x^*(t), x(t + 1))$. In the autonomous case, an element of the state space $\mathcal{S}_\Delta$ at time t was denoted as $(x^*(t), x(t))$.

2.2. Multivariate Markov chains. We now aim at generalizing the constructions of Sect. 2.1 to more than two state spaces.

   Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be discrete sets, $P_1, \ldots, P_n$ be stochastic matrices defining Markov chains on them, and let $\Lambda_1^2, \ldots, \Lambda_{n-1}^n$ be stochastic links between these sets:

$$P_k : \mathcal{S}_k \times \mathcal{S}_k \to [0, 1], \quad \sum_{y \in \mathcal{S}_k} P_k(x, y) = 1, \quad x \in \mathcal{S}_k, \quad k = 1, \ldots, n;$$
$$(2.19)$$
$$\Lambda_{k-1}^k : \mathcal{S}_k \times \mathcal{S}_{k-1} \to [0, 1], \quad \sum_{y \in \mathcal{S}_{k-1}} \Lambda_{k-1}^k(x, y) = 1, \quad x \in \mathcal{S}_k, \quad k = 2, \ldots, n.$$

   Assume that these matrices satisfy the commutation relations

$$\Delta_{k-1}^k := \Lambda_{k-1}^k P_{k-1} = P_k \Lambda_{k-1}^k, \quad k = 2, \ldots, n. \quad (2.20)$$

   The state spaces for our multivariate Markov chains are defined as follows:

$$\mathcal{S}_\Lambda^{(n)} = \left\{ (x_1, \ldots, x_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \mid \prod_{k=2}^{n} \Lambda_{k-1}^k(x_k, x_{k-1}) \neq 0 \right\},$$
$$(2.21)$$
$$\mathcal{S}_\Delta^{(n)} = \left\{ (x_1, \ldots, x_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \mid \prod_{k=2}^{n} \Delta_{k-1}^k(x_k, x_{k-1}) \neq 0 \right\}.$$

The transition probabilities for the Markov chains $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ are defined as (we use the notation $X_n = (x_1, \ldots, x_n)$, $Y_n = (y_1, \ldots, y_n)$)

$$P_\Lambda^{(n)}(X_n, Y_n) = \begin{cases} P_1(x_1, y_1) \prod_{k=2}^n \frac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, y_{k-1})}{\Lambda_{k-1}^k(x_k, y_{k-1})}, & \prod_{k=2}^n \Lambda_{k-1}^k(x_k, y_{k-1}) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

(2.22)

$$P_\Delta^{(n)}(X_n, Y_n) = P(x_1, y_1) \prod_{k=2}^n \frac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, x_{k-1})}{\Delta_{k-1}^k(x_k, x_{k-1})}.$$

(2.23)

One way to think of $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ is as follows. For $P_\Lambda^{(n)}$, starting from $X_n = (x_1, \ldots, x_n)$, we first choose y_1 according to the transition matrix $P(x_1, y_1)$, then choose y_2 using $\frac{P_2(x_2, y_2) \Lambda_1^2(y_2, y_1)}{\Lambda_1^2(x_2, y_1)}$, which is the conditional distribution of the middle point in the successive application of P_2 and Λ_1^2 provided that we start at x_2 and finish at y_1, after that we choose y_3 using the conditional distribution of the middle point in the successive application of P_3 and Λ_2^3 provided that we start at x_3 and finish at y_2, and so on. Thus, one could say that Y_n is obtained by the *sequential update*.

For $P_\Delta^{(n)}$, starting from $X_n = (x_1, \ldots, x_n)$ we *independently* choose $y_1, \ldots, y_n$ according to $P_1(x_1, y_1)$ for y_1 and $\frac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, x_{k-1})}{\Delta_{k-1}^k(x_k, x_{k-1})}$, for y_k, $k = 2, \ldots, n$. The latter formula is the conditional distribution of the middle point in the successive application of P_k and Λ_{k-1}^k provided that we start at x_k and finish at x_{k-1}. Thus, it is natural to say that this Markov chain corresponds to the *parallel update*.

We aim at proving the following generalization of Proposition 2.2.

Proposition 2.5. *Let $m_n(x_n)$ be a probability measure on $\mathcal{S}_n$. Consider the evolution of the measure*

$$m_n(x_n) \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1)$$

(2.24)

on $\mathcal{S}_\Lambda^{(n)}$ under the Markov chain $P_\Lambda^{(n)}$, and denote by $(x_1(j), \ldots, x_n(j))$ the result after $j = 0, 1, 2, \ldots$ steps. Then for any $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$ the joint distribution of

$$(x_n(0), \ldots, x_n(k_n), x_{n-1}(k_n), x_{n-1}(k_n + 1), \ldots, x_{n-1}(k_{n-1}),$$
$$x_{n-2}(k_{n-1}), \ldots, x_2(k_2), x_1(k_2), \ldots, x_1(k_1))$$

coincides with the stochastic evolution of m_n under transition matrices

$$(\underbrace{P_n, \ldots, P_n}_{k_n}, \Lambda_{n-1}^n, \underbrace{P_{n-1}, \ldots, P_{n-1}}_{k_{n-1} - k_n}, \Lambda_{n-2}^{n-1}, \ldots, \Lambda_1^2, \underbrace{P_1, \ldots, P_1}_{k_1 - k_2}).$$

(2.25)

Exactly the same statement holds for the Markov chain $P_\Delta^{(n)}$ and the initial condition

$$m(x_n) \Delta_{n-1}^n(x_n, x_{n-1}) \cdots \Delta_1^2(x_2, x_1)$$

(2.26)

with Λ's replaced by Δ's in the above sequence of matrices.

The following lemma is useful.

Lemma 2.6. *Consider the matrix* $\Lambda : \mathcal{S}_n \times \mathcal{S}_\Lambda^{(n-1)} \to [0, 1]$ *given by*

$$\Lambda(x_n, (x_1, \ldots, x_{n-1})) := \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1). \tag{2.27}$$

Then $\Lambda P_\Lambda^{(n-1)} = P_n \Lambda$. *If we denote this matrix by* Δ *then*

$$P_\Lambda^{(n)}(X_n, Y_n) = \begin{cases} \frac{P_\Lambda^{(n-1)}(X_{n-1}, Y_{n-1}) P_n(x_n, y_n) \Lambda(y_n, Y_{n-1})}{\Delta(x_n, Y_{n-1})}, & \Delta(x_n, Y_{n-1}) > 0, \\ 0, & otherwise. \end{cases} \tag{2.28}$$

Also, using the same notation,

$$P_\Lambda^{(n)}(X_n, Y_n) = \frac{P_\Lambda^{(n-1)}(X_{n-1}, Y_{n-1}) P_n(x_n, y_n) \Lambda(y_n, X_{n-1})}{\Delta(x_n, X_{n-1})}. \tag{2.29}$$

Proof of Lemma 2.6. Let us check the commutation relation $\Lambda P_\Lambda^{(n-1)} = P_n \Lambda$. We have

$$\Lambda P_\Lambda^{(n-1)}(x_n, Y_{n-1}) = \sum_{x_1, \ldots, x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1)$$

$$\times P_1(x_1, y_1) \prod_{k=2}^{n-1} \frac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, y_{k-1})}{\Delta_{k-1}^k(x_k, y_{k-1})}, \tag{2.30}$$

where the sum is taken over all $x_1, \ldots, x_{n-1}$ such that $\prod_{k=2}^{n-1} \Delta_{k-1}^k(x_k, y_{k-1}) > 0$. Computing the sum over x_1 and using the relation $\Lambda_1^2 P_1 = \Delta_1^2$ we obtain

$$\Lambda P_\Lambda^{(n-1)}(x_n, Y_{n-1}) = \sum_{x_2, \ldots, x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_2^3(x_3, x_2)$$

$$\times P_2(x_2, y_2) \Lambda_1^2(y_2, y_1) \prod_{k=3}^{n-1} \frac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, y_{k-1})}{\Delta_{k-1}^k(x_k, y_{k-1})}. \tag{2.31}$$

Now we need to compute the sum over x_2. If $\Delta_1^2(x_2, y_1) = 0$ then $P_2(x_2, y_2) = 0$ because otherwise the relation $\Delta_1^2 = P_2 \Lambda_1^2$ implies that $\Lambda_1^2(y_2, y_1) = 0$, which contradicts the hypothesis that $Y_{n-1} \in \mathcal{S}_\Lambda^{(n-1)}$. Thus, we can extend the sum to all $x_2 \in \mathcal{S}_2$, and the relation $\Lambda_2^3 P_2 = \Delta_2^3$ gives

$$\Lambda P_\Lambda^{(n-1)}(x_n, Y_{n-1}) = \sum_{x_3, \ldots, x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_3^4(x_4, x_3) P_3(x_3, y_3)$$

$$\times \Lambda_2^3(y_3, y_2) \Lambda_1^2(y_2, y_1) \prod_{k=4}^{n-1} \frac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, y_{k-1})}{\Delta_{k-1}^k(x_k, y_{k-1})}. \tag{2.32}$$

Continuing like that we end up with

$$\Lambda_{n-2}^{n-1}(y_{n-1}, y_{n-2}) \cdots \Lambda_1^2(y_2, y_1) \sum_{x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) P_{n-1}(x_{n-1}, y_{n-1}), \tag{2.33}$$

which, by $\Lambda^n_{n-1} P_{n-1} = P_n \Lambda^n_{n-1}$ is exactly $P_n \Lambda(x_n, Y_{n-1})$. Let us also note that

$$\Delta(x_n, Y_{n-1}) = \Delta^n_{n-1}(x_n, y_{n-1}) \Lambda^{n-1}_{n-2}(y_{n-1}, y_{n-2}) \cdots \Lambda^2_1(y_2, y_1). \qquad (2.34)$$

The needed formulas for $P^{(n)}_\Lambda$ and $P^{(n)}_\Delta$ are now verified by straightforward substitution.
□

Proof of Proposition 2.5. Let us give the argument for $P^{(n)}_\Lambda$; for $P^{(n)}_\Delta$ the proof is literally the same. By virtue of Lemma 2.6, we can apply Proposition 2.2 by taking

$$\mathcal{S}^* = \mathcal{S}_n, \quad \mathcal{S} = \mathcal{S}^{(n-1)}_\Lambda, \quad P^* = P_n, \quad P = P^{(n-1)}_\Lambda, \quad k = k_n, \quad l = k_1 - k_n, \quad (2.35)$$

and $\Lambda(x_n, X_{n-1})$ as in Lemma 2.6. Proposition 2.2 says that the joint distribution

$$(x_n(0), x_n(1), \ldots, x_n(k_n), X_{n-1}(k_n), X_{n-1}(k_n + 1), \ldots, X_{n-1}(k_1)) \qquad (2.36)$$

is the evolution of m_n under

$$(\underbrace{P_n, \ldots, P_n}_{k_n}, \Lambda, \underbrace{P^{(n-1)}_\Lambda, \ldots, P^{(n-1)}_\Lambda}_{k_1 - k_n}). \qquad (2.37)$$

Induction on n completes the proof. □

As in the previous section, Proposition 2.5 can be also proved in the nonautonomous situation. Let us give the necessary definitions.

We now have a time variable $t \in \mathbb{Z}$, and our state spaces as well as transition matrices depend on t:

$$\mathcal{S}_k(t), \quad P_k(x, y \mid t), \quad k = 1, \ldots, n, \qquad \Lambda^k_{k-1}(x_k, x_{k-1} \mid t), \quad k = 2, \ldots, n. \qquad (2.38)$$

The commutation relations are

$$\Delta^k_{k-1}(t) := \Lambda^k_{k-1}(t) P_{k-1}(t) = P_k(t) \Lambda^k_{k-1}(t + 1), \qquad k = 2, \ldots, n. \qquad (2.39)$$

The multivariate state spaces are defined as

$$\mathcal{S}^{(n)}_\Lambda(t) = \left\{ (x_1, \ldots, x_n) \in \mathcal{S}_1(t) \times \cdots \times \mathcal{S}_n(t) \mid \prod_{k=2}^n \Lambda^k_{k-1}(x_k, x_{k-1} \mid t) \neq 0 \right\},$$

$$\mathcal{S}^{(n)}_\Delta(t) = \left\{ (x_1, \ldots, x_n) \in \mathcal{S}_1(t + n - 1) \times \cdots \times \mathcal{S}_n(t) \mid \right.$$

$$\left. \prod_{k=2}^n \Delta^k_{k-1}(x_k, x_{k-1} \mid t + n - k) \neq 0 \right\}.$$

Then the transition matrices for $P^{(n)}_\Lambda$ and $P^{(n)}_\Delta$ are defined as

$$P^{(n)}_\Lambda(X_n, Y_n \mid t) = P_1(x_1, y_1 \mid t) \prod_{k=2}^n \frac{P_k(x_k, y_k \mid t) \Lambda^k_{k-1}(y_k, y_{k-1} \mid t + 1)}{\Delta^k_{k-1}(x_k, y_{k-1} \mid t)} \qquad (2.40)$$

if $\prod_{k=2}^n \Delta^k_{k-1}(x_k, y_{k-1} \mid t) > 0$ and 0 otherwise; and

$$P^{(n)}_\Delta(X_n, Y_n) = P(x_1, y_1 \mid t + n - 1)$$

$$\times \prod_{k=2}^n \frac{P_k(x_k, y_k \mid t + n - k) \Lambda^k_{k-1}(y_k, x_{k-1} \mid t + n - k + 1)}{\Delta^k_{k-1}(x_k, x_{k-1} \mid t + n - k)}. \qquad (2.41)$$

Proposition 2.7. *Fix $t_0 \in \mathbb{Z}$, and let $m_n(x_n)$ be a probability measure on $\mathcal{S}_n(t_0)$. Consider the evolution of the measure*

$$m_n(x_n)\Lambda_{n-1}^n(x_n, x_{n-1} \mid t_0) \cdots \Lambda_1^2(x_2, x_1 \mid t_0) \tag{2.42}$$

on $\mathcal{S}_\Lambda^{(n)}(t_0)$ under $P_\Lambda^{(n)}(t)$. Denote by $(x_1(t_0 + j), \ldots, x_n(t_0 + j))$ the result after $j = 0, 1, 2, \ldots$ steps. Then for any $k_1 \geq k_2 \geq \cdots \geq k_n \geq t_0$ the joint distribution of

$$(x_n(t_0), \ldots, x_n(k_n), x_{n-1}(k_n), x_{n-1}(k_n + 1), \ldots, x_{n-1}(k_{n-1}),$$
$$x_{n-2}(k_{n-1}), \ldots, x_2(k_2), x_1(k_2), \ldots, x_1(k_1))$$

coincides with the stochastic evolution of m_n under transition matrices

$$P_n(t_0), \ldots, P_n(k_n - 1), \Lambda_{n-1}^n(k_n), P_{n-1}(k_n), \ldots, P_{n-1}(k_{n-1} - 1),$$
$$\Lambda_{n-2}^{n-1}(k_{n-1}), \ldots, \Lambda_1^2(k_2), P_1(k_2), \ldots, P_1(k_1 - 1).$$

A similar statement holds for the Markov chain $P_\Delta^{(n)}(t)$ and the initial condition

$$m(x_n)\Delta_{n-1}^n(x_n, x_{n-1} \mid t_0) \cdots \Delta_1^2(x_2, x_1 \mid t_0 + n - 2). \tag{2.43}$$

For any $k_1 > k_2 > \cdots > k_n \geq t_0$ the joint distribution of

$$(x_n(t_0), \ldots, x_n(k_n), x_{n-1}(k_n + 1), x_{n-1}(k_n + 2), \ldots, x_{n-1}(k_{n-1}),$$
$$x_{n-2}(k_{n-1} + 1), \ldots, x_2(k_2), x_1(k_2 + 1), \ldots, x_1(k_1))$$

coincides with the stochastic evolution of m_n under transition matrices

$$P_n(t_0), \ldots, P_n(k_n - 1), \Delta_{n-1}^n(k_n), P_{n-1}(k_n + 1), \ldots, P_{n-1}(k_{n-1} - 1),$$
$$\Delta_{n-2}^{n-1}(k_{n-1}), \ldots, \Delta_1^2(k_2), P_1(k_2 + 1), \ldots, P_1(k_1 - 1).$$

The proof is very similar to that of Proposition 2.5.

2.3. Toeplitz-like transition probabilities.
The goal of this section is to provide some general recipe on how to construct commuting stochastic matrices.

Proposition 2.8. *Let $\alpha_1, \ldots, \alpha_n$ be nonzero complex numbers, and let $F(x)$ be an analytic function in an annulus A centered at the origin that contains all α_j^{-1}'s. Assume that $F(\alpha_1^{-1}) \cdots F(\alpha_n^{-1}) \neq 0$. Then*

$$\frac{1}{F(\alpha_1^{-1}) \cdots F(\alpha_n^{-1})} \sum_{y_1 < \cdots < y_n \in \mathbb{Z}} \det [\alpha_i^{y_j}]_{i,j=1}^n \det [f(x_j - y_i)]_{i,j=1}^n = \det [\alpha_i^{x_j}]_{i,j=1}^n, \tag{2.44}$$

where

$$f(m) = \frac{1}{2\pi i} \oint \frac{F(z)\mathrm{d}z}{z^{m+1}}, \tag{2.45}$$

and the integral is taken over any positively oriented simple loop in A.

Proof of Proposition 2.8. Since the left-hand side is symmetric with respect to permutations of y_j's and it vanishes when two y_j's are equal, we can extend the sum to $\mathbb{Z}^n$ and divide the result by $n!$. We obtain

$$\sum_{y_1,\ldots,y_n \in \mathbb{Z}} \det [\alpha_i^{y_j}]_{i,j=1}^n \det [f(x_j - y_i)]_{i,j=1}^n = n! \det \Big[\sum_{y=-\infty}^{+\infty} \alpha_k^y f(x_j - y) \Big]_{k,j=1}^n.$$
(2.46)

Further,

$$\sum_{y=-\infty}^{+\infty} \alpha_k^y f(x_j - y) = \sum_{y=-\infty}^{+\infty} \frac{1}{2\pi i} \oint \frac{\alpha_k^y F(z) dz}{z^{x_j - y + 1}}$$

$$= \frac{1}{2\pi i} \oint_{|z| = c_1 < |\alpha_k|^{-1}} F(z) dz \sum_{y=x_j+1}^{+\infty} \frac{\alpha_k^y}{z^{x_j - y + 1}} + \frac{1}{2\pi i} \oint_{|z| = c_2 > |\alpha_k|^{-1}} F(z) dz \sum_{y=-\infty}^{x_j} \frac{\alpha_k^y}{z^{x_j - y + 1}}$$

$$= \frac{1}{2\pi i} \oint_{|z| = c_1 < |\alpha_k|^{-1}} \frac{\alpha_k^{x_j+1} F(z)}{1 - \alpha_k z} dz - \frac{1}{2\pi i} \oint_{|z| = c_2 > |\alpha_k|^{-1}} \frac{\alpha_k^{x_j+1} F(z)}{1 - \alpha_k z} dz = \alpha_k^{x_j} F(\alpha_k^{-1}).$$

$\square$

Proposition 2.9. *In the notation of Proposition 2.8, assume that the variable y_n is virtual, $y_n = \text{virt}$, and set $f(x_k - \text{virt}) = \alpha_n^{x_k}$ for any $k = 1, \ldots, n$. Then*

$$\frac{1}{F(\alpha_1^{-1}) \cdots F(\alpha_{n-1}^{-1})} \sum_{y_1 < \cdots < y_{n-1} \in \mathbb{Z}} \det [\alpha_i^{y_j}]_{i,j=1}^{n-1} \det [f(x_j - y_i)]_{i,j=1}^n = \det [\alpha_i^{x_j}]_{i,j=1}^n.$$
(2.47)

Proof of Proposition 2.9. Expansion of $\det [f(x_j - y_i)]_{i,j=1}^n$ along the last row gives

$$\det [f(x_j - y_i)]_{i,j=1}^n = \sum_{k=1}^n (-1)^{n-k} \alpha_n^{x_k} \cdot \det [f(x_j - y_i)]_{\substack{i=1,\ldots,n-1 \\ j=1,\ldots,k-1,k+1,\ldots,n}}.$$
(2.48)

The application of Proposition 2.8 to each of the resulting summands in the left-hand side of the desired equality produces the expansion of $\det [\alpha_i^{x_j}]_{i,j=1}^n$ along the last row.

$\square$

For $n = 1, 2, \ldots$, denote

$$\mathfrak{X}_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_1 < \cdots < x_n\}.$$
(2.49)

In what follows we assume that the (nonzero) complex parameters $\alpha_1, \alpha_2, \ldots$ are such that the ratios $\det[\alpha_i^{x_j}]_{i,j=1}^n / \det [\alpha_i^{j-1}]_{i,j=1}^n$ are nonzero for all $n = 1, 2, \ldots$ and all $(x_1 \ldots, x_n)$ in $\mathfrak{X}^n$. This holds, for example, when all α_j's are positive. The Vandermonde determinant in the denominator is needed to make sense of $\det[\alpha_i^{x_j}]_{i,j=1}^n$ when some of the α_j's are equal.

Under this assumption, define the matrices $\mathfrak{X}_n \times \mathfrak{X}_n$ and $\mathfrak{X}_n \times \mathfrak{X}_{n-1}$ by

$$T_n(\alpha_1, \ldots, \alpha_n; F)(X, Y) = \frac{\det [\alpha_i^{y_j}]_{i,j=1}^n}{\det [\alpha_i^{x_j}]_{i,j=1}^n} \frac{\det [f(x_i - y_j)]_{i,j=1}^n}{\prod_{j=1}^n F(\alpha_j^{-1})}, \quad X, Y \in \mathfrak{X}_n,$$

$$T_{n-1}^n(\alpha_1, \ldots, \alpha_n; F)(X, Y) = \frac{\det [\alpha_i^{y_j}]_{i,j=1}^{n-1}}{\det [\alpha_i^{x_j}]_{i,j=1}^n} \frac{\det [f(x_i - y_j)]_{i,j=1}^n}{\prod_{j=1}^{n-1} F(\alpha_j^{-1})}, \quad X \in \mathfrak{X}_n, \ Y \in \mathfrak{X}_{n-1},$$

where in the second formula $y_n = \text{virt}$. By Propositions 2.8 and 2.9, the sums of entries of these matrices along rows are equal to 1. We will often omit the parameters α_j from the notation so that the above matrices will be denoted as $T_n(F)$ and $T_{n-1}^n(F)$.

We are interested in these matrices because they have nice commutation relations, as the following proposition shows.

Proposition 2.10. *Let F_1 and F_2 be two functions holomorphic in an annulus containing α_j^{-1}'s, that are also nonzero at these points. Then*

$$\begin{aligned} T_n(F_1)T_n(F_2) &= T_n(F_2)T_n(F_1) = T_n(F_1 F_2), \\ T_n(F_1)T_{n-1}^n(F_2) &= T_{n-1}^n(F_1)T_{n-1}(F_2) = T_{n-1}^n(F_1 F_2). \end{aligned} \tag{2.50}$$

Proof of Proposition 2.10. The first line and the relation $T_{n-1}^n(F_1)T_{n-1}(F_2) = T_{n-1}^n(F_1 F_2)$ are proved by straightforward computations using the fact the Fourier transform of $F_1 F_2$ is the convolution of those of F_1 and F_2. The only additional ingredient in the proof of the relation $T_n(F_1)T_{n-1}^n(F_2) = T_{n-1}^n(F_1 F_2)$ is

$$\sum_{y \in \mathbb{Z}} f_1(x - y) f_2(y - \text{virt}) = \sum_{y \in \mathbb{Z}} f_1(x - y) \alpha_n^y = F_1(\alpha_n^{-1}) \alpha_n^x. \tag{2.51}$$

$\square$

Remark 2.11. In the same way one proves the commutation relation

$$T_{n-1}^n(F_1)T_{n-2}^{n-1}(F_2) = T_{n-1}^n(F_2)T_{n-2}^{n-1}(F_1) \tag{2.52}$$

but we will not need it later.

2.4. Minors of some simple Toeplitz matrices. The goal of the section is to derive explicit formulas for $T_n(F)$ and $T_{n-1}^n(F)$ from the previous section for some simple functions F.

Lemma 2.12. *Consider $F(z) = 1 + pz$, that is*

$$f(m) = \begin{cases} p, & m = 1, \\ 1, & m = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.53}$$

Then for integers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$,

$$\det [f(x_i - y_j)]_{i,j=1}^n = \begin{cases} p^{\sum_{i=1}^n (x_i - y_i)}, & \text{if } y_i - x_i \in \{-1, 0\} \text{ for all } 1 \le i \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.54}$$

Proof of Lemma 2.12. If $x_i < y_i$ for some i then $x_k < y_l$ for $k \leq i$ and $l \geq i$, which implies that $f(x_k - y_l) = 0$ for such k, l, and thus the determinant in question vanishes. If $x_i > y_i + 1$ then $x_k > y_l + 1$ for $k \geq i$ and $l \leq i$, which means $f(x_k - y_l) = 0$, and the determinant vanishes again. Hence, it remains to consider the case when $x_i - y_i \in \{0, 1\}$ for all $1 \leq i \leq n$.

Split $\{x_i\}_{i=1}^n$ into blocks of neighboring integers with distance between blocks being at least 2. Then it is easy to see that $\det[f(x_i - y_j)]$ splits into the product of determinants corresponding to blocks. Let $(x_k, \ldots, x_{l-1})$ be such a block. Then there exists m, $k \leq m < l$, such that $x_i = y_i + 1$ for $k \leq i < m$, and $x_i = y_i$ for $m \leq i < l$. The determinant corresponding to this block is the product of determinants of two triangular matrices, one has size $m - k$ and diagonal entries equal to p, while the other one has size $l - m$ and diagonal entries equal to 1. Thus, the determinant corresponding to this block is equal to p^{m-k}, and collecting these factors over all blocks yields the result. □

Lemma 2.13. *Consider $F(z) = (1 - qz)^{-1}$, that is*

$$f(m) = \begin{cases} q^m, & m \geq 0, \\ 0, & otherwise. \end{cases} \tag{2.55}$$

(i) For integers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$,

$$\det[f(x_i - y_j)]_{i,j=1}^n = \begin{cases} q^{\sum_{i=1}^n (x_i - y_i)}, & x_{i-1} < y_i \leq x_i, \ 1 \leq i \leq n, \\ 0, & otherwise. \end{cases} \tag{2.56}$$

(The condition $x_0 < y_1$ above is empty.)
(ii) For integers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_{n-1}$, and with virtual variable $y_n = $ virt such that $f(x - \text{virt}) = q^x$,

$$\det[f(x_i - y_j)]_{i,j=1}^n$$
$$= \begin{cases} (-1)^{n-1} q^{\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} y_i}, & x_i < y_i \leq x_{i+1}, \ 1 \leq i \leq n-1, \\ 0, & otherwise. \end{cases} \tag{2.57}$$

Proof of Lemma 2.13. (i) Let us first show that the needed inequalities are satisfied. Indeed, if $x_i < y_i$ for some i then $\det[f(x_i - y_j)] = 0$ by the same reasoning as in the previous lemma. On the other hand, if $x_{i-1} \geq y_i$ then $x_k \geq y_l$ for $k \geq i - 1$, $l \leq i$. Let i be the smallest number such that $x_{i-1} \geq y_i$. Then columns i and $i + 1$ have the form

$$\begin{bmatrix} 0 \ldots & 0 & q^{x_{i-1} - y_{i-1}} & q^{x_i - y_{i-1}} & * * \ldots \\ 0 \ldots & 0 & q^{x_{i-1} - y_i} & q^{x_i - y_i} & * * \ldots \end{bmatrix}^T, \tag{2.58}$$

where the 2×2 block with powers of q is on the main diagonal. This again implies that the determinant vanishes. On the other hand, if the interlacing inequalities are satisfied then the matrix $[f(x_i - y_j)]$ is triangular, and computing the product of its diagonal entries yields the result.

(ii) The statement follows from (i). Indeed, we just need to multiply both sides of (i) by q^{y_1}, denote $y_1(\leq x_1)$ by virt, and then cyclically permute y_j's. □

Lemma 2.14. *Consider* $F(z) = p + qz(1 - qz)^{-1}$, *that is*

$$f(m) = \begin{cases} p, & m = 0, \\ q^m, & m \geq 1, \\ 0, & otherwise. \end{cases} \tag{2.59}$$

(i) For integral $x_1 < \cdots < x_n$ *and* $y_1 < \cdots < y_n$,

$$\det \left[f(x_i - y_j) \right]_{i,j=1}^n = q^{\sum_{i=1}^n (x_i - y_i)} p^{\#\{i \,|\, x_i = y_i\}} (1 - p)^{\#\{i \,|\, x_{i-1} = y_i\}} \tag{2.60}$$

if $x_{i-1} \leq y_i \leq x_i$ *for all* $1 \leq i \leq n$, *and* 0 *otherwise.*
(ii) For integral $x_1 < \cdots < x_n$ *and* $y_1 < \cdots < y_{n-1}$, *and with virtual variable* $y_n = $ virt *such that* $f(x - $ virt$) = q^x$,

$$\det \left[f(x_i - y_j) \right]_{i,j=1}^n = (-1)^{n-1} q^{\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} y_i} p^{\#\{i \,|\, x_{i+1} = y_i\}} (1 - p)^{\#\{i \,|\, x_i = y_i\}} \tag{2.61}$$

if $x_i \leq y_i \leq x_{i+1}$ *for all* $1 \leq i \leq n - 1$, *and* 0 *otherwise.*

Proof of Lemma 2.14. (i) The interlacing conditions are verified by the same argument as in the proof of Lemma 2.13(i) (although the conditions themselves are slightly different). Assuming that they are satisfied, we observe that the matrix elements of $[f(x_i - y_j)]$ are zero for $j \geq i + 2$ because $x_i \leq y_{i+1} < y_{i+2}$ and $f(m) = 0$ for $m < 0$. Further, the $(i, i+1)$-element is equal to p if $x_i = y_{i+1}$ or 0 if $x_i < y_{i+1}$. Thus, the matrix is block-diagonal, with blocks being either of size 1 with entry $f(x_i - y_i)$, or of larger size having the form

$$\begin{pmatrix} q^{x_k - y_k} & p & 0 & \ldots & 0 \\ q^{x_{k+1} - y_k} & q^{x_{k+1} - y_{k+1}} & p & \ldots & 0 \\ q^{x_{k+2} - y_k} & q^{x_{k+2} - y_{k+1}} & q^{x_{k+2} - y_{k+2}} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ q^{x_l - y_k} & q^{x_l - y_{k+1}} & q^{x_l - y_{k+2}} & \ldots & q^{x_l - y_l} \end{pmatrix} \tag{2.62}$$

with $x_k = y_{k+1}, \ldots, x_{l-1} = y_l$, and $x_{k-1} < y_k$, $x_l < y_{l+1}$. The determinant of (2.62) is computable via Lemma 1.2 of [9], and it is equal to

$$q^{x_l - y_k} (1 - p)^{l-k} = q^{x_k + \cdots + x_l - (y_k + \cdots + y_l)} (1 - p)^{l-k}. \tag{2.63}$$

Collecting all the factors yields the desired formula.
The proof of (ii) is very similar to that of Lemma 2.13(ii). $\quad\square$

2.5. Examples of bivariate Markov chains. We now use the formulas from the previous two sections to make the constructions of the first two sections more explicit.

Let us start with bivariate Markov chains. Set $\mathcal{S}^* = \mathfrak{X}_n$ and $\mathcal{S} = \mathfrak{X}_{n-1}$, where the sets $\mathfrak{X}_m$, $m = 1, 2, \ldots$, were introduced in Sect. 2.3. We will also take

$$\Lambda = T_{n-1}^n(\alpha_1, \ldots, \alpha_n; (1 - \alpha_n z)^{-1}) \tag{2.64}$$

for some fixed $\alpha_1, \ldots, \alpha_n > 0$.

The first case we consider is

$$P = T_{n-1}(\alpha_1, \ldots, \alpha_{n-1}; 1 + \beta z), \quad P^* = T_n(\alpha_1, \ldots, \alpha_n; 1 + \beta z), \quad \beta > 0. \quad (2.65)$$

Then Proposition 2.10 implies that

$$\Delta = \Lambda P = P^* \Lambda = T_{n-1}^n(\alpha_1, \ldots, \alpha_n; (1 + \beta z)/(1 - \alpha_n z)). \quad (2.66)$$

According to (2.22), (2.23), we have to compute expressions of the form

$$\frac{P^*(x^*, y^*)\Lambda(y^*, y)}{\Delta(x^*, y)}, \qquad \frac{P^*(x^*, y^*)\Lambda(y^*, x)}{\Delta(x^*, x)}$$

for the sequential and parallel updates, respectively.

We start with the condition probability needed for the Markov chain P_Λ.

Proposition 2.15. *Assume that $x^* \in \mathcal{S}^*$ and $y \in \mathcal{S}$ are such that $\Delta(x^*, y) > 0$, that is, $x_k^* \le y_k \le x_{k+1}^*$ for all $1 \le k \le n - 1$. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, y)}{\Delta(x^*, y)}, \qquad y^* \in \mathcal{S}^*, \quad (2.67)$$

has nonzero weights iff

$$y_k^* - x_k^* \in \{-1, 0\}, \qquad y_{k-1} \le y_k^* < y_k, \qquad k = 1, \ldots, n, \quad (2.68)$$

(equivalently, $\max(x_k^* - 1, y_{k-1}) \le y_k^* \le \min(x_k^*, y_k - 1)$ for all k), and these weights are equal to

$$\prod_{\substack{\max(x_k^* - 1, y_{k-1}) < \min(x_k^*, y_k - 1) \\ k=1,\ldots,n}} \left(\frac{\beta}{\alpha_n + \beta}\right)^{x_k^* - y_k^*} \left(\frac{\alpha_n}{\alpha_n + \beta}\right)^{1 - x_k^* + y_k^*} \quad (2.69)$$

with empty product equal to 1.

Remark 2.16. One way to think about the distribution of $y^* \in \mathcal{S}^*$ is as follows. For each k there are two possibilities for y_k^*: Either $\max(x_k^* - 1, y_{k-1}) = \min(x_k^*, y_k - 1)$, in which case y_k^* is forced to be equal to this number, or $\max(x_k^* - 1, y_{k-1}) = x_k^* - 1$ and $\min(x_k^*, y_k - 1) = x_k^*$, in which case y_k^* is allowed to take one of the two values x_k^* or $x_k^* - 1$. Then in the latter case, $x_k^* - y_k^*$ are i. i. d. Bernoulli random variables with the probability of the value 0 equal to $\alpha_n/(\alpha_n + \beta)$.

Proof of Proposition 2.15. The conditions for non-vanishing of the weights follow from those of Lemmas 2.12 and 2.13, namely from (2.54) and (2.57). Using these formulas we extract the factors of $P^*(x^*, y^*)\Lambda(y^*, y)$ that depend on y^*. This yields $(\alpha_n/\beta)^{\sum_{i=1}^n y_i^*}$. Normalizing these weights so that they provide a probability distribution leads to the desired formula. $\square$

Let us now look at the conditional distribution involved in the definition of the Markov chain P_Δ. The following statement is a direct consequence of Proposition 2.15.

Corollary 2.17. *Assume that $x^* \in \mathcal{S}^*$ and $x \in \mathcal{S}$ are such that $\Delta(x^*, x) > 0$, that is, $x_k^* \leq x_k \leq x_{k+1}^*$ for all $1 \leq k \leq n - 1$. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, x)}{\Delta(x^*, x)}, \qquad y^* \in \mathcal{S}^*, \tag{2.70}$$

has nonzero weights iff $\max(x_k^* - 1, x_{k-1}) \leq y_k^* \leq \min(x_k^*, x_k - 1)$, *and these weights are equal to*

$$\prod_{\substack{\max(x_k^*-1,x_{k-1})<\min(x_k^*,x_k-1) \\ k=1,\dots,n}} \left(\frac{\beta}{\alpha_n + \beta}\right)^{x_k^*-y_k^*} \left(\frac{\alpha_n}{\alpha_n + \beta}\right)^{1-x_k^*+y_k^*}. \tag{2.71}$$

Let us now proceed to the case

$$P = T_{n-1}(\alpha_1, \dots, \alpha_{n-1}; (1 - \gamma z)^{-1}), \quad P^* = T_n(\alpha_1, \dots, \alpha_n; (1 - \gamma z)^{-1}). \tag{2.72}$$

We assume that $0 < \gamma < \min\{\alpha_1, \dots, \alpha_n\}$.
By Proposition 2.10,

$$\Delta = \Lambda P = P^* \Lambda = T_{n-1}^n(\alpha_1, \dots, \alpha_n; 1/((1 - \alpha_n z)(1 - \gamma z))). \tag{2.73}$$

Again, let us start with P_Λ.

Proposition 2.18. *Assume that $x^* \in \mathcal{S}^*$ and $y \in \mathcal{S}$ are such that $\Delta(x^*, y) > 0$, that is, $x_{k-1}^* < y_k - 1 < x_{k+1}^*$ for all k. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, y)}{\Delta(x^*, y)}, \qquad y^* \in \mathcal{S}^*, \tag{2.74}$$

has nonzero weights iff

$$x_{k-1}^* < y_k^* \leq x_k^*, \qquad y_{k-1} \leq y_k^* < y_k, \qquad k = 1, \dots, n - 1, \tag{2.75}$$

(equivalently, $\max(x_{k-1}^*+1, y_{k-1}) \leq y_k^* \leq \min(x_k^*, y_k - 1)$ for all k), and these weights are equal to

$$\prod_{k=1}^n \frac{(\alpha_n/\gamma)^{y_k^*}}{\sum_{l=\max(x_{k-1}^*+1,y_{k-1})}^{\min(x_k^*,y_k-1)} (\alpha_n/\gamma)^l}. \tag{2.76}$$

Here $\max(x_0^* + 1, y_0)$ is assumed to denote $-\infty$.

Remark 2.19. Less formally, these formulas state the following: Each y_k^* has to belong to the segment $[\max(x_{k-1}^*+1, y_{k-1}), \min(x_k^*, y_k-1)]$, and the restriction that $\Delta(x^*, y) > 0$ guarantees that these segments are nonempty. Then the claim is that y_k^*'s are independent, and the distribution of y_k^* in the corresponding segment is proportional to the weights $(\alpha_n/\gamma)^{y_k^*}$. In other words, this is the geometric distribution with ratio α_n/γ conditioned to live in the prescribed segment.

Proof of Proposition 2.18. Similarly to the proof of Proposition 2.15, we use Lemma 2.13 to derive the needed inequalities and to single out the part of the ratio $P^*(x^*, y^*)\Lambda(y^*, y)/\Delta(x^*, y)$ that depends on y^*. One readily sees that it is equal to $(\alpha_n/\gamma)^{\sum_{k=1}^n y_k^*}$, and this concludes the proof. □

Let us state what this computation means in terms of the conditional distribution used in the construction of P_Δ.

Corollary 2.20. *Assume that $x^* \in \mathcal{S}^*$ and $x \in \mathcal{S}$ are such that $\Delta(x^*, x) > 0$, that is, $x_{k-1}^* < x_k - 1 < x_{k+1}^*$ for all k. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, x)}{\Delta(x^*, x)}, \qquad y^* \in \mathcal{S}^*, \tag{2.77}$$

has nonzero weights iff $\max(x_{k-1}^* + 1, x_{k-1}) \le y_k^* \le \min(x_k^*, x_k - 1)$ for all k, and these weights are equal to

$$\prod_{k=1}^n \frac{(\alpha_n/\gamma)^{y_k^*}}{\sum_{l=\max(x_{k-1}^*+1, x_{k-1})}^{\min(x_k^*, x_k-1)} (\alpha_n/\gamma)^l}. \tag{2.78}$$

In the four statements above we computed the ingredients needed for the constructions of the bivariate Markov chains for the simplest possible Toeplitz-like transition matrices. In these examples we always had $x_k^* \ge y_k^*$, or, informally speaking, "particles jump to the left". Because of the previous works on the subject, it is more convenient to deal with the case when particles "jump to the right". The arguments are very similar, so let us just state the results.

Consider

$$P = T_{n-1}(\alpha_1, \ldots, \alpha_{n-1}; 1 + \beta z^{-1}), \quad P^* = T_n(\alpha_1, \ldots, \alpha_n; 1 + \beta z^{-1}), \quad \beta > 0. \tag{2.79}$$

- For P_Λ, we have $\max(x_k^*, y_{k-1}) \le y_k^* \le \min(x_k^* + 1, y_k - 1)$. This segment consists of either 1 or 2 points, in the latter case $y_k^* - x_k^*$ are i. i. d. Bernoulli random variables with the probability of 0 equal to $(1 + \alpha_n\beta)^{-1}$.
- For P_Δ, we have $\max(x_k^*, x_{k-1}) \le y_k^* \le \min(x_k^* + 1, x_k - 1)$, and the rest is the same as for P_Λ.

Now consider

$$P = T_{n-1}(\alpha_1, \ldots, \alpha_{n-1}; (1 - \gamma z^{-1})^{-1}), \quad P^* = T_n(\alpha_1, \ldots, \alpha_n; (1 - \gamma z^{-1})^{-1}), \tag{2.80}$$

for $0 < \gamma < \min\{\alpha_1^{-1}, \ldots, \alpha_n^{-1}\}$.

- For P_Λ, we have $\max(x_k^*, y_{k-1}) \le y_k^* \le \min(x_{k+1}^*, y_k) - 1$, and y_k^* are independent geometrically distributed with ratio $(\alpha_n\gamma)$ random variables conditioned to stay in these segments.
- For P_Δ, we have $\max(x_k^*, x_{k-1}) \le y_k^* \le \min(x_{k+1}^*, x_k) - 1$, and the rest is the same as for P_Λ.

Thus, we have so far considered eight bivariate Markov chains. It is natural to denote them as

$$P_\Lambda(1 + \beta z^{\pm 1}), \quad P_\Delta(1 + \beta z^{\pm 1}), \quad P_\Lambda((1 - \gamma z^{\pm 1})^{-1}), \quad P_\Delta((1 - \gamma z^{\pm 1})^{-1}).$$
(2.81)

Observe that although all four chains of type P_Λ live on one and the same state space, all four chains of type P_Δ live on different state spaces. For the sake of completeness, let us list those state spaces:

$$S_\Lambda = \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \mid x_k^* + 1 \le x_k \le x_{k+1}^* \text{ for all } k\},$$
$$S_\Delta(1 + \beta z) = \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \mid x_k^* \le x_k \le x_{k+1}^* \text{ for all } k\},$$
$$S_\Delta(1 + \beta z^{-1}) = \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \mid x_k^* + 1 \le x_k \le x_{k+1}^* + 1 \text{ for all } k\},$$
$$S_\Delta((1 - \gamma z)^{-1}) = \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \mid x_{k-1}^* + 2 \le x_k \le x_{k+1}^* \text{ for all } k\},$$
$$S_\Delta((1 - \gamma z^{-1})^{-1}) = \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \mid x_k^* + 1 \le x_k \le x_{k+2}^* - 1 \text{ for all } k\}.$$

In the above formulas we always use the convention that if an inequality involves a nonexistent variable (like x_0 or x_{n+1}^*), it is omitted.

2.6. Examples of multivariate Markov chains.
Let us now use some of the examples of the bivariate Markov chains from the previous section to construct explicit examples of multivariate (not necessarily autonomous) Markov chains following the recipe of Sect. 2.1.

For any $m \ge 0$ we set $\mathcal{S}_m = \mathfrak{X}_m$, which is the set of strictly increasing m-tuples of integers. In this section we will denote these integers by $x_1^m < \cdots < x_m^m$.

Fix an integer $n \ge 1$, and choose n positive real numbers $\alpha_1, \ldots, \alpha_n$. We take the maps Λ_{k-1}^k to be

$$\Lambda_{k-1}^k = T_{k-1}^k(\alpha_1, \ldots, \alpha_k; (1 - \alpha_k z)^{-1}), \qquad k = 2, \ldots, n. \quad (2.82)$$

We consider the Markov chain $P_\Lambda^{(n)}$, i.e., the sequential update, first. Its state space has the form

$$S_\Lambda^{(n)} = \left\{ (x^1, \ldots, x^n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \mid \prod_{m=2}^n \Lambda_{m-1}^m(x^m, x^{m-1}) > 0 \right\}$$
$$= \left\{ \{x_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_k^{m+1} < x_k^m \le x_{k+1}^{m+1} \text{ for all } k, m \right\}. \quad (2.83)$$

In other words, this is the space of n interlacing integer sequences of length $1, \ldots, n$.

Let t be an integer time variable. We now need to choose the transition probabilities $P_m(t)$, $m = 1, \ldots, n$.

Let $\{F_t(z)\}_{t \ge t_0}$ be a sequence of functions each of which has one of the four possibilities:

$$F_t(z) = (1 + \beta_t^+ z) \text{ or } (1 + \beta_t^-/z) \text{ or } (1 - \gamma_t^+ z)^{-1} \text{ or } (1 - \gamma_t^-/z)^{-1}. \quad (2.84)$$

Here we assume that

$$\beta_t^\pm, \gamma_t^\pm > 0, \qquad \gamma_t^+ < \min\{\alpha_1, \ldots, \alpha_n\}, \quad \gamma_t^- < \min\{\alpha_1^{-1}, \ldots, \alpha_n^{-1}\}. \quad (2.85)$$

We set

$$P_m(t) = T_m(\alpha_1, \ldots, \alpha_m; F_t(z)), \qquad m = 1, \ldots, n. \tag{2.86}$$

Then all needed commutation relations are satisfied, thanks to Proposition 2.10.

The results of Sect. 2.5 enable us to describe the resulting Markov chain on $\mathcal{S}_\Lambda^{(n)}$ as follows.

At time moment t we observe a (random) point $\{x_k^m(t)\} \in S_\Lambda^{(n)}$. In order to obtain $\{x_k^m(t+1)\}$, we perform the sequential update from level 1 to level n. When we are at level m, $1 \leq m \leq n$, the new positions of the particles $x_1^m < \cdots < x_m^m$ are decided independently.

(1) For $F_t(z) = 1 + \beta_t^+ z$, the particle x_k^m is either forced to stay where it is if $x_{k-1}^{m-1}(t+1) = x_k^m(t)$, or it is forced to jump to the left by 1 if $x_k^{m-1}(t+1) = x_k^m(t)$, or it chooses between staying put or jumping to the left by 1 with probability of staying equal to $1/(1 + \beta_t^+ \alpha_m^{-1})$. This follows from Proposition 2.15.

(2) For $F_t(z) = 1 + \beta_t^-/z$, the particle x_k^m is either forced to stay where it is if $x_k^{m-1}(t+1) = x_k^m(t) + 1$, or it is forced to jump to the right by 1 if $x_{k-1}^{m-1}(t+1) = x_k^m(t) + 1$, or it chooses between staying put or jumping to the right by 1 with probability of staying equal to $1/(1 + \beta_t^- \alpha_m)$.

(3) For $F_t(z) = (1 - \gamma_t^+ z)^{-1}$, the particle x_k^m chooses its new position according to a geometric random variable with ratio α_m/γ_t^+ conditioned to stay in the segment

$$[\max(x_{k-1}^m(t) + 1, x_{k-1}^{m-1}(t+1)), \min(x_k^m(t), x_k^{m-1}(t+1) - 1)]. \tag{2.87}$$

In other words, it tries to jump to the left using the geometric distribution of jump length, but it is conditioned not to overcome $x_{k-1}^m(t) + 1$ (in order not to "interact" with the jump of x_{k-1}^m), and it is also conditioned to obey the interlacing inequalities with the updated particles on level $m - 1$. This follows from Proposition 2.18.

(4) For $F_t(z) = (1 - \gamma_t^-/z)^{-1}$, the particle x_k^m chooses its new position according to a geometric random variable with ratio $\alpha_m \gamma_t^-$ conditioned to stay in the segment

$$[\max(x_k^m(t), x_{k-1}^{m-1}(t+1)), \min(x_{k+1}^m(t), x_k^{m-1}(t+1)) - 1]. \tag{2.88}$$

In other words, it tries to jump to the right using the geometric distribution of jump length, but it is conditioned not to overcome $x_{k+1}^m(t) - 1$ (so that it does not interact with jumps of x_{k+1}^m), and it is also conditioned to obey the interlacing inequalities with the updated particles on level $m - 1$.

Projection to $\{x_1^m\}_{m\geq 1}$. A remarkable property of the Markov chain $P_\Lambda^{(n)}$ with steps of the first three types is that its projection onto the n-dimensional subspace $\{x_1^1 > x_1^2 > \cdots > x_1^n\}$ (the smallest coordinates on each level) is also a Markov chain. Moreover, since these are the leftmost particles on each level, they have no interlacing condition on their left to be satisfied, which makes the evolution simpler. Let us describe these Markov chains.

At time moment t we observe $\{x_1^1(t) > x_1^2(t) > \cdots > x_1^n(t)\}$. In order to obtain $\{x_1^m(t + 1)\}_{m=1}^n$, we perform the sequential update from x_1^1 to x_1^n.

(1) For $F_t(z) = 1 + \beta_t^+ z$, the particle x_1^m is either forced to jump (it is being *pushed*) to the left by 1 if $x_1^{m-1}(t+1) = x_1^m(t)$, or it chooses between not moving at all or jumping to the left by 1 with probability of not moving equal to $1/(1 + \beta_t^+ \alpha_m^{-1})$.

(2) For $F_t(z) = 1 + \beta_t^-/z$, the particle x_1^m is either forced to stay where it is if $x_1^{m-1}(t+1) = x_1^m(t) + 1$, or it chooses between staying put or jumping to the right by 1 with probability of staying equal to $1/(1 + \beta_t^- \alpha_m)$.

(3) For $F_t(z) = (1 - \gamma_t^+ z)^{-1}$, the particle x_1^m chooses its new position according to a geometrically distributed with ratio γ_t^+/α_n jump to the left from the point $\min(x_1^m(t), x_1^{m-1}(t+1) - 1)$. That is, if $x_1^m(t) < x_1^{m-1}(t+1)$ then x_1^m simply jumps to the left with the geometric distribution of the jump, while if $x_1^m(t) \geq x_1^{m-1}(t+1)$ then x_1^m is first being pushed to the position $x_1^{m-1}(t+1) - 1$ and then it jumps to the left using the geometric distribution.

(4) For the transition probability with $F_t(z) = (1 - \gamma_t^-/z)^{-1}$, the particle x_1^m is conditioned to stay below $\min(x_2^m(t), x_1^{m-1}(t+1)) - 1$, which involves x_2^m, thus the projection is *not Markovian*.

The Markov chains on $\{x_1^1 > \cdots > x_1^n\}$ corresponding to $1 + \beta_t^+ z$ and $1 + \beta_t^-/z$ are the "Bernoulli jumps with pushing" and "Bernoulli jumps with blocking" chains discussed in [35].

Projection to $\{x_m^m\}_{m \geq 1}$. Similarly, the projection of the "big" Markov chain to $\{x_1^1 \leq x_2^2 \leq \cdots \leq x_n^n\}$ is Markovian for the steps of types one, two, and four, but it is not Markovian for the step of the third type $F_t(z) = (1 - \gamma_t^+ z)^{-1}$.

Let us now consider the parallel update Markov chain $P_\Delta^{(n)}$, or rather one of them. Choose a sequence of functions $G_t(z) = 1 + \beta_t z^{-1}$ with $\beta_t \geq 0$, and set

$$P_m(t) = T_m(\alpha_1, \ldots, \alpha_m; G_t(z)), \qquad m = 1, \ldots, n. \tag{2.89}$$

In case $\beta_t = 0$, $P_m(t)$ is the identity matrix. As before, the needed commutation relations are satisfied by Proposition 2.10.

The (time-dependent) state space of our Markov chain is

$$S_\Delta^{(n)}(t) = \left\{ (x^1, \ldots, x^n) \in S_1 \times \cdots \times S_n \mid \prod_{m=2}^n \Delta_{m-1}^m(x^m, x^{m-1} \mid t + n - m) > 0 \right\}$$

$$= \left\{ \{x_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_k^m < x_k^{m-1} \leq x_{k+1}^m \text{ if } \beta_{t+n-m} = 0, \right.$$

$$\left. x_k^m < x_k^{m-1} \leq x_{k+1}^m + 1 \text{ if } \beta_{t+n-m} > 0 \right\}. \tag{2.90}$$

The update rule follows from the analog of Corollary 2.17 for $(1 + \beta_t z^{-1})$. Namely, assume we have $\{x_k^m(t)\} \in S_\Delta^{(n)}(t)$. Then we choose $\{x_k^m(t)\}$ *independently of each other* as follows. We have

$$\max(x_k^m(t), x_{k-1}^{m-1}(t)) \leq x_k^m(t+1) \leq \min(x_k^m(t) + 1, x_k^{m-1}(t) - 1). \tag{2.91}$$

This segment consists of either 1 or 2 points, and in the latter case $x_k^{m+1}(t+1)$ has probability of not moving equal to $(1 + \alpha_m \beta_{t+n-m})^{-1}$, and it jumps to the right by 1 with remaining probability. In particular, if $\beta_{t+n-m} = 0$ then $x_k^m(t+1) = x_k^m(t)$ for all $k = 1, \ldots, m$.

Less formally, each particle x_k^m either stays put or moves to the right by 1. It is forced to stay put if $x_k^m(t) = x_k^{m-1}(t) - 1$, and it is forced to move by 1 if $x_k^m(t) = x_{k-1}^{m-1}(t) - 1$. Otherwise, it jumps with probability $1 - (1 + \alpha_n \beta_{t+n-m})^{-1}$.

Projection to $\{x_1^m\}_{m \geq 1}$. Once again, the projection of this Markov chain to $\{x_1^1 > \cdots > x_1^n\}$ is also a Markov chain, and its transition probabilities are as follows: Each particle x_1^m at time moment t is either forced to stay if $x_1^m(t) = x_1^{m-1}(t) - 1$ or it stays with probability $(1 + \alpha_n \beta_{t+n-m})^{-1}$ and jumps to the right by 1 with complementary probability. This Markov chain has no pushing because x_1^m's do not have neighbors on the left. This is the "TASEP with parallel update", see e.g. [16].

Projection to $\{x_m^m\}_{m \geq 1}$. We can also restrict our "big" Markov chain to the particles $\{x_1^1, x_2^2, \ldots, x_n^n\}$. Then at time moment t they satisfy the inequalities

$$x_{m-1}^{m-1}(t) \leq x_m^m(t) \quad \text{if} \quad \beta_{t+n-m} = 0, \qquad x_{m-1}^{m-1}(t) \leq x_m^m(t) + 1 \quad \text{if} \quad \beta_{t+n-m} > 0,$$
(2.92)

and the update rule is as follows. If $x_{m-1}^{m-1}(t) = x_m^m(t) + 1$ then x_m^m moves to the right by 1: $x_m^m(t+1) = x_m^m(t)$. However, if $x_{m-1}^{m-1}(t) \leq x_m^m(t)$ then x_m^m stays put with probability $(1 + \alpha_n \beta_{t+n-m})^{-1}$, and it jumps to the right by 1 with the complementary probability.

In the special case when all $\alpha_j = 1$,

$$\beta_k = \begin{cases} \beta, & k \geq n - 1, \\ 0, & k < n - 1, \end{cases}$$
(2.93)

and with the densely packed initial condition $x_k^m(n - m) = k - m - 1$, the Markov chain $P_\Lambda^{(n)}$ discussed above is equivalent to the so-called shuffling algorithm on domino tilings of the Aztec diamonds that at time n produces a random domino tiling of the diamond of size n distributed according to the measure that assigns to a tiling the weight proportional to β raised to the number of vertical tiles, see [59].

2.7. Continuous time multivariate Markov chain.
The (discrete time) Markov chains considered above admit degenerations to continuous time Markov chains. Let us work out one of the simplest examples.

As in the previous sections, we fix an integer $n \geq 1$ and n positive real numbers $\alpha_1, \ldots, \alpha_n$, and take

$$\Lambda_{k-1}^k = T_{k-1}^k(\alpha_1, \ldots, \alpha_k; (1 - \alpha_k z)^{-1}), \qquad k = 2, \ldots, n.$$
(2.94)

We will consider a limit of the Markov chain $P_\Lambda^{(n)}$, so our state space is

$$S_\Lambda^{(n)} = \left\{ \{x_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_k^{m+1} < x_k^m \leq x_{k+1}^{m+1} \text{ for all } k, m \right\}.$$
(2.95)

In the notation of the previous section, let us take $F_t(z) = 1 + \beta^-/z$ for a fixed $\beta_- > 0$ and $t = 1, 2, \ldots$. Thus, we obtain an autonomous Markov chain on $S_\Lambda^{(n)}$, whose transition probabilities are determined by the following recipe.

In order to obtain $\{x_k^m(t+1)\}$ from $\{x_k^m(t)\}$, we perform the sequential update from level 1 to level n. When we are at level m, $1 \leq m \leq n$, for each $k = 1, \ldots, m$ the particle x_k^m is either forced to stay if $x_k^{m-1}(t+1) = x_k^m(t) + 1$, or it is forced to jump to the right by 1 if $x_{k-1}^{m-1}(t+1) = x_k^m(t) + 1$, or it chooses between staying put or jumping to the right by 1 with probability of staying equal to $(1 + \beta^- \alpha_m)^{-1}$. Note that, since particles can only move to the right, it is easy to order the elements of the state space so that the matrix of transition probabilities is triangular.

We are now interested in taking the limit $\beta^- \to 0$.

Lemma 2.21. *Let $A(\epsilon)$ be a (possibly infinite) triangular matrix, whose matrix elements are polynomials in an indeterminate $\epsilon > 0$:*

$$A(\epsilon) = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \ldots, \tag{2.96}$$

and assume that $A_0 = \mathbb{1}$. Then for any $\tau \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} (A(\epsilon))^{\lceil \tau / \epsilon \rceil} = \exp(\tau A_1). \tag{2.97}$$

Proof of Lemma 2.21. For the finite size matrix the claim is standard, and the triangularity assumption reduces the computation of any fixed matrix element of $(A(\epsilon))^{\lceil \tau / \epsilon \rceil}$ to the finite matrix case. □

This lemma immediately implies that the transition probabilities of the Markov chain described above converge, in the limit $\beta^- \to 0$ and time rescaling by β^-, to those of the continuous time Markov chain on $S_A^{(n)}$, whose generator is the linear in β^- term of the generator of the discrete time Markov chain. Let denote this linear term by $L^{(n)}$. Its off-diagonal entries are not hard to compute:

$$L^{(n)} \left(\{x_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}}, \{y_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}} \right) = 1 \tag{2.98}$$

if there exists $1 \leq a \leq b$, $1 \leq b \leq n$, $0 \leq c \leq n - b$ such that

$$x_a^b = x_{a+1}^{b+1} = \cdots = x_{a+c}^{b+c} = x,$$
$$y_a^b = y_{a+1}^{b+1} = \cdots = y_{a+c}^{b+c} = x + 1,$$

and $x_k^m = y_k^m$ for all other values of (k, m), and

$$L^{(n)} \left(\{x_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}}, \{y_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}} \right) = 0 \tag{2.99}$$

in all other cases.

Less formally, this continuous time Markov chain can be described as follows. Each of the particles x_k^m has its own exponential clock, all clocks are independent. When the x_a^b-clock rings, the particle checks if its jump by one to the right would violate the interlacing condition. If no violation happens, that is, if

$$x_a^b < x_a^{b-1} - 1 \quad \text{and} \quad x_a^b < x_{a+1}^{b+1}, \tag{2.100}$$

then this jump takes place. If $x_a^b = x_a^{b-1} - 1$ then the jump is blocked. On the other hand, if $x_a^b = x_{a+1}^{b+1}$ then we find the longest string $x_a^b = x_{a+1}^{b+1} = \cdots = x_{a+c}^{b+c}$ and move

all the particles in this string to the right by one. One could think that the particle x_a^b has pushed the whole string.

We denote this continuous time Markov chain by $\mathcal{P}^{(n)}$.

Similarly to $P_\Lambda^{(n)}$, each of the Markov chains P_m on $\mathcal{S}_m$ also has a continuous limit as $\beta^- \to 0$. Indeed, the transition probabilities of the Markov chain generated by $T_m(\alpha_1, \ldots, \alpha_m; 1 + \beta^-/z)$ converge to ($x^m, y^m \in \mathcal{S}_m$),

$$
\left(\lim_{\beta^- \to 0} \left(T_m(\alpha_1, \ldots, \alpha_m; 1 + \beta^-/z) \right)^{[\tau/\beta^-]} \right) (x^m, y^m)
$$
$$
= \frac{\det [\alpha_i^{y_j^m}]_{i,j=1}^m}{\det [\alpha_i^{x_j^m}]_{i,j=1}^m} \frac{\det [\tau^{y_i^m - x_j^m} \mathbb{1}(y_i^m - x_j^m \geq 0)/(y_i^m - x_j^m)!]_{i,j=1}^m}{\exp(\tau \sum_{j=1}^m \alpha_j)}. \quad (2.101)
$$

Thus, the limit of P_m is the Doob h-transform of m independent Poisson processes by the harmonic function $h(x_1, \ldots, x_m) = \det [\alpha_i^{x_j}]_{i,j=1}^m$, cf. [61]. Let us denote this continuous time Markov chain by $\mathcal{P}_m$, and the above matrix of its transition probabilities over time τ by $\mathcal{P}_m(\tau)$.

Taking the same limit β^- in Proposition 2.5 leads to the following statement.

Proposition 2.22. *Let $m_n(x^n)$ be a probability measure on $\mathcal{S}_n$. Consider the evolution of the measure*

$$
m_n(x^n) \Lambda_{n-1}^n(x^n, x^{n-1}) \cdots \Lambda_1^2(x^2, x^1) \quad (2.102)
$$

on $\mathcal{S}_\Lambda^{(n)}$ under the Markov chain $\mathcal{P}^{(n)}$, and denote by $(x^1(t), \ldots, x^n(t))$ the result after time $t \geq 0$. Then for any

$$
0 = t_n^0 \leq \cdots \leq t_n^{c(n)} = t_{n-1}^0 \leq \cdots \leq t_{n-1}^{c(n-1)} = t_{n-2}^0 \leq \cdots \leq t_2^{c(2)} = t_1^0 \leq \cdots \leq t_1^{c(1)} \quad (2.103)
$$

(here $c(1), \ldots, c(n)$ are arbitrary nonnegative integers) the joint distribution of

$$
x^n(t_n^0), \ldots, x^n(t_n^{c(n)}), x^{n-1}(t_{n-1}^0), x^{n-1}(t_{n-1}^1), \ldots, x^{n-1}(t_{n-1}^{c(n-1)}),
$$
$$
\ldots, x^2(t_2^0), \ldots, x^2(t_2^{c(2)}), x^1(t_1^0), \ldots, x^1(t_1^{c(1)})
$$

coincides with the stochastic evolution of m_n under transition matrices

$$
\mathcal{P}_n(t_n^1 - t_n^0), \ldots, \mathcal{P}_n(t_n^{c(n)} - t_n^{c(n)-1}), \Lambda_{n-1}^n,
$$
$$
\mathcal{P}_{n-1}(t_{n-1}^1 - t_{n-1}^0), \ldots, \mathcal{P}_{n-1}(t_{n-1}^{c(n-1)} - t_{n-1}^{c(n-1)-1}), \Lambda_{n-2}^{n-1},
$$
$$
\ldots, \Lambda_1^2, \mathcal{P}_1(t_1^1 - t_1^0), \ldots, \mathcal{P}_1(t_1^{c(1)} - t_1^{c(1)-1}).
$$

Remark 2.23. It is not hard to see that if in the construction of $P_\Lambda^{(n)}$ we used $F_t(z) = (1 - \gamma^-/z)^{-1}$ and took the limit $\gamma^- \to 0$ then the resulting continuous Markov chains would have been exactly the same. On the other hand, if we used $F_t(z) = (1 + \beta^+ z)$ or $F_t(z) = (1 - \gamma^+ z)^{-1}$ then the limiting continuous Markov chain would have been similar to $\mathcal{P}^{(n)}$, but with particles jumping to the left.

It is slightly technically harder to establish the convergence of Markov chains with alternating steps, for example,

$$F_{2s}(z) = 1 + \beta^+(s)z, \qquad F_{2s+1} = 1 + \beta^-(s)/z, \qquad (2.104)$$

because the transition matrix is no longer triangular (particles jump in both directions). It is possible to prove, however, the following fact:

For any two continuous functions $a(\tau)$ and $b(\tau)$ on $\mathbb{R}_+$ with $a(0) = b(0) = 0$, consider the limit as $\epsilon \to 0$ of the Markov chain $P_\Lambda^{(n)}$ with alternating F_t's as above,

$$\beta^-(s) = \epsilon a(\epsilon s), \qquad \beta^+(s) = \epsilon b(\epsilon s), \qquad (2.105)$$

and the time rescaled by ϵ. Then this Markov chain converges to a continuous time Markov chain, whose generator at time τ is equal to $a(\tau)$ times the generator of $\mathcal{P}^{(n)}$ plus $b(\tau)$ times the generator of the Markov chain similar to $\mathcal{P}^{(n)}$ but with particles jumping to the left.

The statement of Proposition 2.22 also remains true, but in the definition of the Markov chains $\mathcal{P}_m$ one needs to replace the Poisson process by the one-dimensional process whose generator is $a(\tau)$ times the generator of the Poisson process plus $b(\tau)$ times the generator of the Poisson process jumping to the left.

2.8. Determinantal structure of the correlation functions. The goal of this section is to compute certain averages often called correlation functions for the Markov chains $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ with $F_t(z) = (1 + \beta_t^\pm z^{\pm 1})$ or $(1 - \gamma_t^\pm z^{\pm 1})^{-1}$, and their continuous time counterpart $\mathcal{P}^{(n)}$, starting from a certain specific initial condition.

As usual, we begin with $P_\Lambda^{(n)}$. The initial condition that we will use is natural to call a *densely packed initial condition*. It is defined by

$$x_k^m(0) = k - m - 1, \qquad k = 1, \dots, m, \ m = 1, \dots, n. \qquad (2.106)$$

Definition 2.24. *For any $M \geq 1$, pick M points*

$$\varkappa_j = (y_j, m_j, t_j) \in \mathbb{Z} \times \{1, \dots, n\} \times \mathbb{Z}_{\geq 0} \ \ or \ \ \mathbb{Z} \times \{1, \dots, n\} \times \mathbb{R}_{\geq 0}, \qquad (2.107)$$

$j = 1, \dots, M$. *The value of the Mth correlation function ρ_M of $P_\Lambda^{(n)}$ (or $P_\Delta^{(n)}$) at $(\varkappa_1, \dots, \varkappa_M)$ is defined as*

$$\rho_M(\varkappa_1, \dots, \varkappa_M) = \mathrm{Prob}\{For\ each\ j = 1, \dots, M\ there\ exists\ a\ k_j,$$
$$1 \leq k_j \leq m_j,\ such\ that\ x_{k_j}^{m_j}(t_j) = y_j\}. \qquad (2.108)$$

The goal of this section is to partially evaluate the correlation functions corresponding to the densely packed initial condition.

Introduce a partial order on pairs $(m, t) \in \{1, \dots, n\} \times \mathbb{Z}_{\geq 0}$ or $\{1, \dots, n\} \times \mathbb{R}_{\geq 0}$ via

$$(m_1, t_1) \prec (m_2, t_2) \ \ \text{iff} \ \ m_1 \leq m_2,\ t_1 \geq t_2 \text{ and } (m_1, t_1) \neq (m_2, t_2). \qquad (2.109)$$

In what follows we use positive numbers $\alpha_1, \dots, \alpha_n$ that specify the links Λ_{k-1}^k as in Sect. 2.6, and as before we assume that

$$\beta_t^\pm, \gamma_t^\pm > 0, \qquad \gamma_t^+ < \min\{\alpha_1, \dots, \alpha_n\}, \qquad \gamma_t^- < \min\{\alpha_1^{-1}, \dots, \alpha_n^{-1}\}. \qquad (2.110)$$

Theorem 2.25. *Consider the Markov chain $P_\Lambda^{(n)}$ with the densely packed initial condition and $F_t(z) = (1 + \beta_t^\pm z^{\pm 1})$ or $(1 - \gamma_t^\pm z^{\pm 1})^{-1}$. Assume that triplets $\varkappa_j = (y_j, m_j, t_j)$, $j = 1, \ldots, M$, are such that any two distinct pairs (m_j, t_j), $(m_{j'}, t_{j'})$ are comparable with respect to $\prec$. Then*

$$\rho_M(\varkappa_1, \ldots, \varkappa_M) = \det[\mathcal{K}(\varkappa_i, \varkappa_j)]_{i,j=1}^M, \tag{2.111}$$

where

$$\mathcal{K}(y_1, m_1, t_1; y_2, m_2, t_2) = -\frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{w^{y_2 - y_1 + 1}} \frac{\prod_{t=t_2}^{t_1 - 1} F_t(w)}{\prod_{l=m_1+1}^{m_2}(1 - \alpha_l w)} \mathbb{1}_{[(m_1, t_1) \prec (m_2, t_2)]}$$

$$+ \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{\alpha^{-1}}} dz \frac{\prod_{t=0}^{t_1-1} F_t(w)}{\prod_{t=0}^{t_2-1} F_t(z)} \frac{\prod_{l=1}^{m_1}(1 - \alpha_l w)}{\prod_{l=1}^{m_2}(1 - \alpha_l z)} \frac{w^{y_1}}{z^{y_2+1}} \frac{1}{w - z},$$

the contours Γ_0, $\Gamma_{\alpha^{-1}}$ are closed and positively oriented, and they include the poles 0 and $\{\alpha_1^{-1}, \ldots, \alpha_n^{-1}\}$, respectively, and no other poles.

This statement obviously implies

Corollary 2.26. *For the Markov chain $\mathcal{P}^{(n)}$, with the notation of Theorem 2.25 and densely packed initial condition, the correlation functions are given by the same determinantal formula with the kernel*

$$\mathcal{K}(y_1, m_1, \tau_1; y_2, m_2, \tau_2)$$
$$= -\frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{w^{y_2 - y_1 + 1}} \frac{e^{(t_1 - t_2)/w}}{\prod_{l=m_1+1}^{m_2}(1 - \alpha_l w)} \mathbb{1}_{[(m_1, t_1) \prec (m_2, t_2)]}$$
$$+ \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{\alpha^{-1}}} dz \frac{e^{t_1/w}}{e^{t_2/z}} \frac{\prod_{l=1}^{m_1}(1 - \alpha_l w)}{\prod_{l=1}^{m_2}(1 - \alpha_l z)} \frac{w^{y_1}}{z^{y_2+1}} \frac{1}{w - z}.$$

Remark 2.27. For the more general continuous time Markov chain described in Remark 2.23 a similar to Corollary 2.26 result holds true, where one needs to replace the function $e^{t/w}$ by $e^{a(t)/w + b(t)w}$.

Proof of Theorem 2.25. The starting point is Proposition 2.7. The densely packed initial condition is a measure on $\mathcal{S}_\Lambda^{(n)}$ of the form $m_n(x^n) \Lambda_{n-1}^n(x^n, x^{n-1}) \cdots \Lambda_1^2(x^2, x^1)$ with m_n being the delta-measure at the point $(-n, -n+1, \ldots, -1) \in \mathcal{S}_n$.

This delta-measure can be rewritten (up to a constant) as
$\det[\alpha_j^{x_i^n}]_{i,j=1,\ldots,n} \det[\Psi_{n-l}^n(x_k^n)]_{k,l=1,\ldots,n}$ with

$$\Psi_{n-l}^n(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} \prod_{j=l+1}^n (1 - \alpha_j w) w^{x+l} \frac{dw}{w}, \qquad l = 1, \ldots, n. \tag{2.112}$$

Indeed, $\mathrm{Span}(\Psi_{n-l}^n \mid l = 1, \ldots, n)$ is exactly the space of all functions on $\mathbb{Z}$ supported by $\{-1, \ldots, -n\}$.

We are then in a position to apply Theorem 4.2 of [13]. For convenience of the reader, this theorem can be found in Appendix A. (In fact, the change of notation that facilitates the application was already used in Proposition 2.22 above.) The computation of the matrix M^{-1} of that theorem follows verbatim the computation in the proof of

Theorem 3.2 of [22], where θ_j of [22] has to be replaced by α_j^{-1} for all $j = 1, \ldots, n$. Arguing exactly as in that proof we arrive at the desired integral representation for the correlation kernel. $\square$

Finally, one can also derive similar formulas for the Markov chain $P_\Delta^{(n)}$. As the state space $S_\Delta^{(n)}$ is now

$$S_\Delta^{(n)}(t) = \{(x^n(t), x^{n-1}(t+1), \ldots, x^1(t+n-1)\}, \qquad (2.113)$$

we need to define the densely packed initial condition differently, cf. the end of Sect. 2.6. We set

$$x_k^m(n-m) = k - m - 1, \qquad k = 1, \ldots, m, \ m = 1, \ldots, n, \qquad (2.114)$$

and assume that $F_t(z) \equiv 1$ for $t = 0, \ldots, n-2$. This means that

$$\Delta_{m-1}^m(x^m, x^{m-1} \mid n - m) = \Lambda_{m-1}^m(x^m, x^{m-1}), \qquad m = 2, \ldots, n, \qquad (2.115)$$

and our initial condition is of the form (2.43).

Corollary 2.28. *For the Markov chain $P_\Delta^{(n)}$, with the above assumptions, notation of Theorem 2.25, and densely packed initial condition, under the additional assumption that for any two pairs $(m_j, t_j) \prec (m_{j'}, t_{j'})$ we have*

$$t_j - t_{j'} \geq m_{j'} - m_j, \qquad (2.116)$$

the correlation functions are given by the same determinantal formula as in Theorem 2.25.

Proof of Corollary 2.28. Comparing the formulas for the joint distributions for $P_\Delta^{(n)}$ and $P_\Delta^{(n)}$ in Proposition 2.7 we see that with the densely packed initial conditions they simply coincide. Hence, the correlation functions are the same. $\square$

Note that according to the remark at the end of Sect. 2.6, the correlation functions for the shuffling algorithm of domino tilings of Aztec diamonds can be obtained from Theorem 2.25 and Corollary 2.28.

3. Geometry

3.1. Macroscopic behavior, limit shape. It is more convenient for us to slightly modify the definition of the height function (1.2) by assuming that its first argument varies over $\mathbb{Z}$, and

$$h(x, n, t) = \left| \{k \mid x_k^n(t) > x\} \right|. \qquad (3.1)$$

Clearly, this modification has no effect on asymptotic statements.

We are interested in large time behavior of the interface. The macroscopic choice of variables is

$$x = [(\nu - \eta)L], \quad n = [\eta L], \quad t = \tau L, \qquad (3.2)$$

where $(\nu, \eta, \tau) \in \mathbb{R}_+^3$ and $L \gg 1$ is a large parameter setting the macroscopic scale. For fixed η and τ, $h(x, n, t) = n$ for ν small enough (e.g., $\nu = 0$) and $h(x, n, t) = 0$ for ν large enough. Define the x-density of our system as the local average number of particles on unit length in the x-direction. Then, for large L, one expects that $-L^{-1}\partial h/\partial \nu \simeq x$-density. Thus, our model has facets when the x-density is constant (equal to 0 or 1 in our situation), which are interpolated by curved pieces of the surface, see Fig. 3.

Claim. The domain $\mathcal{D} \subset \mathbb{R}^3_+$, where the x-density of our system is asymptotically strictly between 0 and 1 is given by

$$|\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}. \tag{3.3}$$

Equivalently, x-density $\in (0, 1)$ iff there exists a (non-degenerate) triangle with sides $\sqrt{\nu}, \sqrt{\eta}, \sqrt{\tau}$. Denote by π_ν, π_η and π_τ the angles of this triangle as indicated in Fig. 4. Claim 3.1 follows from Proposition 3.1 below.

The condition (3.3) is also equivalent to saying that the circle centered at 0 of radius $\sqrt{\eta/\tau}$ has two disjoint intersections with the circle centered at 1 of radius $\sqrt{\nu/\tau}$. In that case, the two intersections are complex conjugate. Denote by $\Omega(\nu, \eta, \tau)$ the intersection in

$$\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}. \tag{3.4}$$

Then, we have the following properties:

$$|\Omega|^2 = \frac{\eta}{\tau}, \quad |1 - \Omega|^2 = \frac{\nu}{\tau}, \quad \arg(\Omega) = \pi_\nu, \quad \arg(1 - \Omega) = -\pi_\eta. \tag{3.5}$$

The cosine rule gives the angles π_*'s in $(0, \pi)$ by

$$\pi_\nu = \arccos\left(\frac{\tau + \eta - \nu}{2\sqrt{\tau\eta}}\right),$$

$$\pi_\eta = \arccos\left(\frac{\tau + \nu - \eta}{2\sqrt{\tau\nu}}\right), \tag{3.6}$$

$$\pi_\tau = \arccos\left(\frac{\eta + \nu - \eta}{2\sqrt{\nu\eta}}\right).$$

Proposition 3.1 (Bulk scaling limit). *For any $k = 1, 2, \ldots$, consider*

$$\varkappa_j(L) = (x_j(L), n_j(L), t_j(L)), \quad j = 1, \ldots, k, \tag{3.7}$$

such that for any $i \neq j$ and any $L > 0$ either $(n_i(L), t_i(L)) \prec (n_j(L), t_j(L))$ or $(n_j(L), t_j(L)) \prec (n_i(L), t_i(L))$ (the notation $\prec$ was defined in (1.3)). Assume that

$$\lim_{L \to \infty} \frac{x_j}{L} = \nu, \quad \lim_{L \to \infty} \frac{n_j}{L} = \eta, \quad \lim_{L \to \infty} \frac{t_j}{L} = \tau, \quad j = 1, \ldots, k; \tag{3.8}$$

we have $(\nu, \eta, \tau) \in \mathcal{D}$; and also all the differences $x_i - x_j$, $n_i - n_j$, $t_i - t_j$ do not depend on the large parameter L. Then the k-point correlation function $\rho_k(\varkappa_1, \ldots, \varkappa_k)$ converges to the determinant $\det[K^{\mathrm{bulk}}_{ij}]_{1 \leq i,j \leq k}$, where

$$K^{\mathrm{bulk}}_{i,j} = \frac{1}{2\pi\mathrm{i}} \int_{1-\Omega(\nu,\eta,\tau)}^{1-\overline{\Omega}(\nu,\eta,\tau)} \mathrm{d}w \, \frac{(1 - w)^{n_i - n_j} e^{(t_j - t_i)w}}{w^{x_i - x_j + 1}}, \tag{3.9}$$

where for $(n_i, t_i) \not\prec (n_j, t_j)$ the integration contour crosses $\mathbb{R}_+$, while for $(n_i, t_i) \prec (n_j, t_j)$ the contour crosses $\mathbb{R}_-$. On the other hand, if $(\nu, \eta, \tau) \notin \mathcal{D}$, then

$$\lim_{L \to \infty} \rho_k(\varkappa_1, \ldots, \varkappa_k) = 0, \quad \text{if } \sqrt{\nu} > \sqrt{\eta} + \sqrt{\tau},$$

$$\lim_{L \to \infty} \rho_k(\varkappa_1, \ldots, \varkappa_k) = 0, \quad \text{if } \sqrt{\nu} < \sqrt{\tau} - \sqrt{\eta}, \tag{3.10}$$

$$\lim_{L \to \infty} \rho_k(\varkappa_1, \ldots, \varkappa_k) = 1, \quad \text{if } \sqrt{\nu} < \sqrt{\eta} - \sqrt{\tau}.$$

Proof of Proposition 3.1. One follows exactly the same steps as in Sect. 3.2 of [62], replacing the double integral (35) in there by (4.2). The deformed paths are then like in Fig. 12 but with $z_c = w_c$. The degenerate cases when $\varkappa_j \notin \mathcal{D}$ are treated in the same way with limiting kernel $K_{i,j}$ being either 0 (no residue in the contour integral computation) or triangular ($K_{i,j} = 0$ for $x_i < x_j$) with $K_{i,i} = 1$, when the integral in (3.9) is over a complete circle around the origin. $\square$

Corollary 3.2. *Let ρ denote the asymptotic x-density. Then, in $\mathcal{D}$, it is given by*

$$\rho(\nu, \eta, \tau) = \lim_{L \to \infty} \rho_1([\nu L], [\eta L], \tau L) = \pi_\eta / \pi \in [0, 1]. \tag{3.11}$$

Consequently,

$$\boldsymbol{h}(\nu, \eta, \tau) := \lim_{L \to \infty} \frac{\mathbb{E}h([(\nu - \eta)L], [\eta L], \tau L)}{L} = \frac{1}{\pi} \int_\nu^{(\sqrt{\tau} + \sqrt{\eta})^2} \pi_\eta(\nu', \eta, \tau) d\nu'. \tag{3.12}$$

Below we perform the integral in (3.12) to get an explicit expression for the limit shape $\boldsymbol{h}$. Along the way we derive some interesting geometric relations. First of all, $\boldsymbol{h}$ is homogeneous of degree one (since it is the scaling limit under the same scaling in all directions).

Lemma 3.3. *For any $\alpha > 0$,*

$$\boldsymbol{h}(\alpha\nu, \alpha\eta, \alpha\tau) = \alpha\boldsymbol{h}(\nu, \eta, \tau), \tag{3.13}$$

from which it follows

$$\left(\nu \frac{\partial}{\partial \nu} + \eta \frac{\partial}{\partial \eta} + \tau \frac{\partial}{\partial \tau}\right) \boldsymbol{h}(\nu, \eta, \tau) = \boldsymbol{h}(\nu, \eta, \tau). \tag{3.14}$$

Proof of Lemma 3.3. It follows directly from the geometric property $\pi_\eta(\alpha\nu, \alpha\eta, \alpha\tau) = \pi_\eta(\nu, \eta, \tau)$. $\square$

Therefore, we just need to compute the partial derivatives of $\boldsymbol{h}$, then the limit shape $\boldsymbol{h}$ will be determined by the l.h.s. of (3.14).

Proposition 3.4. *The partial derivatives of the limit shape $\boldsymbol{h}$ are given by*

$$\frac{\partial \boldsymbol{h}}{\partial \nu} = -\frac{\pi_\eta}{\pi}, \quad \frac{\partial \boldsymbol{h}}{\partial \eta} = 1 - \frac{\pi_\nu}{\pi}, \quad \frac{\partial \boldsymbol{h}}{\partial \tau} = \frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\pi \sin(\pi_\tau)}. \tag{3.15}$$

Remark 3.5. Another expression for the growth velocity is

$$\frac{\partial \boldsymbol{h}}{\partial \tau} = \frac{1}{\pi} \text{Im}\Omega(\nu, \eta, \tau). \tag{3.16}$$

This can be understood using Proposition 3.1. The macroscopic growth velocity is equal to the average flow of particles, J. It is computed in Sect. 5, see (5.34) with $Q = 0$: $\mathbb{E}(J) = -\partial_2 K(x, n, t; x, n, t)$. Then, by (5.40) we have $\mathbb{E}(J) = K(x, n, t; x + 1, n, t)$. Then, by Proposition 3.1 one gets $\mathbb{E}(J) = \text{Im}(\Omega)/\pi$.

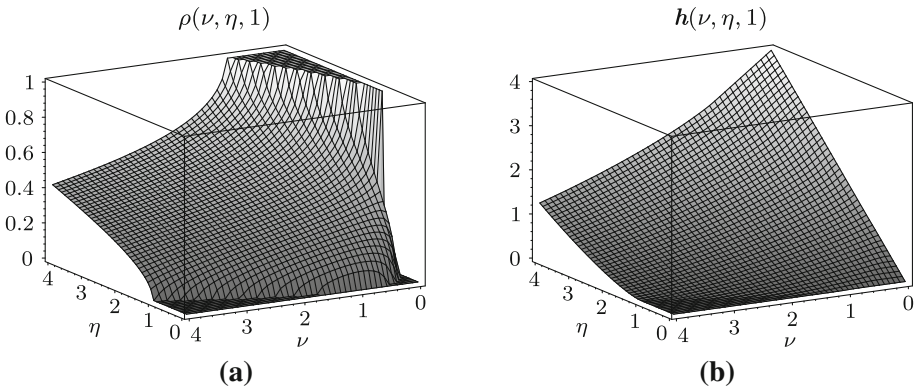As a corollary of Lemma 3.3 and Proposition 3.4, the limit shape is given as follows.

Fig. 5. (**a**) Limiting density of the particles with $\tau = 1$. (**b**) The associated limiting height function. Two facets are visible

Corollary 3.6. *For $(\nu, \eta, \tau) \in \mathcal{D}$, we have*

$$h(\nu, \eta, \tau) = \frac{1}{\pi}\left(-\nu\pi_\eta + \eta(\pi - \pi_\nu) + \tau\,\frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\sin(\pi_\tau)}\right). \tag{3.17}$$

Proof of Proposition 3.4. From (3.12) we immediately have the first relation: $\partial h / \partial \nu = -\pi_\eta / \pi$. In the derivatives of h with respect to τ and η we have one term coming from the boundary term and one from the internal derivative. The boundary terms will actually be zero, since the density at the upper edge is zero. We need to compute (Fig. 5)

$$\frac{\partial \pi_\eta}{\partial \eta} = \frac{1}{\sqrt{4\eta\tau - (\nu - \eta - \tau)^2}}, \quad \frac{\partial \pi_\eta}{\partial \tau} = \frac{\nu - \eta - \tau}{2\tau\sqrt{4\eta\tau - (\nu - \eta - \tau)^2}}. \tag{3.18}$$

Then, we apply the indefinite integrals

$$\int \frac{\mathrm{d}x}{a^2 - x^2} = \arcsin(x/|a|) + C, \quad \int \frac{x\mathrm{d}x}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + C. \tag{3.19}$$

For the derivative with respect to η,

$$\pi\frac{\partial h}{\partial \eta} = \int_\nu^{(\sqrt{\bar{\eta}}+\sqrt{\tau})^2} \frac{\partial \pi_\eta}{\partial \eta}\mathrm{d}\nu' + (1 + \sqrt{\tau/\eta})\pi_\eta\left((\sqrt{\eta} + \sqrt{\tau})^2, \eta, \tau\right)$$

$$= \pi/2 + \arcsin\left(\frac{\eta + \tau - \nu}{2\sqrt{\eta\tau}}\right) = \pi - \arccos\left(\frac{\eta + \tau - \nu}{2\sqrt{\eta\tau}}\right), \tag{3.20}$$

the latter being π_ν. Finally,

$$\pi\frac{\partial h}{\partial \tau} = \int_\nu^{(\sqrt{\bar{\eta}}+\sqrt{\tau})^2} \frac{\partial \pi_\eta}{\partial \tau}\mathrm{d}\nu' + (1 + \sqrt{\eta/\tau})\pi_\eta\left((\sqrt{\eta} + \sqrt{\tau})^2, \eta, \tau\right)$$

$$= \frac{\sqrt{4\eta\tau - (\nu - \eta - \tau)^2}}{2\tau} = \sqrt{\eta/\tau}\,\sin(\pi_\nu), \tag{3.21}$$

and by the sinus theorem for the triangle of Fig. 4 we have $\sqrt{\eta}/\sqrt{\tau} = \sin(\pi_\eta)/\sin(\pi_\tau)$.

$\square$

3.2. Growth model in the anisotropic KPZ class. For fixed τ, the macroscopic slopes of the interface in the ν- and η-directions are given by $\boldsymbol{h}_\nu := \partial_\nu \boldsymbol{h}$ and $\boldsymbol{h}_\eta := \partial_\eta \boldsymbol{h}$. The speed of growth of the surface, $\partial_\tau \boldsymbol{h}$, depends only on these two slopes. Indeed, by (3.15), we can rewrite

$$v = \frac{\partial \boldsymbol{h}}{\partial \tau} = -\frac{1}{\pi} \frac{\sin(\pi \boldsymbol{h}_\nu) \sin(\pi \boldsymbol{h}_\eta)}{\sin(\pi(\boldsymbol{h}_\nu + \boldsymbol{h}_\eta))}. \tag{3.22}$$

Remark that the speed of growth is monotonically decreasing with the slope

$$\frac{\partial v(\boldsymbol{h}_\nu, \boldsymbol{h}_\eta)}{\partial \boldsymbol{h}_\nu} < 0, \quad \frac{\partial v(\boldsymbol{h}_\nu, \boldsymbol{h}_\eta)}{\partial \boldsymbol{h}_\eta} < 0 \tag{3.23}$$

for $\boldsymbol{h}_\nu, \boldsymbol{h}_\eta, \boldsymbol{h}_\nu + \boldsymbol{h}_\eta \in (0, 1)$.

To see which universality class our model belongs to, we need to compute the determinant of the Hessian of $v = v(\boldsymbol{h}_\nu, \boldsymbol{h}_\eta)$. Explicit computations give

$$\begin{vmatrix} \partial_{\boldsymbol{h}_\nu} \partial_{\boldsymbol{h}_\nu} v & \partial_{\boldsymbol{h}_\nu} \partial_{\boldsymbol{h}_\eta} v \\ \partial_{\boldsymbol{h}_\eta} \partial_{\boldsymbol{h}_\nu} v & \partial_{\boldsymbol{h}_\eta} \partial_{\boldsymbol{h}_\eta} v \end{vmatrix} = -4\pi^2 \frac{\sin(\pi \boldsymbol{h}_\nu)^2 \sin(\pi \boldsymbol{h}_\eta)^2}{\sin(\pi(\boldsymbol{h}_\nu + \boldsymbol{h}_\eta))^4} < 0 \tag{3.24}$$

for $\boldsymbol{h}_\nu, \boldsymbol{h}_\eta, \boldsymbol{h}_\nu + \boldsymbol{h}_\eta \in (0, 1)$, i.e., for $(\nu, \eta, \tau) \in \mathcal{D}$. Thus, our model belongs to the *anisotropic* KPZ universality class of growth models in 2 + 1 dimensions.

3.3. A few other geometric properties. During the asymptotic analysis we will use a few more geometric quantities, which we collect in this section. The key function to be analyzed is

$$G(w) \equiv G(w|\nu, \eta, \tau) = \tau w + \nu \ln(1 - w) - \eta \ln(w), \quad w \in \mathbb{C}. \tag{3.25}$$

The critical points of G coincide with Ω as stated below.

Proposition 3.7. *On $\mathbb{C}\backslash\{0, 1\}$, the function G has two critical points (counted with multiplicities). These two points are distinct and complex conjugate if and only if $(\nu, \eta, \tau) \in \mathcal{D}$, in which case the critical points are $\{\Omega, \overline{\Omega}\}$.*

Proof of Proposition 3.7. The derivative of G gives

$$G'(w) = \frac{\tau}{w(w-1)} \left(\left(w - \frac{\eta + \tau - \nu}{2\tau} \right)^2 + \frac{4\eta\tau - (\eta + \tau - \nu)^2}{4\tau^2} \right), \tag{3.26}$$

and we have two distinct complex conjugate solutions iff $4\eta\tau - (\eta + \tau - \nu)^2 > 0$, i.e., iff $(\nu, \eta, \tau) \in \mathcal{D}$. Also, from (3.5) and (3.6) we get

$$\text{Re}(\Omega) = \frac{\eta + \tau - \nu}{2\tau}, \quad \text{Im}(\Omega) = \frac{\sqrt{4\eta\tau - (\eta + \tau - \nu)^2}}{2\tau}. \tag{3.27}$$

Thus, Ω and $\overline{\Omega}$ are the two solutions of $G'(w) = 0$, i.e., the two critical points. $\quad\square$

The main formulas needed later are the partial derivatives of Ω as well as $G''(\Omega)$.

Proposition 3.8. *Denote* $\kappa = 2\tau\,\mathrm{Im}(\Omega) = \sqrt{4\eta\tau - (\eta + \tau - \nu)^2}$. *Then we have*

$$G''(\Omega) = \frac{-\mathrm{i}\kappa}{\Omega(1 - \Omega)}, \tag{3.28}$$

which implies

$$|G''(\Omega)| = \frac{\kappa}{|\Omega(1 - \Omega)|}, \quad \arg(G''(\Omega)) = -\frac{\pi}{2} - \pi_\nu + \pi_\eta. \tag{3.29}$$

Moreover,

$$\frac{\partial\Omega}{\partial\nu} = \frac{\mathrm{i}\Omega}{\kappa}, \quad \frac{\partial\Omega}{\partial\eta} = \frac{\mathrm{i}(1 - \Omega)}{\kappa}, \quad \frac{\partial\Omega}{\partial\tau} = \frac{-\mathrm{i}\Omega(1 - \Omega)}{\kappa}. \tag{3.30}$$

Proof of Proposition 3.8. From (3.26) we get

$$G''(\Omega) = \frac{2\tau}{\Omega(\Omega - 1)}(\Omega - \mathrm{Re}(\Omega)) = \frac{2\mathrm{i}\tau\,\mathrm{Im}(\Omega)}{\Omega(\Omega - 1)}. \tag{3.31}$$

The modulus is immediate, while the argument is obtained using (3.5).

Since Ω is the intersection point of the circles $|z| = \sqrt{\eta/\tau}$ and $|1 - z| = \sqrt{\nu/\tau}$, the direction of $\partial_\nu\Omega$ is orthogonal to the vector Ω and $\partial_\eta\Omega$ is orthogonal to $1 - \Omega$. Therefore, for some $c_1, c_2 \in \mathbb{R}$,

$$\frac{\partial\Omega}{\partial\nu} = c_1\Omega\mathrm{i}, \quad \frac{\partial\Omega}{\partial\eta} = c_2(1 - \Omega)\mathrm{i}. \tag{3.32}$$

Looking at the real part of these equations, we get $\partial_\nu\mathrm{Re}(\Omega) = -c_1\mathrm{Im}(\Omega)$, and $\partial_\nu\mathrm{Re}(\Omega) = c_2\mathrm{Im}(\Omega)$. On the other hand,

$$\mathrm{Re}(\Omega) = \frac{\eta + \tau - \nu}{2\tau} \quad \Rightarrow \quad \partial_\nu\mathrm{Re}(\Omega) = -\frac{1}{2\tau}, \quad \partial_\eta\mathrm{Re}(\Omega) = \frac{1}{2\tau}. \tag{3.33}$$

From this we conclude that

$$\partial_\nu\Omega = \frac{\mathrm{i}\Omega}{2\tau\,\mathrm{Im}(\Omega)}, \quad \partial_\eta\Omega = \frac{\mathrm{i}(1 - \Omega)}{2\tau\,\mathrm{Im}(\Omega)}. \tag{3.34}$$

To get $\partial_\tau\Omega$, we can use the following property: $\Omega(a\nu, a\eta, a\tau) = \Omega(\nu, \eta, \tau)$ for any $a > 0$, which implies

$$\left(\nu\partial_\nu + \eta\partial_\eta + \tau\partial_\tau\right)\Omega = 0. \tag{3.35}$$

This equation leads to

$$\partial_\tau\Omega = -\frac{\mathrm{i}}{2\tau\,\mathrm{Im}(\Omega)}\left(\frac{\nu}{\tau}\Omega + \frac{\eta}{\tau}(1 - \Omega)\right) = -\frac{\mathrm{i}\Omega(1 - \Omega)}{2\tau\,\mathrm{Im}(\Omega)}, \tag{3.36}$$

using $|\Omega|^2 = \eta/\tau$ and $|1 - \Omega|^2 = \nu/\tau$, see (3.5). $\quad\square$

Another important function appearing in the asymptotics of the kernel is the imaginary part of $G(\Omega)$ (and its derivatives).

Proposition 3.9. *We have*

$$\gamma(\nu, \eta, \tau) := \text{Im}(G(\Omega)) = \tau \text{Im}(\Omega) - \nu \pi_\eta - \eta \pi_\nu. \tag{3.37}$$

Its derivatives are

$$\frac{\partial \text{Im}(G(\Omega))}{\partial \nu} = -\pi_\eta, \quad \frac{\partial \text{Im}(G(\Omega))}{\partial \eta} = -\pi_\nu, \tag{3.38}$$

and

$$\frac{\partial^2 \text{Im}(G(\Omega))}{\partial \nu \partial \eta} = -\frac{1}{\kappa}, \quad \kappa = 2\tau \text{Im}(\Omega). \tag{3.39}$$

Proof of Proposition 3.9. The relation (3.37) is a direct consequence of (3.5). The rest are just simple computations. □

4. Gaussian Fluctuations

In this section we first state a couple of equivalent forms of the correlation kernel. In particular, the kernel for a fixed (n, t) has a Christoffel-Darboux representation in terms of Charlier polynomials. In the second part of the section we prove Theorem 1.2 on the Gaussian fluctuations.

4.1. Kernel representations. For the analysis of the variance we will use a representation in terms of Charlier polynomials. These polynomials are defined on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, while our particles at level n live on $\{-n, -n+1, \ldots\}$. Thus, it is convenient to shift the position at level n by n, i.e., the positions of particles at level n will be denoted by $-n + x$, $x \geq 0$. Finally, we also conjugate by a factor $(-1)^{n_1-n_2}$. More precisely, the relation between the shifted and conjugate kernel K and the kernel $\mathcal{K}$ in Theorem 1.1, is the following,

$$K(x_1, n_1, t_1; x_2, n_2, t_2) = (-1)^{n_1-n_2} \mathcal{K}(x_1 - n_1, n_1, t_1; x_2 - n_2, n_2, t_2). \tag{4.1}$$

For later use, we give the explicit double integral representation of K which will be used in the asymptotic analysis.

Corollary 4.1. *The extended kernel K can be expressed as*

$$K(x_1, n_1, t_1; x_2, n_2, t_2)$$
$$= \begin{cases} \frac{e^{t_1-t_2}}{(2\pi i)^2} \oint_{\Gamma_1} dz \oint_{\Gamma_0} dw \frac{z^{n_1}}{e^{t_1 z}(1-z)^{x_1+1}} \frac{e^{t_2 w}(1-w)^{x_2}}{w^{n_2}} \frac{1}{w-z}, & (n_1, t_1) \not\prec (n_2, t_2) \\ \frac{e^{t_1-t_2}}{(2\pi i)^2} \oint_{\Gamma_1} dz \oint_{\Gamma_{0,z}} dw \frac{z^{n_1}}{e^{t_1 z}(1-z)^{x_1+1}} \frac{e^{t_2 w}(1-w)^{x_2}}{w^{n_2}} \frac{1}{w-z}, & (n_1, t_1) \prec (n_2, t_2) \end{cases}. \tag{4.2}$$

Proof of Corollary 4.1. The kernel (4.2) is obtained by substituting into (4.1) the expression for $\mathcal{K}$ from (1.6), and applying the change of variables $z \to 1/(1-w)$ and $w \to 1/(1-z)$. □

It is instructive to see the structure of the kernel that leads the above integral representation.

Proposition 4.2. *The extended kernel K is given by*

$$K(x_1, n_1, t_1; x_1, n_2, t_2) = -\phi^{((n_1,t_1),(n_2,t_2))}(x_1, x_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1,t_1}(x_1) \Phi_{n_2-k}^{n_2,t_2}(x_2) \tag{4.3}$$

with

$$\Psi_k^{n,t}(x) = \frac{1}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{e^{tw}(1-w)^k}{w^{x+1}},$$

$$\Phi_k^{n,t}(x) = \frac{-1}{2\pi i} \oint_{\Gamma_1} dz \frac{z^x e^{-tz}}{(1-z)^{k+1}}, \tag{4.4}$$

$$\phi^{((n_1,t_1),(n_2,t_2))}(x_1, x_2) = \frac{1}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{e^{w(t_1-t_2)}}{w^{x_1-x_2+1}(w-1)^{n_2-n_1}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]},$$

where $\Gamma_{0,1}$ and Γ_1 are any simple anticlockwise oriented contours that include poles $\{0, 1\}$ and $\{1\}$, respectively.

Proof of Proposition 4.2. Using the integral representations for Ψ and Φ one checks that

$$\sum_{k\geq 0} \Psi_k^{n_1,t_1}(x) \Phi_k^{n_2,t_2}(y) = \phi^{((n_1,t_1),(n_2,t_2))}(x, y). \tag{4.5}$$

Thus (4.3) becomes

$$K(x_1, n_1, t_1; x_2, n_2, t_2) = \begin{cases} \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1,t_1}(x_1) \Phi_{n_2-k}^{n_2,t_2}(x_2), & (n_1, t_1) \nprec (n_2, t_2) \\ -\sum_{l=0}^{\infty} \Psi_{n_1+l}^{n_1,t_1}(x_1) \Phi_{n_2+l}^{n_2,t_2}(x_2), & (n_1, t_1) \prec (n_2, t_2) \end{cases}. \tag{4.6}$$

This new expression is good because in (4.4) we never have the case when the pole at $w = 1$ in $\Psi_k^{n,t}$ survives. Then, one has just to take the sums inside the integral. For example, for $(n_1, t_1) \nprec (n_2, t_2)$, we first take the sum inside the integrals and then we extend it to $k = \infty$. This can be done provided $|1 - w| > |1 - z|$. Then, to get the formula (4.2), one just has to rename the variables $z \to 1 - w$ and $w \to 1 - z$. □

For the computation of the variance, we will need only the kernel at fixed (n, t). It is given in terms of the Charlier polynomials, $C_n(x, t)$, given by

$$C_n(x, t) = \frac{n!}{t^n} \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^x e^{wt}}{w^{n+1}}, \tag{4.7}$$

which satisfy $C_n(x, t) = C_x(n, t)$, and are orthogonal with respect to the weight $w_t(x) = \frac{e^{-t}t^x}{x!}$, namely

$$\sum_{x\geq 0} C_n(x, t)C_m(x, t)w_t(x) = \frac{n!}{t^n}\delta_{n,m}, \quad t > 0. \tag{4.8}$$

Corollary 4.3. *The kernel $K(x, n, t; y, n, t)$ is equivalent (conjugate) to the kernel $K_{n,t}(x, y)$, given by*

$$K_{n,t}(x, y) = \sqrt{nt} \frac{q_{n-1}(x, t)q_n(y, t) - q_n(x, t)q_{n-1}(y, t)}{x - y}, \tag{4.9}$$

where

$$q_n(x, t) = w_t(x)^{1/2} \frac{t^{n/2}}{\sqrt{n!}} C_n(x, t). \tag{4.10}$$

Proof of Corollary 4.3. Consider $n_1 = n_2 = n$ and $t_1 = t_2 = t$ in (4.3). Then,

$$K(x, n, t; y, n, t) = \sum_{k=0}^{n-1} \Psi_k^{n,t}(x) \Phi_k^{n,t}(y). \tag{4.11}$$

For all $k \geq 0$, $w = 1$ is not a pole in the integral representation of $\Psi_k^{n,t}$. Using (4.7) and $C_n(x, t) = C_x(n, t)$, we get $\Psi_k^{n,t}(x) = \frac{t^x}{x!} C_k(x, t)$. Also, by the change of variable $z \to 1 - w$ in the integral representation $\Phi_k^{n,t}$ we obtain $\Phi_k^{n,t}(x) = e^{-t} \frac{t^k}{k!} C_k(x, t)$. Thus the kernel is written

$$K(x, n, t; y, n, t) = w_t(x) \sum_{k=0}^{n-1} \frac{t^k}{k!} C_k(x, t) C_k(y, t), \tag{4.12}$$

which is conjugate to the kernel

$$K_{n,t}(x, y) = \sum_{k=0}^{n-1} q_k(x, t) q_k(y, t). \tag{4.13}$$

From $C_n(x, t) = u_n x^n + \cdots$ with $u_n = 1/(-t)^n$, we have $q_n(x, t) = v_n x^n + \cdots$ with $v_n = (-1)^n/\sqrt{t^n n!}$. Then, (4.9) follows from the Christoffel-Darboux formula. $\square$

Remark 4.4. For later use, we rewrite q_n as

$$q_n(x, t) = B_{n,t}(x) I_{n,t}(x), \quad B_{n,t}(x) = \frac{e^{-t/2} t^{x/2}}{\sqrt{x!}} \frac{\sqrt{n!}}{t^{n/2}}, \tag{4.14}$$

and

$$I_{n,t}(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1 - w)^x e^{wt}}{w^{n+1}}. \tag{4.15}$$

4.2. Proof of Theorem 1.2. In this section we look only at the height function at a given time. Therefore, it is convenient to set $\lambda = \nu/\tau$ and $c = \eta/\tau$ so that we have $n = [\eta L] = [ct]$ and $x = [\nu L] = [\lambda t]$. In these variables, the equation for the bulk region given by (3.3) is rewritten as

$$(1 - \sqrt{c})^2 < \lambda < (1 + \sqrt{c})^2. \tag{4.16}$$

First we compute the variance of the height.

Proposition 4.5. *For any* $\lambda \in ((1 - \sqrt{c})^2, (\sqrt{c} + 1)^2)$,

$$\lim_{t \to \infty} \frac{\mathrm{Var}(h([(\lambda - c)t], [ct], t))}{\ln(t)} = \frac{1}{2\pi^2}. \tag{4.17}$$

With this we can prove Theorem 1.2.

Proof of Theorem 1.2. It is a consequence of Proposition 4.5 and [74]. More precisely, in Sect. 2 of [74] the convergence in distribution (a generalization of the result for the sine kernel of [31]) is stated. However, following the proof of the theorem, one realizes that it is done by controlling the cumulants, i.e., also the moments converge. ☐

Proof of Proposition 4.5. The variance can be written in terms of the one and two point correlation functions ρ_1 and ρ_2. Namely,

$$\mathrm{Var}(h([(\lambda - c)t], [ct], t)) = \sum_{x,y > [\lambda t]} \rho_2(x, y) + \sum_{x > [\lambda t]} \rho_1(x) - \left(\sum_{x > [\lambda t]} \rho_1(x) \right)^2, \tag{4.18}$$

where $\rho_2(x, y) = K_{n,t}(x, x) K_{n,t}(y, y) - K_{n,t}(x, y) K_{n,t}(y, x)$ and $\rho_1(x) = K_{n,t}(x, x)$. Using $K_{n,t}^2 = K_{n,t}$ on $\ell_2(\mathbb{Z}_+)$, we have

$$\mathrm{Var}(h([(\lambda - c)t], [ct], t)) = \sum_{x > [\lambda t]} K_{n,t}(x, x) - \sum_{x,y > [\lambda t]} K_{n,t}(x, y) K_{n,t}(y, x)$$

$$= \sum_{x > [\lambda t]} \sum_{y=0}^{\infty} K_{n,t}(x, y) K_{n,t}(y, x) - \sum_{x,y > [\lambda t]} K_{n,t}(x, y) K_{n,t}(y, x)$$

$$= \sum_{x > [\lambda t]} \sum_{y \leq [\lambda t]} \left(K_{n,t}(x, y) \right)^2, \qquad n = [ct]. \tag{4.19}$$

We use the expression (4.9) for the kernel $K_{n,t}$. We decompose the sum in (4.19) into the following three sets:

$$M = \{x, y \in \mathbb{Z}_+^2 | x > [\lambda t], y \leq [\lambda t], y - x \leq \varepsilon_1 t\},$$
$$R_1 = \{x, y \in \mathbb{Z}_+^2 | x > [\lambda t], y \leq [\lambda t], \varepsilon_1 t < y - x < \varepsilon_2 t\}, \tag{4.20}$$
$$R_2 = \{x, y \in \mathbb{Z}_+^2 | x > [\lambda t], y \leq [\lambda t], \varepsilon_2 t \leq y - x\},$$

where the parameter $\varepsilon_2 = \frac{1}{2} \min\{(1 + \sqrt{c})^2 - \lambda, \lambda - (1 - \sqrt{c})^2\}$ is chosen so that R_1 is a subset of the bulk. Thus

$$\mathrm{Var}(h([(\lambda - c)t], [ct], t)) = M_t + R_{t,1} + R_{t,2}, \tag{4.21}$$

with

$$M_t = \sum_{x,y \in M} \left| K_{n,t}(x, y) \right|^2, \quad R_{t,k} = \sum_{x,y \in R_k} \left| K_{n,t}(x, y) \right|^2. \tag{4.22}$$

Remark. The parameter ε_1, small, will be chosen t-dependent in the end.

(1) Bound on $R_{t,2}$. For x, $y \in R_2$, we use $y - x \geq \varepsilon_2 t$, and extend the sum to infinities

$$R_{t,2} \leq \frac{1}{\varepsilon_2^2} \sum_{x \geq \lambda t} \sum_{y \leq \lambda t} \left(|q_{[ct]}(x, t)|^2 |q_{[ct]-1}(y, t)|^2 + |q_{[ct]-1}(x, t)|^2 |q_{[ct]}(y, t)|^2 \right|$$

$$+ 2|q_{[ct]-1}(x, t) q_{[ct]}(x, t)| |q_{[ct]-1}(y, t) q_{[ct]}(y, t)| \right) \leq \frac{4}{\varepsilon_2^2}. \tag{4.23}$$

The last inequality follows from Cauchy-Schwarz and the property

$$\sum_{x \geq 0} |q_k(x, t)|^2 = \langle \Psi_k^{n,t}, \Phi_k^{n,t} \rangle = 1, \quad \text{for all } k. \tag{4.24}$$

(2) Bound on $R_{t,1}$. Since this time x, $y \in R_1$ are always in the bulk, we just use the bound of Lemma 6.8 and get

$$R_{t,1} \leq \text{const} \sum_{x, y \in R_1} \frac{1}{(x - y)^2} = \text{const} \sum_{z=[\varepsilon_1 t]}^{[\varepsilon_2 t]} \frac{1}{z}$$

$$= \Psi([\varepsilon_2 t] + 1) - \Psi([\varepsilon_1 t]), \tag{4.25}$$

where $\Psi(x)$ is the *digamma* function, which has the Taylor expansion at infinity given by

$$\Psi(x) = \ln(x) - 1/(2x) + \mathcal{O}(1/x^2). \tag{4.26}$$

Thus

$$R_{t,1} \leq \text{const} \ln(1/\varepsilon_1), \tag{4.27}$$

with const t-independent, as long as $z, t \to \infty$ as $t \to \infty$.

(3) Limit value for M_t. This time we need more than just a bound. Recall that $n = ct$ and set $x = [\lambda t] + \xi_1$, $y = [\lambda t] - \xi_2$. We have $1 \leq \xi_1 + \xi_2 \leq \varepsilon_1 t$. Lemma 6.4 gives

$$q_{[ct]-\ell}(\lambda t + \xi, t) = \frac{1}{\sqrt{\pi}} \frac{t^{-1/2}}{\sqrt[4]{c - \frac{(1+c-\lambda)^2}{4}}} \left[\mathcal{O}(t^{-1/2}) + \mathcal{O}(\varepsilon_1) \right.$$

$$\left. + \cos\left[t\alpha(c, \lambda + \xi/t) + \beta(c, \lambda) - \ell\partial_c\alpha(c, \lambda) \right] \right]. \tag{4.28}$$

We use it with $\ell = 0, 1$, together with the trigonometric identity

$$\cos(b_1 + \delta)\cos(b_2) - \cos(b_1)\cos(b_2 + \delta) = \sin(\delta)\sin(b_2 - b_1), \tag{4.29}$$

with $\delta = -\partial_c\alpha(c, \lambda)$, $b_1 = t\alpha(c, \lambda+\xi_1/t)+\beta(c, \lambda)$, $b_2 = t\alpha(c, \lambda-\xi_1/t)+\beta(c, \lambda)$. The factor $\sin^2(\delta)$ cancels the $\sqrt[4]{\cdots}$ term exactly. We obtain (using (4.9))

$$M_t = \sum_{\xi_1=1}^{[\varepsilon_1 t]} \sum_{\xi_2=0}^{\xi_1-1} \frac{1}{\pi^2} \frac{1}{(\xi_1 + \xi_2)^2} \left[\mathcal{O}(t^{-1/2}) + \mathcal{O}(\varepsilon_1) \right.$$

$$\left. + \sin^2\left[t(\alpha(c, \lambda - \xi_2/t) - \alpha(c, \lambda + \xi_1/t)) \right] \right]. \tag{4.30}$$

The contribution of the error terms can be bounded by $\ln(\varepsilon_1 t)\mathcal{O}(t^{-1/2}, \varepsilon_1)$, and the remainder is

$$\sum_{\xi_1=1}^{[\varepsilon_1 t]} \sum_{\xi_2=0}^{\xi_1-1} \frac{1}{\pi^2} \frac{1}{(\xi_1+\xi_2)^2} \sin^2\left[t(\alpha(c, \lambda-\xi_2/t) - \alpha(c, \lambda+\xi_1/t))\right]. \quad (4.31)$$

Let $b(\lambda) = -\alpha(c, \lambda)$, then

$$b'(\lambda) = \arccos\left(\frac{1+\lambda-c}{2\sqrt{\lambda}}\right) \in (0, \pi), \quad \text{for } (1-\sqrt{c})^2 < \lambda < (1+\sqrt{c})^2. \quad (4.32)$$

By Lemma 4.6 below (one needs to shift the argument of $b(\lambda \pm \xi_j/t)$ by λ to apply it), for large t the leading term in the sum is identical to the one where $\sin^2(\cdots)$ is replaced by its mean, i.e., $1/2$. Thus

$$(4.31) = (1 + \mathcal{O}(\varepsilon_1, (\varepsilon_1^2\sqrt{t})^{-1})) \sum_{\xi_1=1}^{[\varepsilon_1 t]} \sum_{\xi_2=0}^{\xi_1-1} \frac{1}{2\pi^2} \frac{1}{(\xi_1+\xi_2)^2}$$

$$= \frac{1}{2\pi^2} \ln(\varepsilon_1 t)(1 + \mathcal{O}(\varepsilon_1, (\varepsilon_1^2\sqrt{t})^{-1})). \quad (4.33)$$

Thus,

$$M_t = \ln(\varepsilon_1 t)\left(\frac{1}{2\pi^2} + \mathcal{O}(t^{-1/2}, \varepsilon_1, (\varepsilon_1^2\sqrt{t})^{-1})\right). \quad (4.34)$$

Now we choose $\varepsilon_1 = 1/\ln(t)$. Then,

$$\text{Var}(h([(\lambda-c)t], [ct], t)) = \frac{1}{2\pi^2} \ln(t) + \mathcal{O}\left(1, \ln(\ln(t)), (\ln(t))^3/\sqrt{t}\right), (4.35)$$

which implies (4.17). Modulo Lemma 4.6, the proof of Theorem 1.2 is complete.

$\square$

Lemma 4.6. *Let $b(x)$ be a smooth function (C^2 is enough) on a neighborhood of the origin with $b'(0) \in (0, \pi)$. Then*

$$\sum_{\xi_1=1}^{[\varepsilon t]} \sum_{\xi_2=0}^{\xi_1-1} \frac{\sin^2[tb(\xi_1/t) - tb(-\xi_2/t)]}{(\xi_1+\xi_2)^2} = \sum_{\xi_1=1}^{[\varepsilon t]} \sum_{\xi_2=0}^{[\varepsilon t]-1} \frac{1}{2(\xi_1+\xi_2)^2}\left(1 + \mathcal{O}\left(\varepsilon, \frac{1}{\varepsilon^2\sqrt{t}}\right)\right) \quad (4.36)$$

uniformly for $\varepsilon > 0$ small enough.

Proof of Lemma 4.6. We divide the sum into two regions

$$I_1 = \{\xi_1 \geq 1, \xi_2 \geq 0 | 1 \leq \xi_1+\xi_2 \leq \varepsilon\sqrt{t}\}, \quad (4.37)$$
$$I_2 = \{\xi_1 \geq 1, \xi_2 \geq 0 | \varepsilon\sqrt{t} < \xi_1+\xi_2 \leq \varepsilon t\}.$$

Let us evaluate the contribution to (4.36) of $(\xi_1, \xi_2) \in I_1$. We set $z = \xi_1 + \xi_2$ and get

$$\sum_{z=1}^{[\varepsilon\sqrt{t}]} \sum_{\xi_1=1}^{z} \frac{1}{z^2} \sin^2[tb(\xi_1/t) - tb((\xi_1-z)/t)]. \quad (4.38)$$

Taylor expansion around zero leads to

$$tb(\xi_1/t) - tb((\xi_1 - z)/t) = zb'(0) + \mathcal{O}(\varepsilon^2). \tag{4.39}$$

Thus

$$(4.38) = \sum_{z}^{[\varepsilon\sqrt{t}]} \frac{1}{z} \left(\sin^2 \left[zb'(0) \right] + \mathcal{O}(\varepsilon^2) \right). \tag{4.40}$$

The sum with the sine squared can be explicitly evaluated:

$$\sum_{z=1}^{P} \frac{\sin^2(\sigma z)}{z} = \frac{1}{2} \ln(P) + \mathcal{O}(1), \quad \text{as } P \to \infty, \tag{4.41}$$

provided $0 < \sigma < \pi$. Since $\sum_{z=1}^{P} 1/z = \ln(P)/2 + \mathcal{O}(1/P)$, we have

$$\sum_{z=1}^{P} \frac{\sin^2(\sigma z)}{z} = \sum_{z=1}^{P} \frac{1}{2z} \left(1 + \mathcal{O}(1/P) \right). \tag{4.42}$$

Using $P = [\varepsilon\sqrt{t}]$ and going back to the original variables (ξ_1, ξ_2) we have

$$\sum_{(\xi_1, \xi_2) \in I_1} \frac{\sin^2 [tb(\xi_1/t) - tb(-\xi_2/t)]}{(\xi_1 + \xi_2)^2} = \sum_{(\xi_1, \xi_2) \in I_1} \frac{1}{2(\xi_1 + \xi_2)^2} \left(1 + \mathcal{O}\left(\frac{1}{\varepsilon\sqrt{t}}, \varepsilon^2 \right) \right). \tag{4.43}$$

Now we evaluate the contribution to (4.36) of $(\xi_1, \xi_2) \in I_2$. Let $(X, Y) \in I_2$, then we have $X + Y \geq \varepsilon\sqrt{t}$. We consider a neighborhood of size $M = [\varepsilon^2\sqrt{t}]$ around (X, Y), namely the contribution

$$\sum_{x,y=0}^{M} \frac{1}{(X + Y + x + y)^2} \sin^2 [tb((X + x)/t) - tb(-(Y + y)/t)]. \tag{4.44}$$

Since $\sin^2(\cdots) \geq 0$ and $\frac{1}{(X+Y)^2} - \frac{1}{(X+Y+x+y)^2} \geq 0$, if we replace $\frac{1}{(X+Y+x+y)^2}$ by $\frac{1}{(X+Y)^2}$ in (4.44) the error made is bounded by

$$\sum_{x,y=0}^{M} \left(\frac{1}{(X + Y)^2} - \frac{1}{(X + Y + x + y)^2} \right)$$

$$= \sum_{x,y=0}^{M} \frac{1}{(X + Y)^2} \left(1 - \frac{1}{(1 + \mathcal{O}(\varepsilon))^2} \right) = \sum_{x,y=0}^{M} \frac{1}{(X + Y)^2} \mathcal{O}(\varepsilon), \tag{4.45}$$

because $(x + y)/(X + Y) \leq 2\varepsilon$. This relation can be inverted and we also get

$$\sum_{x,y=0}^{M} \frac{1}{(X + Y)^2} = \sum_{x,y=0}^{M} \frac{1}{(X + Y + x + y)^2} \left(1 + \mathcal{O}(\varepsilon) \right). \tag{4.46}$$

Therefore we have

$$
(4.44) = \sum_{x,y=0}^{M} \frac{\mathcal{O}(\varepsilon)}{(X+Y+x+y)^2} + \sum_{x,y=0}^{M} \frac{\sin^2\left[tb((X+x)/t) - tb(-(Y+y)/t)\right]}{(X+Y)^2}.
$$

$$(4.47)$$

Now we apply the Taylor expansion to the argument in the sine squared. Denote $\kappa_1 = tb(X/t) - tb(-Y/t)$, $\theta_1 = b'(X/t)$ and $\theta_2 = b'(-Y/t)$. Then the argument in the $\sin^2(\cdots)$ is $\kappa_1 + \theta_1 x + \theta_2 y + \mathcal{O}(\varepsilon^2)$. The ε^2 error term is smaller than the $\mathcal{O}(\varepsilon)$ in (4.47), thus

$$
(4.47) = \sum_{x,y=0}^{M} \frac{\mathcal{O}(\varepsilon)}{(X+Y+x+y)^2} + \sum_{x,y=0}^{M} \frac{\sin^2\left[\kappa_1 + \theta_1 x + \theta_2 y\right]}{(X+Y)^2}. \tag{4.48}
$$

Since b is smooth and $b'(0) \in (0, \pi)$, in a neighborhood of 0 we also have $b' \in (0, \pi)$. Thus, for ε small enough, $0 < \theta_1, \theta_2 < \pi$ uniformly in t, because $|Y|/t \le \varepsilon$ and $|X|/t \le \varepsilon$. The second sum in (4.48) can be computed explicitly. For $0 < \theta_1, \theta_2 < \pi$ we have the identity

$$
\sum_{x,y=0}^{M} \sin^2\left[\kappa_1 + \theta_1 x + \theta_2 y\right]
$$

$$
= \frac{(M+1)^2}{2} - \frac{\cos(2\kappa_1 + \theta_1 M + \theta_2 M)\sin(\theta_1(M+1))\sin(\theta_2(M+1))}{2\sin(\theta_1)\sin(\theta_2)}
$$

$$
= \sum_{x,y=0}^{M} \frac{1}{2}(1 + \mathcal{O}(1/M^2)). \tag{4.49}
$$

We replace (4.49) into (4.48) and finally obtain

$$
\sum_{x,y=0}^{M} \frac{\sin^2\left[tb((X+x)/t) - tb(-(Y+y)/t)\right]}{(X+Y+x+y)^2}
$$

$$
= \sum_{x,y=0}^{M} \frac{1}{2(X+Y+x+y)^2}\left(1 + \mathcal{O}(\varepsilon, (\varepsilon^4 t)^{-1})\right). \tag{4.50}
$$

This estimate holds for all the region I_2, thus

$$
\sum_{(\xi_1,\xi_2)\in I_2} \frac{\sin^2\left[tb(\xi_1/t) - tb(-\xi_2/t)\right]}{(\xi_1+\xi_2)^2} = \sum_{(\xi_1,\xi_2)\in I_2} \frac{1}{2(\xi_1+\xi_2)^2}\left(1 + \mathcal{O}(\varepsilon, (\varepsilon^4 t)^{-1})\right).
$$

$$(4.51)$$

The estimates of (4.43) and (4.51) imply the statement of the lemma. □

5. Correlations along Space-like Paths

In this section we present an extension of Theorem 1.1 to the three types of lozenges (see Fig. 6). Then we explain the three different ways of computing height differences. These are then used in the proof of Theorem 1.3.
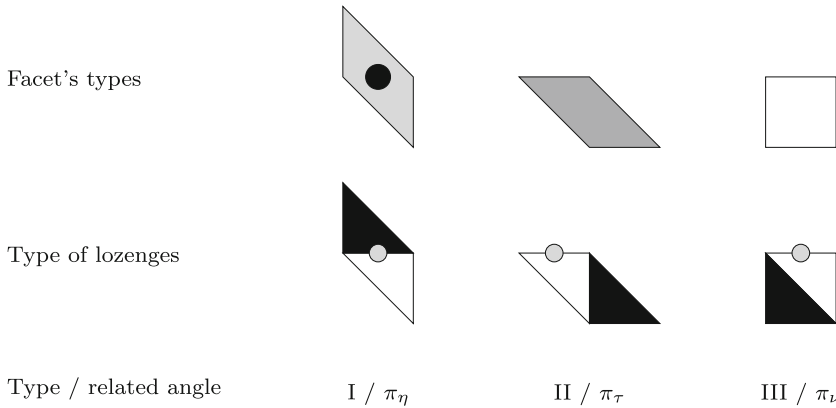
Fig. 6. Facet's types of Fig. 2, their associated lozenges and angles. The gray circle is at position $= (x, n)$

5.1. Joint distribution of the three types of lozenges. As we saw in the Introduction, particle configurations can also be interpreted as lozenge tiling (see Fig. 2) of a half-plane. One can draw the corresponding triangular lattice by associating to the three types of facets three lozenges made by one black and one white triangle as indicated in Fig. 6. We define the position of a black / white triangle to be an (x, n)-coordinate on the mid-point of its horizontal side.

Thus, in our system of coordinates, these positions are pairs of integers (x, n) with $x \in \mathbb{Z}$, $n \in \{0, 1, \ldots\}$ for black and $n \in \{1, 2, \ldots\}$ for white triangles. We first state the result in the common way from the tiling point of view, and then we will reformulate it by using the kernel K defined in (4.2).

For any pair of black and white triangles with space-time coordinates (x, n, t) and (x', n', t'), define the kernel

$$\widetilde{\mathcal{K}}(\blacktriangle\, (x, n, t); \diagdown(x', n', t')) = (-1)^{x-x'+n-n'} \mathcal{K}(x, n, t; x', n', t'), \qquad (5.1)$$

where $\mathcal{K}$ is the kernel defined in (1.6).

Theorem 5.1. *Consider a finite set of lozenges at time moments* $t_1 \leq t_2 \leq \cdots \leq t_M$, *consisting of triangles*

$$(b_i, w_i) := (\blacktriangle\, (x_i, n_i, t_i), \diagdown(x_i', n_i', t_i)). \qquad (5.2)$$

Assume that $n_i \geq n_j$ *if* $t_i < t_j$. *Then*

$$\mathbb{P}\{\text{There is a lozenge } (b_i, w_i) \text{ at time } t_i,$$
$$\text{for every } i = 1, \ldots, M\} = \det\left[\widetilde{\mathcal{K}}(b_i, w_j)\right]_{1 \leq i, j \leq M}. \qquad (5.3)$$

Proof of Theorem 5.1. We prove the statement by induction on the number of lozenges (b_i, w_i) which are not of the form $\diagdown\!\!\blacktriangle$. When this number is zero, then the statement is Theorem 1.1, which is the base of the induction.

Consider any set $\mathcal{S}$ of lozenges at any time moments, plus another lozenge. Then, the l.h.s. of (5.3) obviously satisfies

$$\mathbb{P}\left(\mathcal{S}\cup\diagdown\right)+\mathbb{P}\left(\mathcal{S}\cup\diagdown\right)+\mathbb{P}\left(\mathcal{S}\cup\blacksquare\right)=\mathbb{P}\left(\mathcal{S}\right), \tag{5.4}$$

where in the l.h.s we either keep the white triangle fixed, or we keep the black triangle fixed (and we assume that $\mathcal{S}$ does not contain the fixed triangle). Next we verify that the same relation holds for the r.h.s. of (5.3).

Case (a): the **fixed** triangle is **white**. From the explicit formula for the kernel, we get

$$\widetilde{\mathcal{K}}(\blacktriangleleft(x,n,t);\triangledown)+\widetilde{\mathcal{K}}(\blacktriangleleft(x,n-1,t);\triangledown)+\widetilde{\mathcal{K}}(\blacktriangleleft(x+1,n-1,t);\triangledown)$$
$$= \begin{cases} 1, & \text{if } \triangledown = \triangledown(x,n,t), \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

This implies relation (5.4) for the r.h.s. of (5.3).

Case (b): the **fixed** triangle is **black**. There are two possibilities: (i) the black triangle is on the lower boundary $\{(x,n,t)\,|\,n=0\}$, or (ii) it is not on the boundary. In case (ii), the relevant relation on the kernel is

$$\widetilde{\mathcal{K}}(\blacktriangleleft;\triangledown(x',n',t))+\widetilde{\mathcal{K}}(\blacktriangleleft;\triangledown(x',n'+1,t))+\widetilde{\mathcal{K}}(\blacktriangleleft;\triangledown(x'-1,n'+1,t))$$
$$= \begin{cases} 1, & \text{if } \blacktriangleleft = \blacktriangleleft(x',n',t), \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

In case (i), our assumption $n_i \geq n'_j$ whenever $t_i < t'_j$ implies that we are considering the last time moment, t_M. Then, in the formula for the kernel (1.6), the first residue term drops out and the second term vanishes on $\triangledown(x',0,t)$ (since at $z=1$ there is no pole anymore). Thus (5.6) still gives the needed relation:

$$\mathbb{P}\left(\mathcal{S}\cup\diagdown\right)+\mathbb{P}\left(\mathcal{S}\cup\blacksquare\right)=\mathbb{P}\left(\mathcal{S}\right). \tag{5.7}$$

With the relation (5.4) verified (which, in one case, degenerates to (5.7)), let us explain the induction step. Let us take a lozenge in the set $\{(b_i,w_i),1\leq i\leq M\}$ which is not of the type $\diagdown$. For example, consider $\blacksquare$ and denote by $\mathcal{S}$ the set of remaining $M-1$ lozenges. Then

$$\mathbb{P}\left(\mathcal{S}\cup\blacksquare\right)=\mathbb{P}\left(\mathcal{S}\right)-\mathbb{P}\left(\mathcal{S}\cup\diagdown\right)-\mathbb{P}\left(\mathcal{S}\cup\diagdown\right), \quad \text{with } \triangledown \text{ fixed.} \tag{5.8}$$

So, we have a linear combination of two terms with one less lozenge of type different from $\diagdown$, plus the third term with $\diagdown$ whose black triangle is one position on the right with respect to the $\blacksquare$. For this term we use

$$\mathbb{P}\left(\mathcal{S}\cup\diagdown\right)=\mathbb{P}\left(\mathcal{S}\right)-\mathbb{P}\left(\mathcal{S}\cup\diagdown\right)-\mathbb{P}\left(\mathcal{S}\cup\diagdown\blacksquare\right), \quad \text{with } \blacktriangleleft \text{ fixed.} \tag{5.9}$$

So, the third term in (5.8) is rewritten as a linear combination of two terms with one less lozenge of type different from $\diagdown$, plus a third term with a lozenge of type $\blacksquare$, and this lozenge is one position to the right from the initial $\blacksquare$ in the l.h.s. of (5.8). This can be continued iteratively. A similar argument holds for lozenges of type $\diagdown$ with (5.8) and (5.9) applied in the opposite order.

Thus, we can represent the r.h.s. of (5.3) as a linear combination of those with fewer lozenges of type ◥, ◩, plus an expression of the same kind but with one of the ◥ or ◩ lozenges far to the right.

We still have to verify that the formula with one more lozenge of type ◥ or ◩ agrees when such a lozenge moves to $+\infty$. Since the determinant in (5.3) is invariant with respect to conjugation, consider the kernel $2^{x-x'}\widetilde{\mathcal{K}}(\blacktriangleleft(x, n, t); \triangledown(x', n', t'))$ instead. Then, one verifies that

$$
\begin{aligned}
2^{x-x'}\widetilde{\mathcal{K}}(\blacktriangleleft(x, n, t); \triangledown(x', n', t')) &\to 0 \quad \text{as } x' \to +\infty, \\
2^{x-x'}\widetilde{\mathcal{K}}(\blacktriangleleft(x, n, t); \triangledown(x', n', t')) &\to 0 \quad \text{as } x \to +\infty,
\end{aligned}
\tag{5.10}
$$

and also for a lozenge (b, w) far to the right (i.e., when $x \to +\infty$) we have

$$
2^{x-x'}\widetilde{\mathcal{K}}(b; w) \to
\begin{cases}
1, & \text{if } (b, w) = \text{◩}, \\
0, & \text{if } (b, w) = \text{◥}.
\end{cases}
\tag{5.11}
$$

Therefore, if in the r.h.s. of (5.3) there is one lozenge ◥ that is far to the right, the determinant tends to zero, which agrees with

$$
\mathbb{P}(\mathcal{S} \cup \text{◥}) \to 0 \quad \text{as } \text{◥} \to +\infty.
\tag{5.12}
$$

On the other hand, if in the r.h.s. of (5.3) there is one lozenge ◩ that is far to the right, the determinant tends to the determinant of its minor corresponding to $\mathcal{S}$, which is in agreement with

$$
\mathbb{P}(\mathcal{S} \cup \text{◩}) \to \mathbb{P}(\mathcal{S}) \quad \text{as } \text{◩} \to +\infty.
\tag{5.13}
$$

This completes the induction step. $\square$

In the next section we will describe height function differences as a sum over lozenges of type ◤ or ◥. To each lozenge one can associate a position. We decided to set the position of a lozenge to be equal to the position of the **white** triangle, see Fig. 6. Now we restate Theorem 5.1 in a slightly different form.

Theorem 5.2. *For any $N = 1, 2, \ldots$, pick N triples*

$$
\varkappa_j = (x_j, n_j, t_j) \in \mathbb{Z} \times \mathbb{Z}_{>0} \times \mathbb{R}_{\geq 0}
$$

such that $x_j + n_j \geq 0$ and

$$
t_1 \leq t_2 \leq \cdots \leq t_N, \qquad n_1 \geq n_2 \geq \cdots \geq n_N.
\tag{5.14}
$$

Then

$$
\begin{aligned}
\mathbb{P}\{&\textit{For each } j = 1, \ldots, N \textit{ at } (x_j, n_j, t_j) \textit{ there is a lozenge} \\
&\textit{of type } \theta_j \in \{\text{I, II, III}\}\} = \det\left[K_\theta(\varkappa_i, \theta_i; \varkappa_j, \theta_j)\right]_{i,j=1}^{N},
\end{aligned}
\tag{5.15}
$$

where

$$
\begin{aligned}
&K_\theta(x_1, n_1, t_1, \theta_1; x_2, n_2, t_2, \theta_2) \\
&= \begin{cases}
K(x_1 + n_1, n_1, t_1; x_2 + n_2, n_2, t_2), & \text{if } \theta_1 = \text{I}, \\
-K(x_1 + n_1, n_1 - 1, t_1; x_2 + n_2, n_2, t_2), & \text{if } \theta_1 = \text{II}, \\
K(x_1 + n_1 - 1, n_1 - 1, t_1; x_2 + n_2, n_2, t_2), & \text{if } \theta_1 = \text{III},
\end{cases}
\end{aligned}
\tag{5.16}
$$

with K as defined in Sect. 4.1.

Proof of Theorem 5.2. The proof is simple. One just applies the correspondence

$$\text{Type I at } (x, n, t) \Leftrightarrow (\blacktriangle\,(x, n, t), \triangledown(x, n, t)), \tag{5.17}$$

$$\text{Type II at } (x, n, t) \Leftrightarrow (\blacktriangle\,(x + 1, n - 1, t), \triangledown(x, n, t)), \tag{5.18}$$

$$\text{Type III at } (x, n, t) \Leftrightarrow (\blacktriangle\,(x, n - 1, t), \triangledown(x, n, t)) \tag{5.19}$$

to (5.3) and then rewrites $\widetilde{\mathcal{K}}$ in terms of $\mathcal{K}$. Then, using the relation (4.1), we get the expression in terms of K. Finally, one conjugates the kernel by $(-1)^{x_1 - x_2}$ and obtains the desired kernel K_θ. $\square$

5.2. Height differences as time integration of fluxes.
To determine the height function at a position (m, n) at a given time t, one can act in three different ways:

(a) Sum along the x-direction:

$$h(m, n, t) = \sum_{x > m} \mathbb{1} \text{ (lozenge of type I at } (x, n, t)). \tag{5.20}$$

(b) Sum along the n-direction: for $n' > n$,

$$h(m, n, t) = h(m, n', t) + H_{n,n'}(m, t), \tag{5.21}$$

where

$$H_{n,n'}(m, t) = - \sum_{p=n+1}^{n'} \mathbb{1} \text{ (lozenge of type II at } (m, p, t)). \tag{5.22}$$

(c) Integrate the current over time: for $t > t'$,

$$h(m, n, t) = h(m, n, t') + J_{t',t}(m, n), \tag{5.23}$$

where $J_{t',t}(m, n)$ is the number of particles (= lozenges of type I) which jumped from site (m, n) to site $(m + 1, n)$ during the time interval $[t', t]$.

In principle, one could use (a) alone to determine the height. However, this turns out to be not very practical when dealing with joint distributions of height functions at different points $(m_1, n, t), \ldots, (m_K, n, t)$. The reason is that the height functions are linear functions of lozenges of type I but the same lozenges appear in several of them. The result is a very tedious computation. This can be avoided by using (b) and (c) depending on the cases, see Fig. 7 for an illustration.

Therefore, the expression

$$\mathbb{E}\left(\prod_{k=1}^{N} [h(m_k, n_k, t_k) - \mathbb{E}(h(m_k, n_k, t_k))] \right) \tag{5.24}$$
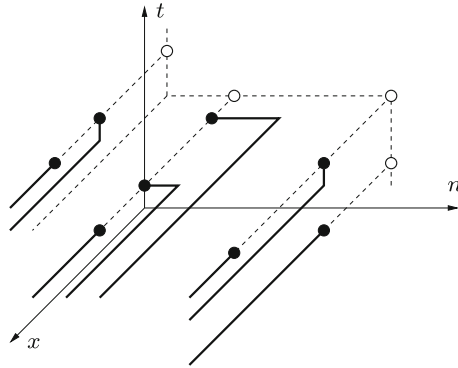
Fig. 7. The *black dots* represent the space-time positions where we want to study the height functions. They live on a space-like surface (i.e., for any two points (x_1, n_1, t_1), (x_2, n_2, t_2) on it, either $(n_1, t_1) \prec (n_2, t_2)$ or $(n_2, t_2) \prec (n_1, t_1)$). The *white dots* represents the projection of the *black dots* onto the (n, t)-plane

can be expressed as a sum of terms of the form

$$
\mathbb{E}\Bigg(\prod_{k=1}^{M} [h(m_k, n_k, t_k) - \mathbb{E}(h(m_k, n_k, t_k))]
$$

$$
\times \prod_{\ell=M+1}^{R} \left[H_{n_\ell, n'_\ell}(m_\ell, t_\ell) - \mathbb{E}(H_{n_\ell, n'_\ell}(m_\ell, t_\ell)) \right]
$$

$$
\times \prod_{j=R+1}^{N} \left[J_{t'_j, t_j}(m_j, n_j) - \mathbb{E}(J_{t'_j, t_j}(m_j, n_j)) \right] \Bigg). \tag{5.25}
$$

We now derive a formula for (5.25).

Lemma 5.3. *Assume that the following paths do not intersect and lie on a space-like surface:*

$$
\begin{aligned}
&\{(x, n_k, t_k)\,|\, x > m_k\}, && k = 1, \ldots, M, \\
&\{(m_\ell, p, t_\ell)\,|\, p = n_\ell + 1, \ldots, n'_\ell\}, && \ell = M + 1, \ldots, R, \\
&\{(m_j, n_j, t)\,|\, t \in [t'_j, t_j]\}, && j = R + 1, \ldots, N.
\end{aligned} \tag{5.26}
$$

Then

$$
(5.25) = \sum_{x_1 > m_1} \cdots \sum_{x_M > m_M} \sum_{p_{M+1} = n_{M+1}+1}^{n'_{M+1}} \cdots \sum_{p_R = n_R+1}^{n'_R}
$$

$$
\int_{t'_{R+1}}^{t_{R+1}} ds_{R+1} \cdots \int_{t'_N}^{t_N} ds_N \det \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix}, \tag{5.27}
$$

with the matrix blocks $A_{i,j}$ as follows:

$$
A_{1,1} = \left[(1 - \delta_{i,j}) K(x_i + n_i, n_i, t_i; x_j + n_j, n_j, t_j) \right]_{1 \le i, j \le M},
$$

$$
A_{2,1} = \left[K(m_i + p_i, p_i - 1, t_i; x_j + n_j, n_j, t_j) \right]_{M+1 \le i \le R,\, 1 \le j \le M},
$$

$$
\begin{aligned}
A_{3,1} &= \big[K(m_i + n_i, n_i, s_i; x_j + n_j, n_j, t_j) \big]_{R+1 \le i \le N, \, 1 \le j \le M}, \\
A_{1,2} &= \big[K(x_i + n_i, n_i, t_i; m_j + p_j, p_j, t_j) \big]_{1 \le i \le M, \, M+1 \le j \le R}, \\
A_{2,2} &= \big[(1 - \delta_{i,j}) K(m_i + p_i, p_i - 1, t_i; m_j + p_j, p_j, t_j) \big]_{M+1 \le i, j \le R}, \quad (5.28) \\
A_{3,2} &= \big[K(m_i + n_i, n_i, s_i; m_j + p_j, p_j, t_j) \big]_{R+1 \le i \le N, \, M+1 \le j \le R}, \\
A_{1,3} &= \big[-\partial_{s_j} K(x_i + n_i, n_i, t_i; m_j + n_j, n_j, s_j) \big]_{1 \le i \le M, \, R+1 \le j \le N}, \\
A_{2,3} &= \big[\partial_{s_j} K(m_i + p_i, p_i - 1, t_i; m_j + n_j, n_j, s_j) \big]_{M+1 \le i \le R, \, R+1 \le j \le N}. \\
A_{3,3} &= \big[-(1 - \delta_{i,j}) \partial_{s_j} K(m_i + n_i, n_i, s_i; m_j + n_j, n_j, s_j) \big]_{R+1 \le i, j \le N}.
\end{aligned}
$$

Proof of Lemma 5.3. Below we prove that

$$
\mathbb{E}\left(\prod_{k=1}^{M} h(m_k, n_k, t_k) \prod_{\ell=M+1}^{R} H_{n_\ell, n'_\ell}(m_\ell, t_\ell) \prod_{j=R+1}^{N} J_{t'_j, t_j}(m_j, n_j) \right) \quad (5.29)
$$

is equal to (5.27) but without the $1 - \delta_{i,j}$ terms in $A_{1,1}$, $A_{2,2}$, and $A_{3,3}$. The fact that the subtraction of the averages is given by putting zeros on the diagonal is a simple but important property, which was noticed for example in [51] (see the proof of Theorem 7.2).

For $N = R$, (5.29) is a direct application of Theorem 5.2 to the formulas (5.20) and (5.22). The absence of the minus sign in $A_{2,*}$ is a consequence of the minus in the definition of the H's in (5.22). Next we extend the result when $N > R$, by first considering $N = R + 1$ for clarity. Denote by $\eta(x, n, t, \theta)$ the random variable that is equal to 1 if there is a type θ lozenge at (x, n, t) and 0 otherwise. Recall that lozenges of type I are exactly what we call particles. The flux of particles can be written as

$$
J_{t', t}(m, n) = \lim_{D \to \infty} \sum_{\ell=1}^{D} \eta(m, n, \tau_{i-1}, \mathrm{I})(1 - \eta(m, n, \tau_i, \mathrm{I})) \quad (5.30)
$$

with $\tau_i = t' + i\Delta\tau$, $i = 0, \ldots, D$, $\Delta\tau = (t - t')/D$.

The quantity $\eta(m, n, \tau_{i-1}, \mathrm{I})(1 - \eta(m, n, \tau_i, \mathrm{I}))$ equals 1 iff the site (m, n) was occupied by a particle at time τ_{i-1} and empty at time τ_i. Each particle tries to jump independently with an exponential waiting time. Every time a particle moves, it can also push other particles, but no more than one on each (higher) level $n = const$. In any case, since on each level there is a finite number of particles, the probability that a particle has more than one jump during time $\Delta\tau$ is $\mathcal{O}(\Delta\tau^2)$. Thus, the limit $\Delta\tau \to 0$ is straightforward.

To obtain (5.29) we have to determine the expression at first order in $\Delta\tau$ of

$$
\mathbb{E}\left(\eta(m, n, \tau_{i-1}, \mathrm{I})(1 - \eta(m, n, \tau_i, \mathrm{I})) \prod_{j=1}^{Q} \eta(m_j, n_j, t_j, \theta_j) \right). \quad (5.31)
$$

Then, in the $\Delta\tau \to 0$ limit we will get an integral from t' to t.

Set $\overline{K}_{x,n}(t_1; t_2) = \sum_{k=0}^{n-1} \Psi_k^{n,t_1}(x + n) \Phi_k^{n,t_2}(x + n)$. Remark that in (4.3), $\phi^{((n,\tau_i),(n,\tau_{i-1}))}(x, x) = 1$. Then, since $\tau_i > \tau_{i-1}$, from (4.3) we obtain

$$(5.31) = \det \begin{bmatrix} \overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) & \overline{K}_{m,n}(\tau_{i-1};\tau_i) & K_\theta(m,n,\tau_{i-1},\mathrm{I};q) \\ 1 - \overline{K}_{m,n}(\tau_i;\tau_{i-1}) & 1 - \overline{K}_{m,n}(\tau_i;\tau_i) & -K_\theta(m,n,\tau_i,\mathrm{I};q) \\ K_\theta(q;m,n,\tau_{i-1},\mathrm{I}) & K_\theta(q;m,n,\tau_i,\mathrm{I}) & K(q,q) \end{bmatrix},$$

$$(5.32)$$

where by q we denoted the quadruples $(m_j, n_j, t_j, \theta_j)$, for $j \in \{1, \dots, Q\}$. The kernel K_θ is a simple function of the kernel K, see (5.16). The second line is just one in the diagonal minus the entries of the kernel (cf. complementation principle in the Appendix of [25]). Written in terms of $\overline{K}$ it becomes as above, since the $(2, 1)$ entry has a 1 coming from ϕ. Next we perform two operations keeping the determinant invariant:

$$\text{Second row} \to \text{Second row} + \text{First row}$$
$$\text{Second column} \to \text{Second column} - \text{First column}.$$

We get that (5.32) is equal to

$$\det \begin{bmatrix} \overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) & \Delta\tau\,\partial_2\overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) + \mathcal{O}(\Delta\tau^2) & K_\theta(m,n,\tau_{i-1},\mathrm{I};q) \\ 1 - \mathcal{O}(\Delta\tau) & \mathcal{O}(\Delta\tau^2) & \mathcal{O}(\Delta\tau) \\ K_\theta(q;m,n,\tau_{i-1},\mathrm{I}) & \Delta\tau\,\partial_2 K_\theta(q;m,n,\tau_{i-1},\mathrm{I}) + \mathcal{O}(\Delta\tau^2) & K_\theta(q,q) \end{bmatrix}$$

$$= -\Delta\tau \det \begin{bmatrix} \partial_2\overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) & K_\theta(m,n,\tau_{i-1},\mathrm{I};q) \\ \partial_2 K_\theta(q;m,n,\tau_{i-1},\mathrm{I}) & K_\theta(q,q) \end{bmatrix} + \mathcal{O}(\Delta\tau^2), \qquad (5.33)$$

where ∂_2 means the derivative with respect to τ_{i-1} in the second entry of the kernel. This formula and (5.30) imply

$$\mathbb{E}\left(J_{t',t}(m,n)\prod_{j=1}^{Q}\eta(m_j,n_j,t_j;\theta_j)\right)$$

$$= \int_{t'}^{t} ds \det \begin{bmatrix} -\partial_2 K_\theta(m,n,s,\mathrm{I};m,n,s,\mathrm{I}) & K_\theta(m,n,s,\mathrm{I};m_j,n_j,t_j,\theta_j) \\ -\partial_2 K_\theta(m_i,n_i,t_i,\theta_i;m,n,s,\mathrm{I}) & K_\theta(m_i,n_i,t_i,\theta_i;m_j,n_j,t_j,\theta_j) \end{bmatrix}_{1\le i,j\le Q}.$$

$$(5.34)$$

The case of several factors J is obtained by induction. Expressing K_θ for $\theta \in \{\mathrm{I}, \mathrm{II}\}$ in terms of K only, and considering the fact that H is *minus* the sum of lozenges of type II, we obtain the result. $\quad\square$

5.3. Proof of Theorem 1.3.

Consider the expectation

$$\mathbb{E}\left(\prod_{k=1}^{N}[h(m_k,n_k,t_k) - \mathbb{E}(h(m_k,n_k,t_k))]\right). \qquad (5.35)$$

Our goal is to determine its limit as $L \to \infty$ under the macroscopic scaling: $t_k = \tau_k L$, $n_k = [\eta_k L]$, $m_k = [(\nu_k - \eta_k)L]$, with $\nu_k \in ((\sqrt{\eta_k} - \sqrt{\tau_k})^2, (\sqrt{\eta_k} + \sqrt{\tau_k})^2)$.

The expression (5.35) is a linear combination of expressions from Lemma 5.3. The r.h.s. of (5.27) contains an $N \times N$ determinant; let us write it as the sum over permutations $\sigma \in \mathcal{S}_N$ of terms each of which is $\operatorname{sgn}\sigma$ times the product of matrix elements (i, σ_i), $i = 1, \dots, N$.

The contribution of all permutations with fixed points is zero (because the diagonal matrix elements are zeroes). All other permutations can be written as unions of several cycles of length $\ell \geq 2$. The contributions of the permutations with only cycles of length 2 lead to the final result, i.e., to prove the theorem we first need to show that the sum of the contributions of permutations with cycles of length $\ell \geq 3$ vanishes in the $L \to \infty$ limit.

Consider a cycle of length $\ell \geq 3$ and use the indices $1, \ldots, \ell$ for the corresponding points (m_i, n_i, t_i). Let us order them so that

$$\eta_1 \geq \eta_2 \geq \cdots \geq \eta_\ell, \quad \tau_1 \leq \tau_2 \leq \cdots \leq \tau_\ell, \quad \text{no double points}, \tag{5.36}$$

i.e., $(\eta_j, \tau_j) \prec (\eta_{j-1}, \tau_{j-1})$.

For an ℓ-cycle we need to take the product of the kernels (or their time derivatives depending on the case), and do the summation over $x_k > [(-\eta_k + v_k)L]$, or over $p_\ell \in [[\eta_\ell L] + 1, [\eta'_\ell L]]$, or the integration over $[\tau'_i L, \tau_i L]$ depending on whether in (5.27) we have a sum or an integral.

We first collect all the factors related to a fixed index i. There are three possible cases:

(a) The index i is related to a sum over $[(-\eta_i + v_i)L, \infty)$. Then we have to analyze

$$\sum_{x > [v_i L]} K(x, [\eta_i L], \tau_i L; \varkappa_{\sigma_i}) K(\varkappa_{\sigma_i^{-1}}; x, [\eta_i L], \tau_i L); \tag{5.37}$$

(b) The index i is related to a sum over $[[\eta_i L] + 1, [\eta'_i L]]$. Then we have to analyze:

$$\sum_{p=[\eta_i L]+1}^{[\eta'_i L]} K([v_i L] + p - [\eta_i L], p - 1, \tau_i L; \varkappa_{\sigma_i})$$
$$\times K(\varkappa_{\sigma_i^{-1}}; [v_i L] + p - [\eta_i L], p, \tau_i L); \tag{5.38}$$

(c) The index i is related to an integrated variable. We have in this case

$$\int_{\tau'_i L}^{\tau_i L} dt \, K([v_i L], [\eta_i L], t; \varkappa_{\sigma_i}) K(\varkappa_{\sigma_i^{-1}}; [v_i L] + 1, [\eta_i L], t). \tag{5.39}$$

The second kernel has a shift by one in the second x-entry. This comes from the identity

$$-\partial_{s'} K(m, n, s; m', n', s') = K(m, n, s; m' + 1, n', s'), \tag{5.40}$$

which immediately follows from (4.2).

We analyze these three expressions in the $L \to \infty$ limit using results of Sect. 6.3. First of all, since $w_c - z_c$ remains bounded away from zero all along the integrals/sums, the bounds of Sect. 6.3 imply that the contributions of the error term $\mathcal{O}(L^{-1/8})$ in (6.56) are of the same order, namely $\mathcal{O}(L^{-1/8})$. Therefore we can get rid of them immediately and we will not write them explicitly in what follows.

Case (a) We divide the sum in three parts for which we use Propositions 6.9–6.13.

Case (a/1) Sum in the interval

$$I_1 = \{x \in \mathbb{N}, x \geq (\sqrt{\tau_i} + \sqrt{\eta_i})^2 L - \ell L^{1/3}\}. \tag{5.41}$$

Then, by Propositions 6.12–6.13,

$$\left| \sum_{x \in I_1} K(x, [\eta_i L], \tau_i L; \varkappa_{\sigma_i}) K(\varkappa_{\sigma_i}^{-1}; x, [\eta_i L], \tau_i L) \right|$$

$$\leq \sum_{x \in I_1} \frac{\text{const}}{L^{2/3}} \exp\left(-2\frac{x - (\sqrt{\tau_i} + \sqrt{\eta_i})^2 L}{(\tau_i L)^{1/3}} \right) \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i}^{-1}$$

$$\leq \frac{\text{const}}{L^{1/3}} \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i}^{-1}. \tag{5.42}$$

Therefore, as $L \to \infty$, the contribution of this sum goes to zero.

Case (a/2) Sum in the interval

$$I_2 = \{x \in \mathbb{N}, (\sqrt{\tau_i} + \sqrt{\eta_i})^2 L - L^{2/3} < x < (\sqrt{\tau_i} + \sqrt{\eta_i})^2 L - \ell L^{1/3}\}. \tag{5.43}$$

By Propositions 6.11–6.12,

$$\left| \sum_{x \in I_2} K(x, [\eta_i L], \tau_i L; \varkappa_{\sigma_i}) K(\varkappa_{\sigma_i}^{-1}; x, [\eta_i L], \tau_i L) \right|$$

$$\leq \sum_{x \in I_2} \frac{\text{const}}{L\sqrt{\eta_i \tau_i - \frac{1}{4}(\tau_i + \eta_i - x/L)^2}} \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i}^{-1}$$

$$\leq \frac{\text{const}}{L^{1/6}} \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i}^{-1}. \tag{5.44}$$

Therefore, as $L \to \infty$, this contribution is also infinitesimally small.

In the following (Cases (a/3), (b), and (c)) we will assume that all the entries $\varkappa_i$'s of the kernel are in $\mathcal{D}$ and apply Proposition 6.9 and its Corollary 6.10. Let us justify it. The variables corresponding to time integration (Case (c)) and sum over the p variables (Case (b)) in (5.27) are always in $\mathcal{D}$. Therefore, the only $\varkappa_i$'s which are not in $\mathcal{D}$ correspond to Cases (a/1) and (a/2) above. From Propositions 6.9–6.12, the contributions in the $\varkappa_i$ variable are of order

$$\frac{\mathcal{O}(1)}{L\sqrt{\eta_i \tau_i - \frac{1}{4}(\eta_i + \tau_i - \nu_i)^2}}, \tag{5.45}$$

if $\varkappa_i \in \mathcal{D}$. The sum in Case (a/3) is then bounded by $\mathcal{O}(1)$ because the sum is over $\mathcal{O}(L)$ sites and the square-root singularity is integrable. Even simpler is Case (b) where we never come close to the singularity and the sum is over $\mathcal{O}(L)$ sites. Finally, in Case (c), the integration is over a time span $\mathcal{O}(L)$. Therefore, the contributions of the terms of Cases (a/3), (b), and (c) are $\mathcal{O}(1)$, and for every sum reaching the edge we get a factor $\mathcal{O}(L^{-1/6})$. Thus, in the following we need to determine the asymptotics of Cases (a/3), (b), and (c) in the case where all the entries $\varkappa_i$'s are in $\mathcal{D}$.

Case (a/3) Sum in the interval

$$I_3 = \{x \in \mathbb{N}, [\nu_i L] < x \leq (\sqrt{\tau_i} + \sqrt{\eta_i})^2 L - L^{2/3}\}. \tag{5.46}$$

Define the functions

$$A(v, \eta, \tau) = \frac{1}{2\pi |G''(\Omega(v, \eta, \tau))|\sqrt{v/\tau}} \tag{5.47}$$

and

$$F(v, \eta, \tau) = L \operatorname{Im}(G(\Omega(v, \eta, \tau))). \tag{5.48}$$

Then, by Proposition 6.9 we have

$$\sum_{x \in I_3} K(x, [\eta_i L], \tau_i L; \varkappa_{\sigma_i}) K(\varkappa_{\sigma_i^{-1}}; x, [\eta_i L], \tau_i L)$$

$$= \sum_{x \in I_3} \frac{A(x/L, \eta_i, \tau_i)}{L} \left[\frac{e^{-i\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{i\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} \right.$$

$$+ \frac{e^{-i\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{-i\beta_2(i)}}{\bar{\omega}(i) - \omega(\sigma_i^{-1})} e^{-2iF(x/L, \eta_i, \tau_i)}$$

$$+ \frac{e^{i\beta_1(i)}}{\omega(\sigma_i) - \bar{\omega}(i)} \frac{e^{i\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} e^{2iF(x/L, \eta_i, \tau_i)}$$

$$\left. + \frac{e^{i\beta_1(i)}}{\omega(\sigma_i) - \bar{\omega}(i)} \frac{e^{-i\beta_2(i)}}{\bar{\omega}(i) - \omega(\sigma_i^{-1})} + \circlearrowleft \right] \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i^{-1}}, \tag{5.49}$$

where we used the notation $\omega(i) = \Omega(v_i, \eta_i, \tau_i)$ and $\circlearrowleft$ means the other 12 terms obtained by replacing $\omega(\sigma_i)$ by $\bar{\omega}(\sigma_i)$ and/or $\omega(\sigma_i^{-1})$ by $\bar{\omega}(\sigma_i^{-1})$.

First we want to show that the terms with F in the exponential are irrelevant in the $L \to \infty$ limit. For that, we sum over $N = L^{1/3}$ positions around any vL in the bulk. Then, for $0 \le x \le L^{1/3}$,

$$F(v + x/L, \eta, \tau) = L\gamma(v, \eta, \tau) + x\partial_v \gamma(v, \eta, \tau) + \mathcal{O}(L^{-1/3}) \text{ holds}, \tag{5.50}$$

where $\gamma(v, \eta, \tau) = L^{-1}F(v, \eta, \tau)$. All the other functions $(A, \beta_1(i), \beta_2(i), \text{ and } \omega(i))$ are smooth functions in v_i, i.e., over an interval $L^{1/3}$ vary only by $\sim L^{-2/3}$. Therefore we have to compute an expression of the form

$$\frac{1}{N} \sum_{x=0}^{N-1} e^{2iF(v+x/L, \eta, \tau)} \phi(v + x/L, \eta, \tau), \tag{5.51}$$

where ϕ is a smooth function given in terms of $A, \beta_1(i), \beta_2(i),$ and $\omega(i)$. Thus

$$(5.51) = \phi(v, \eta, \tau) e^{2iL\gamma(v, \eta, \tau)} \frac{1}{N} \sum_{x=0}^{N-1} e^{ibx} + \mathcal{O}(L^{-1/3}) \tag{5.52}$$

with $b = 2\partial_v \gamma(v, \eta, \tau)$. Then, for $0 < b < 2\pi$, we use

$$\frac{1}{N} \sum_{x=0}^{N-1} e^{ibx} = \frac{e^{ibN} - 1}{N(e^{ib} - 1)}. \tag{5.53}$$

In our case, b is strictly between 0 and π as soon as we are away from the facet. When we reach the lower facet, $b \to 0$. However, in the sum over I_3 we are at least at a distance $L^{2/3}$ from the facet, i.e., $b \geq \text{const } L^{-1/6}$. Therefore

$$|(5.53)| \leq \text{const }/(bN) \leq \text{const } L^{-1/6}. \tag{5.54}$$

Since this holds uniformly in the domain I_3, we have shown that the contribution of the terms where the $\exp(\pm 2\mathrm{i}F)$ is present is at worst of order $L^{-1/6}$. Therefore the only non-vanishing terms in (5.49) are

$$\sum_{x \in I_3} \frac{A(x/L, \eta_i, \tau_i)}{L} \left[\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} \right.$$
$$\left. + \frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} + \circlearrowleft \right] \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i^{-1}}. \tag{5.55}$$

All the functions appearing now are smooth and changing over distances $x \sim L$. Thus, defining $x = vL$, the sum becomes, up to an error of order $\mathcal{O}(L^{-1/3})$, the integral

$$\int_{v_i}^{(\sqrt{\tau_i}+\sqrt{\eta_i})^2} dv\, A(v, \eta_i, \tau_i) \left[\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} \right.$$
$$\left. + \frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} + \circlearrowleft \right] \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i^{-1}}. \tag{5.56}$$

The final step is a change of variable. For the term with $\omega(i)$, we set $z_+^i = \omega(i) = \Omega(v, \eta_i, \tau_i)$. Denote the new integration path by $\Gamma_+^i = \{\Omega(v, \eta_i, \tau_i), v : (\sqrt{\tau_i}+\sqrt{\eta_i})^2 \to v_i\}$. The Jacobian was computed in Proposition 3.8, namely

$$\frac{\partial \omega(i)}{\partial v} = \frac{\mathrm{i}\omega(i)}{\kappa} = 2\pi \mathrm{i} A\, e^{\mathrm{i}\beta_2(i)} e^{-\mathrm{i}\beta_1(i)}. \tag{5.57}$$

For the term with $\bar\omega(i)$ we set $z_-^i = \bar\omega(i) = \overline{\Omega}(v, \eta_i, \tau_i)$ and $\Gamma_-^i = \overline{\Gamma_+^i}$. Then (5.56) becomes

$$\frac{-1}{2\pi \mathrm{i}} \sum_{\varepsilon_i = \pm} \varepsilon_i \int_{\Gamma_{\varepsilon_i}^i} dz_{\varepsilon_i}^i \left[\frac{1}{z_{\varepsilon_i}^i - \omega(\sigma_i)} \frac{1}{\omega(\sigma_i^{-1}) - z_{\varepsilon_i}^i} + \circlearrowleft \right] \times \text{terms in } \sigma_i, \sigma_i^{-1}. \tag{5.58}$$

The factor -1 comes from the orientation of $\Gamma_{\varepsilon_i}^i$, see Fig. 8.

Case (b) We sum in the n-direction from $[\eta_i L] + 1$ to $[\eta_i' L]$ for some $\eta_i' > \eta_i$. While doing this, we do not exit the domain $\mathcal{D}$ remaining in the bulk. Therefore, the computations are just a small variation of the sum over I_3 of Case (a). The minor difference comes from the -1 shift in p in the entries of the first kernel. By changing the variable $\alpha = p/L$, we then obtain

$$\lim_{L \to \infty} \sum_{p=[\eta_i L]+1}^{[\eta_i' L]} K([v_i L]+p-[\eta_i L], p-1, \tau_i L; \varkappa_{\sigma_i}) K(\varkappa_{\sigma_i^{-1}}; [v_i L]+p-[\eta_i L], p, \tau_i L)$$

$$= \int_{\eta_i}^{\eta_i'} d\alpha\, A(v_i - \eta_i + \alpha, \alpha, \tau_i) \left[\frac{e^{-\mathrm{i}\beta_1(i)}\omega(i)^{-1}}{\omega(\sigma_i) - \omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} \right.$$
$$\left. + \frac{e^{\mathrm{i}\beta_1(i)}\bar\omega(i)^{-1}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} + \circlearrowleft \right] \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i^{-1}} \tag{5.59}$$
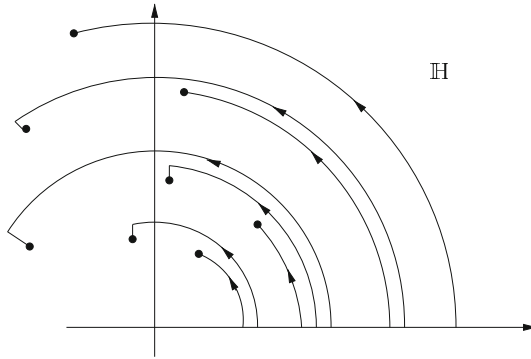
Fig. 8. The image of paths of Fig. 7 under the map Ω

with $\omega(i) = \Omega(\nu_i - \eta_i + \alpha, \alpha, \tau_i)$. For the term with $\omega(i)$, we set $z_i^+ = \omega(i)$ and denote the new integration path by $\Gamma_+^i = \{\Omega(\nu_i - \eta_i + \alpha, \alpha, \tau_i), \alpha : \eta_i' \to \eta_i\}$ (we set the orientation of the path as in Fig. 8). By Proposition 3.8 we get

$$\frac{\partial \omega(i)}{\partial \alpha} = \frac{\mathrm{i}}{\kappa} = 2\pi \mathrm{i} A\, e^{\mathrm{i}\beta_2(i)} e^{-\mathrm{i}\beta_1(i)} \omega(i)^{-1}. \tag{5.60}$$

The change of variable for the term with $\bar\omega(i)$ is similar. The result is the same formula as (5.58) (of course, with the new $\Gamma_{\varepsilon_i}^i$'s).

Case (c) The last case is when we do an integration over a time interval. Similarly to Case (b), we do not have to deal with the edges, since, by assumption, we remain in the bulk of the system. We need to compute

$$\int_{\tau_i' L}^{\tau_i L} \mathrm{d}t\, K([\nu_i L], [\eta_i L], t; \varkappa_{\sigma_i}) K(\varkappa_{\sigma_i^{-1}}; [\nu_i L] + 1, [\eta_i L], t)$$

$$= \int_{\tau_i'}^{\tau_i} \mathrm{d}\tau\, A(\nu_i, \eta_i, \tau) \Bigg[\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} (1 - \omega(i))$$

$$+ \frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} (1 - \bar\omega(i)) e^{-2\mathrm{i}F(\nu_i, \eta_i, \tau)}$$

$$+ \frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} (1 - \omega(i)) e^{2\mathrm{i}F(\nu_i, \eta_i, \tau)}$$

$$+ \frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} (1 - \bar\omega(i)) + \circlearrowleft \Bigg] \times \text{terms in } \varkappa_{\sigma_i}, \varkappa_{\sigma_i^{-1}}. \tag{5.61}$$

The only rapidly changing function is F, which, as for the sum, makes the contributions of the term with it vanishing small as $L \to \infty$. We do the same change of variable as above, i.e., $z_i^+ = \omega(i) = \Omega(\nu_i, \eta_i, \tau)$. Denote the new integration path by $\Gamma_+^i = \{\Omega(\nu_i, \eta_i, \tau), \tau \in [\tau_i', \tau_i]\}$. The Jacobian is computed in Proposition 3.8, namely

$$\frac{\partial \omega(i)}{\partial \tau} = \frac{-\mathrm{i}\omega(i)(1 - \omega(i))}{\kappa} = -2\pi \mathrm{i} A e^{\mathrm{i}\beta_2(i)} e^{-\mathrm{i}\beta_1(i)} (1 - \omega(i)). \tag{5.62}$$

Thus, we obtain again (5.58).

Thus, after summing / integrating all the ℓ variables, we get the contribution of the ℓ-cycle, namely

$$
\frac{(-1)^\ell}{(2\pi i)^\ell} \sum_{\varepsilon_1,\ldots,\varepsilon_\ell=\pm} \prod_{i=1}^\ell \varepsilon_i \int_{\Gamma_{\varepsilon_1}^1} dz_1^{\varepsilon_1} \cdots \int_{\Gamma_{\varepsilon_\ell}^\ell} dz_\ell^{\varepsilon_\ell} \prod_{i=1}^\ell \frac{1}{z_i^{\varepsilon_i} - z_{\sigma_i}^{\varepsilon_{\sigma_i}}},
$$

$$
= \frac{(-1)^\ell}{(2\pi i)^\ell} \sum_{\varepsilon_1,\ldots,\varepsilon_\ell=\pm} \prod_{i=1}^\ell \varepsilon_i \int_{\Gamma_{\varepsilon_1}^1} dz_1^{\varepsilon_1} \cdots \int_{\Gamma_{\varepsilon_\ell}^\ell} dz_\ell^{\varepsilon_\ell} \prod_{i=1}^\ell \frac{1}{z_{\sigma_i}^{\varepsilon_{\sigma_i}} - z_{\sigma_{i-1}}^{\varepsilon_{\sigma_{i-1}}}}, \quad (5.63)
$$

where we set $\sigma_0 := \sigma_\ell$. By Lemma 7.3 in [51], which refers back to [28],

$$
\sum_{\sigma=\ell\text{–cycle in } S_\ell} \prod_{i=1}^\ell \frac{1}{Y_{\sigma_i} - Y_{\sigma_{i-1}}} = 0, \quad \text{for } \ell \geq 3. \quad (5.64)
$$

Therefore, the sum of (5.63) over all possible ℓ-cycles on the same set of indices gives zero for $\ell \geq 3$.

We have shown that we have a Gaussian type formula (sum over all couplings) for points macroscopically away. We still need to compute explicitly the covariance for such points. The covariance is obtained from (5.63) for $\ell = 2$. We need now to consider the signature of the permutation, which for a 2-cycle is -1. We thus obtain a sum of 4 terms which can be put together into (see the end of Sect. 7 in [51] too)

$$
\frac{1}{(2\pi i)^2} \int_{\bar\Omega(v_1,\eta_1,\tau_1)}^{\Omega(v_1,\eta_1,\tau_1)} dz_1 \int_{\bar\Omega(v_2,\eta_2,\tau_2)}^{\Omega(v_2,\eta_2,\tau_2)} dz_2 \frac{1}{(z_1 - z_2)^2}
$$

$$
= \frac{-1}{4\pi^2} \ln\left(\frac{(\Omega(v_1, \eta_1, \tau_1) - \Omega(v_2, \eta_2, \tau_2))(\overline{\Omega}(v_1, \eta_1, \tau_1) - \overline{\Omega}(v_2, \eta_2, \tau_2))}{(\overline{\Omega}(v_1, \eta_1, \tau_1) - \Omega(v_2, \eta_2, \tau_2))(\Omega(v_1, \eta_1, \tau_1) - \overline{\Omega}(v_2, \eta_2, \tau_2))} \right).
$$

$$(5.65)$$

$\square$

5.4. Short and intermediate distance bounds.
Let us first prove the short distance bound (1.18).

Lemma 5.4. *For any $\varkappa_j \in \mathcal{D}$ and any $\varepsilon > 0$, we have*

$$
\mathbb{E}(H_L(\varkappa_1) \cdots H_L(\varkappa_N)) = \mathcal{O}(L^\varepsilon), \quad L \to \infty. \quad (5.66)
$$

Proof of Lemma 5.4. Theorem 1.2 implies, for any integer $m \geq 1$,

$$
\mathbb{E}(H_L(\varkappa_j)^{2m}) = \mathcal{O}(\ln(L)^m). \quad (5.67)
$$

By the Chebyshev inequality,

$$
\mathbb{P}(|H_L(\varkappa_j)| \geq X \ln(L)) = \mathcal{O}(1/X^{2m}), \quad \mathbb{P}(|H_L(\varkappa_j)| \geq Y) = \mathcal{O}(\ln(L)^m/Y^{2m}). \quad (5.68)
$$

The final ingredient is that $|H_L(\varkappa_j)| = \mathcal{O}(L)$, since on level $n = L$ we have only L particles. Therefore, for any Y, we can bound

$$|\mathbb{E}(H_L(\varkappa_1)\cdots H_L(\varkappa_N))| \leq \mathbb{P}(|H_L(\varkappa_1)| \leq Y, \ldots, |H_L(\varkappa_N)| \leq Y)Y^N$$
$$+\mathbb{P}(\exists j \text{ s.t. } |H_L(\varkappa_j)| > Y)\mathcal{O}(L)^N$$
$$= \mathcal{O}(Y^N) + \mathcal{O}(L^N \ln(L)^m / Y^{2m}). \tag{5.69}$$

Taking $Y = L^{\varepsilon/2}$ and $m \gg 1$ large enough, we obtain

$$|\mathbb{E}(H_L(\varkappa_1)\cdots H_L(\varkappa_N))| = \mathcal{O}(L^\varepsilon), \quad \text{for any given } \varepsilon > 0. \tag{5.70}$$

$\square$

Next we give an intermediate distance bound, extending the result of Theorem 1.3.

Lemma 5.5. *Consider the setting as in Theorem 1.3. If the points Ω_i's are not closer than $L^{-1/(8N)}$, then the difference between the expectation $\mathbb{E}(H_L(\varkappa_1)\cdots H_L(\varkappa_N))$ and the r.h.s. of (1.17) is $\mathcal{O}(L^{-1/(12N)})$.*

Proof of Lemma 5.5. It is a small extension of Theorem 1.3. For $N = 1$ the two expressions are identically equal to zero. So, consider $N \geq 2$. We have $|\Omega_i - \Omega_j| \geq L^{-1/16}$, so that the estimate of (6.56) can still be applied in the proof of Theorem 1.3. All the error terms collected are $\mathcal{O}(L^{-1/6})$ (see (5.54)) times at most N factors of order $1/|\Omega_i - \Omega_j| = \mathcal{O}(L^{1/(8N)})$. This accounts into an error $\mathcal{O}(L^{-1/24})$. Now, since the Ω_i's are not away of order one, when one $\mathcal{O}(L^{-1/8})$ in (6.56) is used, it has to be multiplied by at worst $N - 1$ factors of order $1/|\Omega_i - \Omega_j| = \mathcal{O}(L^{1/(8N)})$. Therefore the error is at most $\mathcal{O}(L^{(N-1)/(8N)}L^{-1/8}) = \mathcal{O}(L^{-1/(8N)})$. Similarly, the contribution where $\mathcal{O}(L^{-1/8})$ is used n times is $\mathcal{O}(L^{(N-n)/(8N)}L^{-n/8})$, which is maximal at $n = 1$. Thus, for $N \geq 2$, we get all together $\mathcal{O}(L^{-1/24}) + \mathcal{O}(L^{-1/(8N)}) = \mathcal{O}(L^{-1/(12N)})$. $\square$

5.5. Gaussian Free Field. The Gaussian Free Field on $\mathbb{H}$, see e.g. [73], is a generalized Gaussian process (i.e. it is a probability measure on a suitable class of generalized functions on $\mathbb{H}$) that can be characterized as follows. If we denote by GFF the random generalized function and take any sequence $\{\phi_k\}$ of (compactly supported) test functions, the pairings $\{\mathrm{GFF}(\phi_k)\}$ form a sequence of mean 0 normal variables with covariance matrix

$$\mathbb{E}(\mathrm{GFF}(\phi_k)\,\mathrm{GFF}(\phi_l)) = \int_{\mathbb{H}} |\mathrm{d}z|^2 (\nabla\phi_k(z), \nabla\phi_l(z))$$
$$= \int_{\mathbb{H}^2} |\mathrm{d}z_1|^2|\mathrm{d}z_2|^2\phi_k(z_1)\phi_l(z_2)\mathcal{G}(z_1, z_2),$$

where

$$\mathcal{G}(z, w) = -\frac{1}{2\pi} \ln \left|\frac{z - w}{z - \bar{w}}\right| \tag{5.71}$$

is the Green function of the Laplacian on $\mathbb{H}$ with Dirichlet boundary conditions.

The value of GFF at a point cannot be defined. However, one can think of expectations of products of values of GFF at different points as being finite and equal to

$$\mathbb{E}[\mathrm{GFF}(z_1)\cdots\mathrm{GFF}(z_m)]$$
$$= \begin{cases} 0 & \text{if } m \text{ is odd}, \\ \sum_{\text{pairings } \sigma} \mathcal{G}(z_{\sigma(1)}, z_{\sigma(2)})\cdots\mathcal{G}(z_{\sigma(m-1)}, z_{\sigma(m)}) & \text{if } m \text{ is even}. \end{cases} \tag{5.72}$$

The justification for the notation is the fact that for any finite number of test functions,

$$\mathbb{E}(\mathrm{GFF}(\phi_1)\cdots\mathrm{GFF}(\phi_m)) = \int_{\mathbb{H}^m} \prod_{k=1}^{m} |\mathrm{d}z_k|^2 \phi_k(z_k) \mathbb{E}\left[\mathrm{GFF}(z_1)\cdots\mathrm{GFF}(z_m)\right]. \quad (5.73)$$

The moments (5.73) uniquely determine the Gaussian Free Field.

To state the convergence results, we consider any (smooth) space-like surface $\mathcal{U} \subset \mathbb{R}^3$ in the rounded part of the surface. Namely $\mathcal{U} \subset \mathcal{D}$, and for any two triples $(\nu_i, \eta_i, \tau_i) \in \mathcal{U}$, $i = 1, 2$, $\eta_1 \leq \eta_2$ implies $\tau_1 \geq \tau_2$.

Clearly, the mapping Ω restricted to $\mathcal{U}$ is a bijection. Consider any smooth parametrization $u = (u_1, u_2)$ of $\mathcal{U}$. Denote by $\Omega_{\mathcal{U}}$ the map from u to $\mathbb{H}$, which is the composition of the map from u to (ν, η, τ) and Ω. Then, for any smooth compactly supported test function f on $\mathcal{U}$, we define

$$\langle f, H_L \rangle := \int_{\mathcal{U}} \mathrm{d}u \, f(u) H_L(u), \quad (5.74)$$

where $H_L(u)$ is as in (1.16). Then

$$\langle f, H_L \rangle = \int_{\mathbb{H}} |\mathrm{d}z|^2 J(z) f(\Omega_{\mathcal{U}}^{-1}(z)) H_L(\Omega_{\mathcal{U}}^{-1}(z)), \quad (5.75)$$

where $J(z)$ is the Jacobian of the change of variables $z \to u$ by $\Omega_{\mathcal{U}}^{-1}$.

Theorem 5.6. *For any* $m \in \mathbb{N}$, *and any smooth compactly supported functions* $f_1, \ldots, f_m$ *on* $\mathcal{U}$,

$$\lim_{L\to\infty} \mathbb{E}\left[\prod_{k=1}^{m}\langle f_k, H_L\rangle\right] = \int_{\mathbb{H}^m} \prod_{k=1}^{m} |\mathrm{d}z_k|^2 f_k^{\mathbb{H}}(z_k) \mathbb{E}\left[\mathrm{GFF}(z_1)\cdots\mathrm{GFF}(z_m)\right], \quad (5.76)$$

where $f_i^{\mathbb{H}}(z) := J(z) f(\Omega_{\mathcal{U}}^{-1}(z))$.

Remark 5.7. Since moment convergence to a (multidimensional) Gaussian implies convergence in distribution, Theorem 5.6 implies that the random vector $(\langle f_k, H_L\rangle)_{k=1}^{m}$ converges in distribution (and with all moments) to the Gaussian vector with mean zero and covariance matrix $\| \int_{\mathbb{H}} |\mathrm{d}z|^2 (\nabla f_k^{\mathbb{H}}(z), \nabla f_l^{\mathbb{H}}(z)) \|_{k,l=1,\ldots,m}$.

Proof of Theorem 5.6. We have

$$\mathbb{E}\left[\prod_{k=1}^{m}\langle f_k, H_L\rangle\right] = \int_{\mathbb{H}^m} \prod_{k=1}^{m} |\mathrm{d}z_k|^2 f_k^{\mathbb{H}}(z_k) \mathbb{E}\left[H_L(\Omega_{\mathcal{U}}^{-1}(z_1))\cdots H_L(\Omega_{\mathcal{U}}^{-1}(z_m))\right].$$
$$(5.77)$$

Theorem 1.3 and Lemma 5.5 allow us to determine the last expected value as soon as $|z_i - z_j|$ are away at least of order $\delta := L^{-1/(8m)}$. Denote by

$$\mathbb{H}_\delta^m = \{(z_1, \ldots, z_m) \in \mathbb{H}^m \text{ s.t. } |z_i - z_j| \leq \delta, 1 \leq i < j \leq m\}. \quad (5.78)$$

Then, as $L \to \infty$, we have

$$\int_{\mathbb{H}_\delta^m} \prod_{k=1}^m |\mathrm{d}z_k|^2 f_k^{\mathbb{H}}(z_k) \mathbb{E}\left[H_L(\Omega_{\mathcal{U}}^{-1}(z_1)) \cdots H_L(\Omega_{\mathcal{U}}^{-1}(z_m)) \right]$$

$$= \int_{\mathbb{H}_\delta^m} \prod_{k=1}^m |\mathrm{d}z_k|^2 f_k^{\mathbb{H}}(z_k) \mathbb{E}\left[\mathrm{GFF}(z_1) \cdots \mathrm{GFF}(z_m) \right] + \mathcal{O}(L^{-1/(12m)}). \quad (5.79)$$

Then, since the logarithm is integrable around zero (in two but also in one dimension), the $L \to \infty$ limit is simply given by

$$\lim_{L \to \infty} (5.79) = \int_{\mathbb{H}^m} \prod_{k=1}^m |\mathrm{d}z_k|^2 f_k^{\mathbb{H}}(z_k) \mathbb{E}\left[\mathrm{GFF}(z_1) \cdots \mathrm{GFF}(z_m) \right]. \quad (5.80)$$

We still need to control the contribution coming from $\mathbb{H}^m \setminus \mathbb{H}_\delta^m$. Using Lemma 5.4, this is bounded by

$$\left| \int_{\mathbb{H}^m \setminus \mathbb{H}_\delta^m} \prod_{k=1}^m |\mathrm{d}z_k|^2 f_k^{\mathbb{H}}(z_k) \mathbb{E}\left[H_L(\Omega_{\mathcal{U}}^{-1}(z_1)) \cdots H_L(\Omega_{\mathcal{U}}^{-1}(z_m)) \right] \right| \le \mathrm{const}\, \delta^2 L^\varepsilon,$$

$$(5.81)$$

where const depends only on the functions $f_1, \ldots, f_m$. Since $\delta^2 = L^{-1/(4m)}$ and $\varepsilon > 0$ can be chosen smaller than $1/(4m)$, in the $L \to \infty$ limit this contribution vanishes. $\quad\square$

We actually have a stronger result. Indeed the same formula holds also for smooth functions living on one-dimensional paths. Consider now any simple path γ on $\mathcal{U}$ and denote by s a coordinate on γ. Denote by Ω_γ the composition of the map from s to (ν, η, τ) and Ω, and by $\gamma_{\mathbb{H}} \subset \mathbb{H}$ the image of γ by Ω_γ. Then, we define

$$\langle f, H_L \rangle_\gamma := \int_\gamma \mathrm{d}s\, f(s) H_L(s) \quad (5.82)$$

and we get

$$\langle f, H_L \rangle_\gamma = \int_{\gamma_{\mathbb{H}}} \mathrm{d}z\, J_\gamma(z) f(\Omega_\gamma^{-1}(z)) H_L(\Omega_\gamma^{-1}(z)), \quad (5.83)$$

with $J_\gamma(z)$ the Jacobian of the change of variables from z back to s by Ω_γ^{-1}.

Theorem 5.8. *For any $m \in \mathbb{N}$, consider any smooth functions $f_1, \ldots, f_m$ of compact support on γ. Then*

$$\lim_{L \to \infty} \mathbb{E}\left[\prod_{k=1}^m \langle f_k, H_L \rangle_\gamma \right] = \int_{\gamma_{\mathbb{H}}^m} \prod_{k=1}^m |\mathrm{d}z_k| f_k^\gamma(z_k) \mathbb{E}\left[\mathrm{GFF}(z_1) \cdots \mathrm{GFF}(z_m) \right], \quad (5.84)$$

where $f_i^\gamma(z) := J_\gamma(z) f(\Omega_\gamma^{-1}(z))$.

Proof of Theorem 5.8. The strategy is the same as in the proof of Theorem 5.6. The main difference is that the contribution at small distances will be of order δL^ε. However, this is fine, since we can choose $\varepsilon < 1/(8m)$. $\quad\square$

6. Asymptotics Analysis

In this section we do the asymptotic analysis of the functions q_n's at the (upper) edge and at the bulk. These are used to obtain the Gaussian fluctuations in Sect. 4. Then, we do the asymptotic analysis of the extended kernel in the bulk and provide some bounds at the (upper) edge, needed to prove the Gaussian Free Field correlations in Sect. 5.

6.1. Asymptotics at the edge. First we will determine the upper edge asymptotic of $I_{n,t}$ defined in (4.15), for which we apply exactly the same strategy as in previous papers (Lemma 6.1 and 6.2 are almost identical to the computations of Propositions 15 and 17 in [16]). We first explain the strategy and then give the relevant details.

Lemma 6.1 (Upper edge). *Let $n = ct$ and $x = (1 + \sqrt{c})^2 t + st^{1/3}$, for any $c > 0$. Then,*

$$\lim_{t \to \infty} t^{1/3} I_{n,t}(x) \frac{(-\sqrt{c})^n}{e^{-\sqrt{c}t}(1 + \sqrt{c})^x} = \tilde{\kappa}_2 \mathrm{Ai}(\kappa_2 s), \tag{6.1}$$

uniformly for s in bounded sets, with $\kappa_2 = c^{1/6}(1+\sqrt{c})^{-2/3}$, and $\tilde{\kappa}_2 = (1+\sqrt{c})^{1/3}c^{-1/3}$. Here $\mathrm{Ai}(\cdot)$ is the classical Airy function.

Proof of Lemma 6.1. The strategy is the following. With the replacements $n = ct$ and $x = (1 + \sqrt{c})^2 t + st^{1/3}$ in (4.15), we have an integral of the form

$$\frac{1}{2\pi i} \oint_{\Gamma_0} dz e^{tf_0(z) + t^{2/3} f_1(z) + t^{1/3} f_2(z) + f_3(z)} \tag{6.2}$$

for some functions $f_k(z)$, $k = 0, 1, 2, 3$. The s-dependence is in $f_2(z)$.

Step 1: Find a steep descent path[1] for the function $f_0(z)$, passing through the double critical point z_c given by the condition $f_0'(z_c) = f_0''(z_c) = 0$. In particular, the steep descent path will be chosen so that close to the critical point the descent is the steepest. Then, uniformly for s in a bounded set, the contribution coming from the integration path away from a δ-neighborhood of z_c is of order $\mathcal{O}(e^{-\mu t})$ with $\mu \sim \delta^3$.

Step 2: Consider the contribution of the integration only on $|z - z_c| \leq \delta$, with δ which can still be chosen small enough, but t-independent. In a neighborhood of the critical point, we can use the Taylor expansion of the functions $f_0, \ldots, f_3$ and get

$$\exp(t f_0(z_c) + t^{2/3} f_1(z_c) + t^{1/3} f_2(z_c) + f_3(z_c))$$
$$\times \frac{1}{2\pi i} \int_{\Gamma_0 \cap |z - z_c| \leq \delta} dz \exp(t\kappa_0(z - z_c)^3/3 + t^{2/3}\kappa_1(z - z_c)^2 + t^{1/3}\kappa_2(z - z_c))$$
$$\times \exp(\mathcal{O}(t(z - z_c)^4, t^{2/3}(z - z_c)^3, t^{1/3}(z - z_c)^2, (z - z_c))). \tag{6.3}$$

Remark that we do not have a term $t^{2/3}(z - z_c)$ in the exponential. If such a term remains, then the edge scaling in x is not the right one.

Step 3: Estimate the error terms. We do the change of variable $t^{1/3}(z - z_c) = w$ and choose δ small enough, so that the error terms are much smaller than the main ones.

---

[1] For an integral $I = \int_\gamma dz e^{tf(z)}$, we say that γ is a steep descent path if (1) $\mathrm{Re}(f(z))$ reaches the maximum at some $z_0 \in \gamma$: $\mathrm{Re}(f(z)) < \mathrm{Re}(f(z_0))$ for $z \in \gamma \setminus \{z_0\}$, and (2) $\mathrm{Re}(f(z))$ is monotone along γ except at its maximum point z_0 and, if γ is closed, at a point z_1 where the minimum of $\mathrm{Re}(f)$ is reached.

Subsequently, taking t large enough, the cubic term dominates all the others. Applying $|e^y - 1| \le |y|e^{|y|}$ with y standing for the error term $\mathcal{O}(\cdots)$, and changing the variable $t^{1/3}(z - z_c) = w$, one sees that the difference between the integral with and without the error term is of order $\mathcal{O}(t^{-1/3})$.

Step 4: For the integral without errors, we also do the change of variable $t^{1/3}(z - z_c) = w$ and then we extend the integration paths to infinity. This accounts for an error of order $\mathcal{O}(e^{-\mu t})$. The final formula is then

$$t^{-1/3} \exp(t f_0(z_c) + t^{2/3} f_1(z_c) + t^{1/3} f_2(z_c) + f_3(z_c))$$
$$\times \left(\frac{\pm 1}{2\pi i} \int dw \, e^{\kappa_0 w^3/3 + \kappa_1 w^2 + \kappa_2 w} + \mathcal{O}(t^{-1/3}, e^{-\mu t}) \right), \tag{6.4}$$

where the integral goes from $e^{-\pi i/3}\infty$ to $e^{\pi i/3}\infty$ if $\kappa_0 > 0$ and, in case $\kappa_0 < 0$ it goes from $e^{-2\pi i/3}\infty$ to $e^{2\pi i/3}\infty$. The sign ± 1 depends on the position of the critical point: we have $+1$ if $z_c > 0$ and -1 if $z_c < 0$. Finally, the integral can be rewritten in terms of Airy functions using the following identity:

$$\frac{1}{2\pi i} \int_{e^{-\pi i/3}\infty}^{e^{\pi i/3}\infty} e^{az^3/3 + bz^2 + cz} dz = a^{-1/3} \text{Ai}(b^2/a^{4/3} - c/a^{1/3}) e^{2b^3/3a^2 - bc/a}. \tag{6.5}$$

Specialization to our case. In our specific situation, the critical point is $z_c = -\sqrt{c}$, and the functions $f_0, \ldots, f_3$ are

$$\begin{aligned}
f_0(z) &= z + (1 + \sqrt{c})^2 \ln(1 - z) - c \ln(z), \\
f_1(z) &= 0, \\
f_2(z) &= s \ln(1 - z), \\
f_3(z) &= -\ln(z).
\end{aligned} \tag{6.6}$$

The steep descent path used in the analysis is made of pieces of the two following paths, γ_ρ and γ_{loc} (see Fig. 9), given by

$$\gamma_\rho = \{-\rho e^{i\phi}, \phi \in (-\pi, \pi]\}, \quad \gamma_{\text{loc}} = \{-\sqrt{c} + e^{-\pi i/3 \, \text{sgn}(x)}|x|, x \in [0, \sqrt{c}/2]\}. \tag{6.7}$$

For $\rho \in (0, \sqrt{c}]$, γ_ρ is steep descent path for f_0. In fact, we get

$$\frac{d\text{Re}(f_0(z = \rho e^{i\phi}))}{d\phi} = -\frac{\rho \sin \phi}{|1 - z|^2}(c - \rho^2 + 2\sqrt{c} - 2\rho \cos \phi). \tag{6.8}$$

The last term is minimal for $\phi = 0$, where it equals

$$(\sqrt{c} - \rho)(\rho + \sqrt{c} + 2) \ge 0, \tag{6.9}$$

for $\rho \in (0, \sqrt{c}]$. γ_ρ is a steep descent path for f_0 because the value zero is attained only for $\rho = \sqrt{c}$ and, in that case, only at one point, $\phi = 0$. However, close to the critical point it is not optimal, because the steepest descent path leaves z_c at angle $\pm\pi/3$ (there are rays where $\text{Im}(z - z_c)^3 = 0$). By symmetry, we need to consider only $x \ge 0$,

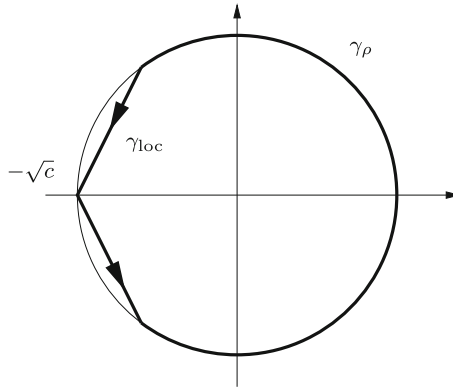$$\frac{d\text{Re}(f_0(z = -\sqrt{c} + e^{-\pi i/3}x))}{dx} = -\frac{x^2 Q(x)}{|z|^2|1 - z|^2}, \tag{6.10}$$

Fig. 9. The steep descent path used in the asymptotic analysis is the bold one

with $Q(x) = \sqrt{c}(1 + \sqrt{c}) - x(1 + x)/2 - \sqrt{c}x$. $Q(0) > 0$, and the computation of the (at most) two zeros of $Q(x)$ shows that none are in the interval $[0, \sqrt{c}/2]$. Thus γ_{loc} is also a steep descent path for f_0. Since this is the steepest descent path for f_0 around the critical point, we choose as path Γ_0 in $I_{n,t}(x)$ the one formed by γ_{loc} close to the critical point, until it intersects $\gamma_{\rho=\sqrt{3c/4}}$, and then we follow $\gamma_{\sqrt{3c/4}}$.

The Taylor expansions near the critical point $z_c = -\sqrt{c}$ of the functions f_k are given by

$$f_0(z) = f_0(-\sqrt{c}) + \tfrac{1}{3}\kappa_0(z + \sqrt{c})^3 + \mathcal{O}((z + \sqrt{c})^4), \quad \kappa_0 = \frac{1}{\sqrt{c}(1 + \sqrt{c})},$$

$$f_2(z) = f_2(-\sqrt{c}) + \kappa_2(z + \sqrt{c}) + \mathcal{O}((z + \sqrt{c})^2), \quad \kappa_2 = -\frac{s}{1 + \sqrt{c}}, \quad (6.11)$$

$$f_3(z) = -\ln(-\sqrt{c}) + \mathcal{O}(z + \sqrt{c}).$$

Thus in our case we have

$$a = \kappa_0 = 1/(\sqrt{c}(1 + \sqrt{c})), \quad b = 0, \quad c = -s/(1 + \sqrt{c}), \quad e^{f_3(z_c)} = -1/\sqrt{c}. \tag{6.12}$$

This, together with the relation

$$\exp(tf_0(z_c) + t^{2/3}f_1(z_c) + t^{1/3}f_2(z_c)) = \frac{(1 + \sqrt{c})^x e^{-\sqrt{c}t}}{(-\sqrt{c})^n} \tag{6.13}$$

proves (6.1). □

Lemma 6.2. *Fix $\ell > 0$ and consider the scaling of Lemma 6.1. Then*

$$\left| t^{1/3} I_{n,t}(x) \frac{(-\sqrt{c})^n}{e^{-\sqrt{c}t}(1 + \sqrt{c})^x} \right| \le \text{const } e^{-s}, \tag{6.14}$$

uniformly for $s \ge -\ell$, where const *is a constant independent of t.*

Proof of Lemma 6.2. For any finite $\tilde{\ell}$, the bound for $s \in [-\ell, \tilde{\ell}]$ is a consequence of Lemma 6.1. The value of $\tilde{\ell}$ can be chosen large but independent of t. The strategy for $s \geq \tilde{\ell}$ is just a small modification of the computation made in Lemma 6.1, and was already used for example in Proposition 17 of [16] and in Proposition 5.3 of [13]. Let us explain it.

In Lemma 6.1 we have seen that γ_ρ is the steep descent path for f_0, for any $\rho \in (0, \sqrt{c}]$. Set $\tilde{s} = (s + \ell + \tilde{\ell})t^{-2/3} \geq \tilde{\ell}t^{-2/3} > 0$ and $\tilde{f}_0(z) = f_0(z) + \tilde{s}\ln(1 - z)$. For any $\tilde{s} \geq 0$, γ_ρ is also a steep descent path for $\tilde{f}_0(z)$. However, for $\tilde{s} > 0$ there are two real critical points for $\tilde{f}_0$, say at $z_c^\pm$ with $|z_c^+| > |z_c^-|$. For $\tilde{s}$ small, we have at lowest order in $\tilde{s}$, $z_c^\pm \simeq -\sqrt{c} \mp \sqrt{\tilde{s}}\sqrt{\kappa_2/\kappa_0}$, with κ_0 and κ_2 given in (6.11). To get the best bound we should pass through z_c^-. However, this precision is not needed to get the exponential bound and we can choose the integration path passing through

$$- \rho = \begin{cases} -\sqrt{c} + (\tilde{s}\kappa_2/\kappa_0)^{1/2}, & \text{if } 0 \leq \tilde{s} \leq \varepsilon, \\ -\sqrt{c} + (\varepsilon\kappa_2/\kappa_0)^{1/2}, & \text{if } \tilde{s} \geq \varepsilon. \end{cases} \tag{6.15}$$

With this choice, for ε small enough, we have $-\sqrt{c} < -\rho < z_c^-$ and in particular, for small $\tilde{s}$, $-\rho$ is very close to the position of the critical point. As in Lemma 6.1, we use the fact that γ_ρ is steep descent to control the contribution away from $|z + \rho| \leq \delta$, while the contribution close to $z = -\rho$ is controlled by the Taylor expansion of $\text{Re}(\tilde{f}_0(z))$, leading to a Gaussian bound. By choosing $\tilde{\ell}$ large enough, all the terms coming from $\text{Re}(f_k(z))$, $k = 1, 2, 3$ are dominated by the leading term of $\text{Re}(f_0(z))$. The final result is that

$$\left| t^{1/3}I_{n,t}(x)\frac{(-\sqrt{c})^n}{e^{-\sqrt{c}t}(1 + \sqrt{c})^x} \right| \leq \text{const } Q(\rho), \tag{6.16}$$

with

$$Q(\rho) = \exp(\text{Re}(t(f_0(-\rho) - f_0(z_c)) + t^{2/3}(f_1(-\rho) - f_1(z_c)) + t^{1/3}(f_2(-\rho) - f_2(z_c))). \tag{6.17}$$

$Q(\rho)$ is decreasing for $-\rho$ from z_c to z_c^- and $-\rho - z_c$ is at most of order $\sqrt{\varepsilon}$. Thus, we can easily bound $Q(\rho)$ by using Taylor expansions. Simple computations lead to the desired exponential bound. □

To get the needed bound on q_n (see (4.10)–(4.14)) around the edge, we use the bound of Lemma 6.2 on $I_{n,t}$ which has still to be multiplied by $B_{n,t}(x)$ given in (4.14).

Lemma 6.3. *Let $n = ct$, $x = (1 + \sqrt{c})^2t + st^{1/3}$ and fix $\ell > 0$. Then*

$$|q_n(x, t)| \leq \text{const } t^{-1/3}e^{-s}, \tag{6.18}$$

for any $s \geq -\ell$, and const *is a t-independent constant.*

Proof of Lemma 6.3. This result follows from Lemma 6.2 if

$$\tilde{B}_{n,t}(x) = \left| B_{n,t}(x)\frac{e^{-\sqrt{c}t}(1 + \sqrt{c})^x}{(-\sqrt{c})^n} \right| \leq \text{const}. \tag{6.19}$$

For the factorials we use the Stirling formula, namely

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{f_n}, \qquad \frac{1}{1 + 12n} \le f_n \le \frac{1}{12n}. \tag{6.20}$$

We obtain

$$\tilde{B}_{ct,t}((1 + \sqrt{c})^2 t) = \left(\frac{c}{(1 + \sqrt{c})^2}\right)^{1/4} (1 + \mathcal{O}(1/t)). \tag{6.21}$$

For $x = \xi t$, $\xi \in [(1 + \sqrt{c})^2, \infty)$, we compute

$$\frac{\tilde{B}_{ct,t}(\xi t)}{\tilde{B}_{ct,t}((1 + \sqrt{c})^2 t)} = \left(\frac{(1 + \sqrt{c})^2}{\xi}\right)^{1/4} (1 + \mathcal{O}(1/t)) e^{th(\xi)}, \tag{6.22}$$

with

$$h(\xi) = \tfrac{1}{2}\xi(1 - \ln(\xi) + 2\ln(1 + \sqrt{c})) - \tfrac{1}{2}(1 + \sqrt{c})^2. \tag{6.23}$$

Since $h'(\xi) = 0$ at $\xi = (1 + \sqrt{c})^2$ and $h''(\xi) = -1/(2\xi) < 0$, we have $e^{th(x)} \le 1$. $\quad \square$

6.2. Asymptotics in the bulk. In this section we derive a precise expansion of $I_{n,t}(x)$ for $x = \lambda t$, with $\lambda \in ((1 - \sqrt{c})^2, (1 + \sqrt{c})^2)$. For any fixed $c > 0$, set $n = ct$ and $x = \lambda t$. Then

$$I_{n,t}(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{w} e^{tg(w)}, \quad g(w) = G(w|\lambda, c, 1), \tag{6.24}$$

see (3.25) for the definition of G. Recall a few results from Sect. 3. For $\lambda \in ((1 - \sqrt{c})^2, (1+\sqrt{c})^2)$, g has two complex conjugate critical points, w_c and $\bar{w}_c$, with $w_c = \Omega(\lambda, c, 1)$. In particular, $|w_c| = \sqrt{c}$, $|1 - w_c| = \sqrt{\lambda}$, and $|g''(w_c)| = \frac{1}{\sqrt{\lambda c}}\sqrt{4c - (1 + c - \lambda)^2}$. When $(\eta, \nu, \tau) = (c, \lambda, 1)$, we denote by π_c the angle π_η and by π_λ the angle π_ν. Then

$$\operatorname{Re}(g(w_c)) = \frac{1 + c - \lambda}{2} - \frac{c}{2}\ln(c) + \frac{\lambda}{2}\ln(\lambda),$$
$$\operatorname{Im}(g(w_c)) = \operatorname{Im}(w_c) - \lambda\pi_c - c\pi_\lambda, \tag{6.25}$$
$$\arg(g(w_c)) = -\frac{\pi}{2} + \pi_c - \pi_\lambda.$$

Lemma 6.4. *Set $\alpha = \alpha(c, \lambda) = \operatorname{Im}(g(w_c))$ and $\beta = \beta(c, \lambda) = -\tfrac{1}{2}(\pi_c + \pi_\lambda + \pi/2)$. Then, as $t \to \infty$,*

$$I_{ct,t}(\lambda t) = \frac{e^{t\operatorname{Re}(g(w_c))}}{\sqrt{|g''(w_c)|t}} \left[\sqrt{\frac{2}{\pi|w_c|^2}} \cos(t\alpha + \beta) + \mathcal{O}(t^{-1/2})\right]. \tag{6.26}$$

For any $\varepsilon_0 > 0$, the errors are uniform for $\lambda \in [(1 - \sqrt{c})^2 + \varepsilon_0, (1 + \sqrt{c})^2 - \varepsilon_0]$.

Remark 6.5. In fact, we prove the bound of (6.26) with error term

$$\mathcal{O}(t^{-1/2}) + \mathcal{O}\left(\sqrt{|g''(w_c)|t}e^{-\text{const}\,|g''(w_c)|\delta^2 t}\right) \tag{6.27}$$

for some $0 < \delta \ll |g''(w_c)|$. In Lemma 6.7 we will have to be careful with the second term of the bound, since $g''(w_c)$ goes to zero at the edge.

Proof of Lemma 6.4. The critical points of g, the points such that $g'(w) = 0$, are w_c and its complex conjugate $\bar{w}_c$. Close to w_c the Taylor expansion of g has a first relevant term which is quadratic,

$$g(w) = g(w_c) + \tfrac{1}{2}g''(w_c)(w - w_c)^2 + \mathcal{O}((w - w_c)^3). \tag{6.28}$$

Now we construct the steep descent path used in the asymptotics. By symmetry we consider only $\text{Im}(w) \geq 0$, the path for $\text{Im}(w) \leq 0$ will be the complex conjugate image of the first one. Let $\gamma_\rho = \{w = \rho e^{i\phi}, \phi \in [0, \pi]\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}\phi}(\text{Re}(g(w = \rho e^{i\phi}))) = \rho \sin(\phi)\left[\frac{\lambda}{|1 - w|^2} - 1\right]. \tag{6.29}$$

This is positive if $|1 - w| < \sqrt{\lambda}$, and negative otherwise.

Locally, consider the path $\gamma_{\text{loc}} = \{w = w_c + \hat{\theta}x, x \in [-\delta, \delta]\}$. Then

$$g(w) = g(w_c) + \tfrac{1}{2}g''(w_c)\hat{\theta}^2 x^2 + \mathcal{O}(x^3), \tag{6.30}$$

where we choose

$$\hat{\theta} = \exp\left(\frac{i\pi}{2} - \frac{i}{2}\arg(g''(w_c))\right) = \exp\left(\frac{3\pi i}{4} + \frac{i(\pi_\lambda - \pi_c)}{2}\right). \tag{6.31}$$

For $-\delta < x < 0$, the path γ_{loc} is closer to 1 than $\sqrt{\lambda}$, while for $0 < x < \delta$ the path γ_{loc} is farther from 1 than $\sqrt{\lambda}$. This is the case since our γ_{loc} has an angle between $\pi/4$ and $3\pi/4$ to the tangent to the circle $|1 - w| = \sqrt{\lambda}$.

So, the steep descent path used is the following: we extend γ_{loc} by adding two circular arcs of type γ_ρ, for adequate ρ, which connect to the real axis; finally we add the complex conjugate image, see Fig. 10 too.

In this way, we have a steep descent path. Thus,

$$I_{n,t}(x) = e^{t\text{Re}(g(w_c))}\mathcal{O}(e^{-\mu t}) + 2\text{Re}\left(\frac{1}{2\pi i}\int_{\gamma_{\text{loc}}}\frac{\mathrm{d}w}{w}e^{tg(w)}\right) \tag{6.32}$$

with $\mu \sim |g''(w_c)|\delta^2$, as soon as $|g''(w_c)| > 0$, i.e., as soon as the second order term dominates all higher order terms in the Taylor expansion.

The second term of (6.32) is given by

$$\frac{1}{2\pi i}\int_{\gamma_{\text{loc}}}\frac{\mathrm{d}w}{w}e^{tg(w)} = \frac{1}{2\pi i}\int_{-\delta}^{\delta}\mathrm{d}x\,\frac{\hat{\theta}}{w_c}e^{tg(w_c)}e^{-\frac{1}{2}t|g''(w_c)|x^2}e^{\mathcal{O}(tx^3)}\mathcal{O}(x)$$

$$= \frac{1}{2\pi i}\frac{\hat{\theta}}{w_c}\int_{-\delta}^{\delta}\mathrm{d}x\,e^{tg(w_c)}e^{-\frac{1}{2}t|g''(w_c)|x^2} + E_1, \tag{6.33}$$

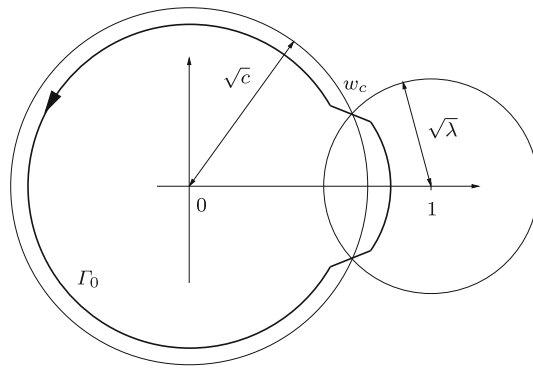**Fig. 10.** Illustration of the steep descent path

where

$$E_1 = \frac{1}{2\pi \mathrm{i}} \frac{\hat{\theta}}{w_c} \int_{-\delta}^{\delta} \mathrm{d}x e^{tg(w_c)} e^{-\frac{1}{2}t|g''(w_c)|x^2} e^{\mathcal{O}(tx^3)} \mathcal{O}(tx^3, x). \tag{6.34}$$

Here we used $|e^x - 1| \le |x|e^{|x|}$. Changing the variable $y = x\sqrt{t}$, we get that

$$|E_1| \le \text{const } e^{t\mathrm{Re}(g(w_c))} \frac{1}{t} \int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} \mathrm{d}y e^{-|g''(w_c)|y^2/2} \mathcal{O}(y, y^3/\sqrt{t}) e^{\mathcal{O}(y^3/\sqrt{t})}$$

$$\le \text{const } \frac{e^{t\mathrm{Re}(g(w_c))}}{t\sqrt{|g''(w_c)|}} \tag{6.35}$$

for δ small enough, i.e., for $0 < \delta \ll |g''(w_c)|$. In this small neighborhood, the quadratic term controls the higher order ones. The final step is to extend the integral on the rest of the r.h.s. of (6.33) from $\pm\delta$ to $\pm\infty$. This can be made up to an error $e^{t\mathrm{Re}(g(w_c))}\mathcal{O}(e^{-\mu t})$ as above.

Resuming we have (counting the contribution from both critical points)

$$I_{n,t} = e^{t\mathrm{Re}(g(w_c))} \left[\mathcal{O}(e^{-\mu t}) + \mathcal{O}(1/(t\sqrt{|g''(w_c)|})) \right]$$

$$+ 2\mathrm{Re}\left(\frac{1}{2\pi \mathrm{i}} \frac{\hat{\theta}}{w_c} \int_{\mathbb{R}} \mathrm{d}x e^{tg(w_c)} e^{-\frac{1}{2}t|g''(w_c)|x^2} \right). \tag{6.36}$$

The error terms are the ones indicated in (6.26), and the Gaussian integral for the last term gives

$$\frac{2e^{t\mathrm{Re}(g(w_c))}}{\sqrt{2\pi t |w_c|^2 |g''(w_c)|}} \mathrm{Re}\left(-\mathrm{i}\hat{\theta} \frac{|w_c|}{w_c} e^{it\mathrm{Im}(g(w_c))} \right). \tag{6.37}$$

We then set $\beta = \arg(-\mathrm{i}\hat{\theta}/w_c) = -\pi/4 - (\pi_c + \pi_\lambda)/2$, so that $-\mathrm{i}\hat{\theta}\frac{|w_c|}{w_c} = e^{\mathrm{i}\beta}$. For λ in a compact subset of $((1 - \sqrt{c})^2, (1 + \sqrt{c})^2)$, $|g''(w_c)|$ is uniformly bounded away from zero and infinity. Thus the lemma is proven. $\square$

The bound of Lemma 6.4 can be easily extended until a position away of order $\mathcal{O}(t^{2/3})$ from the upper edge.

Lemma 6.6. *Set* $\alpha = \text{Im}(g(w_c))$ *and* $\beta = -\frac{1}{2}(\pi_c + \pi_\lambda + \pi/2)$. *Then, for* $\lambda \in [(1 - \sqrt{c})^2 + \varepsilon_0, \, (1 + \sqrt{c})^2 - t^{-1/3}]$, *for any fixed* $\varepsilon_0 > 0$, *we have the uniform estimate*

$$I_{ct,t}(\lambda t) = \frac{e^{t\text{Re}(g(w_c))}}{\sqrt{|g''(w_c)|t}}\left[\sqrt{\frac{2}{\pi |w_c|^2}}\cos(t\alpha + \beta) + \mathcal{O}(t^{-1/2}) + \mathcal{O}\left(\sqrt{t}e^{-\text{const}\,t^{1/3}}\right)\right].$$
(6.38)

Proof of Lemma 6.6. The analysis of Lemma 6.4 can be made also for this case, with only minor differences. Indeed, for $(1+\sqrt{c})^2 - \lambda \sim t^{-1/3}$, we have $|g''(w_c)| \sim t^{-1/6}$ and this time we choose δ going to zero as $t \to \infty$, setting $\delta = t^{-1/4}$. With this choice, (6.32) and (6.35) are still valid because at the border of integration the quadratic term dominates the cubic one. Indeed, with $y = \delta\sqrt{t} = t^{1/4}$, $y^3/\sqrt{t} \sim t^{1/4} \ll t^{1/3} \sim |g''(w_c)|y^2$ holds. Also, the error term coming from steep descent in (6.32) will vanish as $t \to \infty$ but slower than before, with $\mu t \sim t^{1/3}$. $\square$

The results of Lemma 6.4 and Lemma 6.6 imply the following asymptotics for the functions q_n.

Lemma 6.7. *Set* $\alpha = \text{Im}(g(w_c))$ *and* $\beta = -\frac{1}{2}(\pi_c + \pi_\lambda + \pi/2)$ *and fix any* $\varepsilon_0 > 0$. *Then, uniformly in* $\lambda \in [(1 - \sqrt{c})^2 + \varepsilon_0, \, (1 + \sqrt{c})^2 - t^{-1/3}]$, *we have*

$$q_{ct}(\lambda t, t) = \frac{1}{\sqrt{\pi}} \frac{t^{-1/2}}{\sqrt[4]{c - \frac{(1+c-\lambda)^2}{4}}}\left[\cos(t\alpha + \beta) + \mathcal{O}(t^{-1/2})\right].$$
(6.39)

Proof of Lemma 6.7. We just have to compute the prefactor $B_{ct,t}(\lambda t)e^{t\text{Re}(g(w_c))}$. We have (6.25) and applying the Stirling formula for the factorials in $B_{ct,t}(\lambda t)$ we get that

$$B_{ct,t}(\lambda t)e^{t\text{Re}(g(w_c))} = (\lambda/c)^{1/4}(1 + \mathcal{O}(1/t)).$$
(6.40)

$\square$

Now we need to fill the gap between the bulk and the edge. In this region we do not need precise asymptotics, just a bound. Approaching the upper edge, $g''(w_c)$ goes to zero, but then everything can be controlled by the cubic term, because $|g'''(w_c)| \neq 0$ at the edges.

Lemma 6.8. *For* $\varepsilon_0 > 0$ *fixed but small enough, and* $\ell > 0$ *large enough, we have the bound*

$$|q_{ct}(\lambda t, t)| \leq \text{const} \frac{t^{-1/2}}{\sqrt[4]{c - \frac{(1+c-\lambda)^2}{4}}},$$
(6.41)

uniformly for $\lambda \in [(1 + \sqrt{c})^2 - \varepsilon_0, \, (1 + \sqrt{c})^2 - \ell t^{-2/3}]$.

Proof of Lemma 6.8. Consider the ε_0-region close to the upper edge with $\varepsilon_0 > 0$ small enough. We can compute explicitly the direction $\hat{\theta}$, see (6.31). It is a continuous function of λ and, as $\lambda \uparrow (1 + \sqrt{c})^2$, $\hat{\theta} \uparrow e^{i5\pi/4}$ (because $\pi_\lambda \uparrow \pi$ and $\pi_c \downarrow 0$). We need just a bound, so we choose $\hat{\theta} = e^{i5\pi/4}$ and set the local path as

$$\gamma_{\text{loc}} = \{w = w_c + e^{i5\pi/4}x, x \in [-\delta, \text{Im}(w_c)\sqrt{2}]\}. \tag{6.42}$$

The path γ_{loc} reaches at $x = \text{Im}(w_c)\sqrt{2}$ the imaginary axis and this is the reason for the upper edge of γ_{loc}. We have

$$g(w) = g(w_c) + \tfrac{1}{2}g''(w_c)(w - w_c)^2 + \tfrac{1}{6}g'''(w_c)(w - w_c)^3 + \mathcal{O}((w - w_c)^4). \tag{6.43}$$

In a δ-neighborhood of w_c, along the direction $\hat{\theta}$ chosen,

$$\text{Re}(\tfrac{1}{2}g''(w_c)(w - w_c)^2) = -\tfrac{1}{2}|g''(w_c)|x^2(1 + \mathcal{O}(\epsilon_0)) \tag{6.44}$$

and

$$\text{Re}(\tfrac{1}{6}g'''(w_c)(w - w_c)^3) = \tfrac{1}{6}|g'''(w_c)|x^3/\sqrt{2}(1 + \mathcal{O}(\sqrt{\epsilon_0})). \tag{6.45}$$

Therefore, for ε_0 small enough, the quadratic term helps the convergence. For $x \leq 0$, the cubic term helps the convergence, while for $x \in [0, \text{Im}(w_c)\sqrt{2}]$ we will need to control it by the quadratic term. Thus,

$$I_{n,t}(x) = e^{t\text{Re}(g(w_c))}\mathcal{O}(e^{-\mu t}) + 2\text{Re}\left(\frac{1}{2\pi i}\int_{\gamma_{\text{loc}}} \frac{dw}{w}e^{tg(w)}\right), \tag{6.46}$$

with $\mu \simeq |g'''(w_c)|\delta^3$, where $g'''(w_c) \to 2/\sqrt{c}(1 + \sqrt{c})$ as $\lambda \to (1 + \sqrt{c})^2$.

Consider then the contribution coming from the integral over γ_{loc}. We have

$$\left|\int_{\gamma_{\text{loc}}} \frac{dw}{w}e^{tg(w)}\right| \leq \frac{e^{t\text{Re}(g(w_c))}}{|w_c|}\int_{-\delta}^{\text{Im}(w_c)\sqrt{2}} dx \exp\left(-\tfrac{1}{2}t|g''(w_c)|x^2\right)$$
$$\times \exp\left(\tfrac{1}{6}t|g'''(w_c)|x^3/\sqrt{2} + \mathcal{O}(x^4t)\right)(1 + \mathcal{O}(x)), \tag{6.47}$$

the last $1/\sqrt{2}$ coming from $\text{Re}(e^{-i\pi/4}) = 1/\sqrt{2}$. A simple verification gives

$$-\tfrac{1}{2}t|g''(w_c)|x^2 + \tfrac{1}{6}t|g'''(w_c)|x^3/\sqrt{2} \leq -\tfrac{1}{4}t|g''(w_c)|x^2, \quad 0 \leq x \leq \text{Im}(w_c)\sqrt{2}. \tag{6.48}$$

So, for $x \in [0, \text{Im}(w_c)\sqrt{2}]$, the quadratic term is still dominating higher order terms, including the cubic one (the quartic term can be bounded by replacing $\tfrac{1}{4}$ by $\tfrac{1}{6}$ in the above estimate).

On the other hand, for $-\delta \leq x \leq 0$, we have that the cubic term is negative and dominates all higher order terms. More precisely, for δ small enough,

$$\left|\exp\left(\tfrac{1}{6}t|g'''(w_c)|x^3/\sqrt{2} + \mathcal{O}(x^4t)\right)\right| \leq \exp\left(\tfrac{1}{12}t|g'''(w_c)|x^3\right) \leq 1, \tag{6.49}$$

in the region $x \in [-\delta, 0]$.

Using (6.48) for positive x and (6.49) for negative x, we get

$$(6.47) \leq \text{const } e^{t\text{Re}(g(w_c))} \int_{-\delta}^{\text{Im}(w_c)\sqrt{2}} \mathrm{d}x \exp\left(-\tfrac{1}{6}t|g''(w_c)|x^2\right)$$

$$\leq \text{const } e^{t\text{Re}(g(w_c))} \frac{1}{\sqrt{|g''(w_c)|t}}. \tag{6.50}$$

Replacing the value of $|g''(w_c)|$ into this expression ends the proof. $\square$

6.3. Asymptotics of the kernel. In this section we obtain the precise asymptotics of the extended kernel in the bulk first and a bound to control the behavior starting from the upper edge. Here we use several notations introduced in Sect. 3.

As usual, it is convenient to conjugate the kernel before taking the limit. For the upper edge (cf. Lemma 6.2) set

$$W_{i,u} = \exp\left(-\sqrt{n_i t_i} + x_i \ln(1 + \sqrt{n_i/t_i}) - n_i \ln(-\sqrt{n_i/t_i}) - t_i\right), \tag{6.51}$$

and, in the bulk (see Lemma 6.4) set

$$W_{i,b} = \exp\left(\tfrac{1}{2}(t_i + n_i - x_i) - \tfrac{1}{2}n_i \ln(n_i/t_i) + \tfrac{1}{2}x_i \ln(x_i/t_i) - t_i\right). \tag{6.52}$$

Then, define the conjugation as

$$W_i = \begin{cases} W_{i,b}, & \text{for } (\sqrt{t_i} - \sqrt{n_i})^2 \leq x_i \leq (\sqrt{t_i} + \sqrt{n_i})^2, \\ W_{i,u}, & \text{for } x_i \geq (\sqrt{t_i} + \sqrt{n_i})^2. \end{cases} \tag{6.53}$$

Remark that W_i is continuous. Moreover, $|W_{i,u} - W_{i,b}| = \mathcal{O}(L^{-1/3})$ for $|x_i - (\sqrt{t_i} + \sqrt{n_i})^2| = \mathcal{O}(L^{-1/3})$. Therefore in such a neighborhood it is actually irrelevant which formula to use.

Proposition 6.9. *Let us consider two triples (x_1, n_1, t_1) and (x_2, n_2, t_2) parameterized by*

$$x_i = [v_i L], \quad n_i = [\eta_i L], \quad t_i = \tau_i L. \tag{6.54}$$

Assume that they are in the bulk of the system, namely, that $\varepsilon_0 > 0$ exists such that

$$(\sqrt{\tau_i} - \sqrt{\eta_i})^2 + \varepsilon_0 \leq v_i \leq (\sqrt{\tau_i} + \sqrt{\eta_i})^2 - L^{-1/3}. \tag{6.55}$$

Denote $z_c = \Omega(v_1, \eta_1, \tau_1)$, $w_c = \Omega(v_2, \eta_2, \tau_2)$, and assume that these points are not too close: $|z_c - w_c| \geq L^{-1/16}$. Then, the asymptotic expansion

$$(W_1/W_2)K(x_1, n_1, t_1; x_2, n_2, t_2) = \frac{1}{2\pi L \sqrt{|G''(w_c)| |G''(z_c)||1 - z_c|}}$$

$$\times \left[\frac{1}{w_c - z_c} \frac{e^{iL\text{Im}(G(w_c))+i\beta_2}}{e^{iL\text{Im}(G(z_c))+i\beta_1}} + \frac{1}{w_c - \bar{z}_c} \frac{e^{iL\text{Im}(G(w_c))+i\beta_2}}{e^{-iL\text{Im}(G(z_c))-i\beta_1}} \right.$$

$$\left. + \frac{1}{\bar{w}_c - z_c} \frac{e^{-iL\text{Im}(G(w_c))-i\beta_2}}{e^{iL\text{Im}(G(z_c))+i\beta_1}} + \frac{1}{\bar{w}_c - \bar{z}_c} \frac{e^{-iL\text{Im}(G(w_c))-i\beta_2}}{e^{-iL\text{Im}(G(z_c))-i\beta_1}} + \mathcal{O}(L^{-1/8}) \right] \tag{6.56}$$

holds, with the error uniform in L for $L \geq L_0 \gg 1$. The phases β_1 and β_2 are given by

$$\beta_1 = -\frac{5\pi}{4} - \frac{\pi v_1}{2} - \frac{\pi \eta_1}{2}, \quad \beta_2 = \frac{3\pi}{4} + \frac{\pi v_2}{2} - \frac{\pi \eta_2}{2}. \tag{6.57}$$
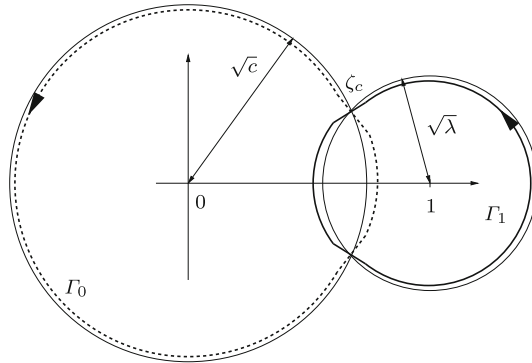
Fig. 11. Illustration of the steep descent paths

Proof of Proposition 6.9. The analysis relies on the double integral representation (4.2) of the kernel. The analysis for the cases $(n_1, t_1) \not\prec (n_2, t_2)$ and $(n_1, t_1) \prec (n_2, t_2)$ are very similar. Let us explain the first case, corresponding to $\eta_1 > \eta_2$, or $\tau_1 < \tau_2$, or $(\eta_1, \tau_1) = (\eta_2, \tau_2)$. The asymptotics employs several ingredients already used in Lemma 6.4 and Lemma 6.6. Thus, we introduce the notations

$$c_i = \eta_i/\tau_i \Rightarrow n_i = [c_i t_i], \quad \lambda_i = \nu_i/\tau_i \Rightarrow x_i = [\lambda_i t_i]. \tag{6.58}$$

The conjugation factor $e^{t_1 - t_2}$ in the kernel representation (4.2) will not appear in the following computations, since it appears automatically in the factors W_1/W_2. Thus, we have to analyze

$$\frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dz e^{t_2 g_2(w) - t_1 g_1(z)} \frac{1}{(1 - z)(w - z)} \tag{6.59}$$

with $g_i(w) = w + \lambda_i \ln(1 - w) - c_i \ln(w) \equiv G(w|\lambda_i, c_i, 1)$, $i = 1, 2$.

The critical points of $g_2(w)$ and $g_1(z)$ are given by

$$w_c = \Omega(\lambda_2, c_2, 1) = \Omega(\nu_2, \eta_2, \tau_2), \quad z_c = \Omega(\lambda_1, c_1, 1) = \Omega(\nu_1, \eta_1, \tau_1). \tag{6.60}$$

The integrals over w are, up to the factor $w/(z - w)$, as in Lemma 6.4. Therefore, the steep descent path Γ_0 is chosen as in Lemma 6.4 and the steep descent path Γ_1 is chosen in a similar way. We illustrate these paths if the critical point is ζ_c, see Fig. 11. In particular, $|w_c| = \sqrt{\eta_2/\tau_2}$ and $|z_c| = \sqrt{\eta_1/\tau_2}$. In our case, we have $|w_c| \leq |z_c|$ and $|w_c - z_c| \geq L^{-1/16}$. The steep descent paths described above actually intersect. Therefore, we have to correct (6.59) by subtracting the residue at $z = w$, as indicated in Fig. 12. We call the "main term" the contribution of the integral with Γ_0 and Γ_1 crossing, while we call the "residual term" the contribution of the residue.

Notice that the integral with the paths Γ_0 and Γ_1 crossing is integrable in the usual sense, because the divergence term $1/(w - z)$ is integrable. The contribution of the main term is the following.

Both integrals can be divided as the part in $\mathbb{H}$ and its complex conjugate. Therefore, in the final expression we get the sum of four terms. Now, we restrict our attention to the integral over the path Γ_0 and Γ_1 on $\mathbb{H}$. The analysis of the integral over Γ_0 is the same as in Lemma 6.4 except for the missing $1/w_c$ factor and that instead of $2\text{Re}(\cdots)$ we just have $(\cdots)$ in (6.36). The integral over Γ_1 is similar.
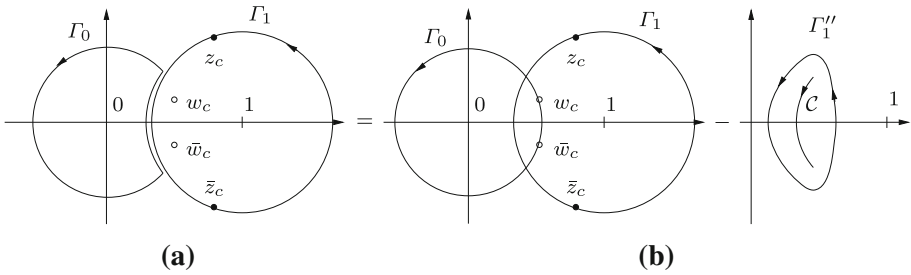
Fig. 12. The subdivision of the integration (6.59). We have $|z_c| \geq |w_c|$ and when $|z_c| = |w_c|$, they are not at the same position

This time we choose the cutoff for the evaluation of the term with the steep descent path equal to $\delta = L^{-1/4}$. There are two reasons. The first one is that we want to get the expansion valid also for ν_i up to $L^{-1/3}$ away from the upper edge, compare with Lemma 6.6. The second reason is that we have the extra factor $1/(w - z)$. The contributions of the steep descent path do not create problems, since the factor is integrable in the usual sense (just need a bound). However, with our choice of δ, in the contribution of the δ-neighborhoods of z_c and w_c we have

$$1/(w - z) = 1/(w_c - z_c) + \mathcal{O}(\delta/|w_c - z_c|^2) = 1/(w_c - z_c) + \mathcal{O}(L^{-1/8}), \quad (6.61)$$

with $\delta = L^{-1/4}$ and $|w_c - z_c| \geq L^{-1/16}$. In the end, the contribution of the main term is given by

$$\frac{1}{w_c - z_c} \frac{e^{t_2 \operatorname{Re}(g_2(w_c))}}{\sqrt{2\pi t_2 |g_2''(w_c)|}} \left[e^{t_2 \operatorname{Im}(g_2(w_c))} \hat{\theta}_2(w_c) + \mathcal{O}\left(L^{-1/2}\right) \right]$$

$$\times \frac{e^{-t_1 \operatorname{Re}(g_1(z_c))}}{\sqrt{2\pi t_1 |g_1''(z_c)|}} \frac{e^{i\pi c_1}}{|1 - z_c|} \left[e^{-t_1 \operatorname{Im}(g_1(z_c))} \hat{\theta}_1(z_c) + \mathcal{O}\left(L^{-1/2}\right) \right]$$

$$+ \frac{e^{t_2 \operatorname{Re}(g_2(w_c))}}{\sqrt{2\pi t_2 |g_2''(w_c)|}} \frac{e^{-t_1 \operatorname{Re}(g_1(z_c))}}{\sqrt{2\pi t_1 |g_1''(z_c)|}} \frac{1}{|1 - z_c|} \mathcal{O}(L^{-1/8}). \quad (6.62)$$

The term $e^{i\pi c_1}$ is the phase of $1/(1 - z_c)$, while $\hat{\theta}_i$ are the directions of the steepest descent paths at the critical points. Explicitly,

$$\hat{\theta}_1(z_c) = \exp(i(\pi - \tfrac{1}{2} \arg(g''(z_c)))) = \exp(i3\pi/4 + i(\pi_{\lambda_1} - \pi_{c_1})/2),$$
$$\hat{\theta}_2(w_c) = \exp(i(\tfrac{\pi}{2} - \tfrac{1}{2} \arg(g''(w_c)))) = \exp(i5\pi/4 + i(\pi_{\lambda_2} - \pi_{c_2})/2). \quad (6.63)$$

Putting together the four terms (two times two critical points) we get the complete contribution of the main term as

$$\frac{e^{t_2 \operatorname{Re}(g_2(w_c)) - t_1 \operatorname{Re}(g_1(z_c))}}{2\pi \sqrt{t_1 t_2 |1 - z_c|^2 |g_2''(w_c)||g_1''(z_c)|}} \left[\mathcal{O}(L^{-1/8}) \right.$$

$$+ \frac{1}{w_c - z_c} \frac{e^{it_2 \operatorname{Im}(g_2(w_c)) + i\beta_2}}{e^{it_1 \operatorname{Im}(g_1(z_c)) + i\beta_1}} + \frac{1}{w_c - \bar{z}_c} \frac{e^{it_2 \operatorname{Im}(g_2(w_c)) + i\beta_2}}{e^{-it_1 \operatorname{Im}(g_1(z_c)) - i\beta_1}}$$

$$\left. + \frac{1}{\bar{w}_c - z_c} \frac{e^{-it_2 \operatorname{Im}(g_2(w_c)) - i\beta_2}}{e^{it_1 \operatorname{Im}(g_1(z_c)) + i\beta_1}} + \frac{1}{\bar{w}_c - \bar{z}_c} \frac{e^{-it_2 \operatorname{Im}(g_2(w_c)) - i\beta_2}}{e^{-it_1 \operatorname{Im}(g_1(z_c)) - i\beta_1}} \right], \quad (6.64)$$

with $\beta_1 = -\arg(\hat{\theta}_1(z_c)) - \pi_{c_1}$ and $\beta_2 = \arg(\hat{\theta}_2(w_c))$. Finally, we replace $g_i(w)t_i = G(w|v_i, \eta_i, \tau_i)L$, $\pi_{\lambda_i} = \pi_{v_i}$, $\pi_{c_i} = \pi_{v_i}$, and

$$e^{t_2 \mathrm{Re}(g_2(w_c)) - t_1 \mathrm{Re}(g_1(z_c))} e^{t_1 - t_2} = W_2/W_1 \qquad (6.65)$$

to get (6.56).

The final step is to estimate the contribution of the residual term (the last case of Fig. 12). It is given by

$$\frac{1}{2\pi i} \int_\zeta^{\bar{\zeta}} dz \frac{e^{(\tau_2 - \tau_1)Lz} e^{(\eta_1 - \eta_2)L \ln(z)}}{(1 - z)^{(v_1 - v_2)L+1}}, \qquad (6.66)$$

where ζ and $\bar{\zeta}$ are the two intersection points of the steep descent path Γ_0 and Γ_1. Since $\tau_2 - \tau_1 \geq 0$, $\eta_1 - \eta_2 \geq 0$, and $|1 - z| = \mathrm{const}$ along the piece of Γ_1 inside Γ_0, we have $\mathrm{Re}(z) \leq \mathrm{Re}(\zeta)$ and $\mathrm{Re}(\ln(z)) \leq \mathrm{Re}(\ln(\zeta))$. Therefore,

$$|(6.66)| \leq e^{t_2 \mathrm{Re}(g_2(\zeta)) - t_1 \mathrm{Re}(g_1(\zeta))} \leq e^{t_2 \mathrm{Re}(g_2(w_c)) - t_1 \mathrm{Re}(g_1(z_c))} \mathcal{O}(e^{-\mu_1 t_1} e^{-\mu_2 t_2}), \quad (6.67)$$

for some positive μ_1, μ_2 and at least one larger than $L^{-1/8}$. This follows from the fact that either one (or both) critical points are away of order $L^{-1/16}$ from ζ, and ζ lies on the steep descent paths of $g_2(w)$ and $-g_1(z)$, with local quadratic behavior.   □

While doing time integration we will also need the following corollary.

Corollary 6.10. *Consider the same setting of Proposition 6.9. Then*

(a) the formula for $K(x_1, n_1, t_1; x_2 + 1, n_2, t_2)$ is the same as (6.56) but with an extra factor $(1 - w_c)$, resp. $(1 - \bar{w}_c)$, to the terms with $e^{i\beta_2}$, resp. $e^{-i\beta_2}$.
(b) the formula for $K(x_1, n_1 - 1, t_1; x_2, n_2, t_2)$ is the same as (6.56) but with an extra factor z_c^{-1}, resp. $(\bar{z}_c)^{-1}$, to the terms with $e^{-i\beta_1}$, resp. $e^{i\beta_1}$.

Proof of Corollary 6.10. The proof is almost identical to the one of Proposition 6.9. The only difference is that in (6.59) we have for (a) an extra term $(1 - w)$ and for (b) an extra $1/z$.   □

At this point we have all the needed estimates in the bulk. However, since our system develops facets, we need to have control at the upper edge. We will just need some bounds and, since the integrals are the same as in Sect. 6.1, apart from the factor $1/(w_c - z_c)$, which we will assume bounded away from zero.

Proposition 6.11. *Consider the setting of Proposition 6.9, but with one or both of the v_i close to the upper edge,*

$$(\sqrt{\tau_i} + \sqrt{\eta_i})^2 - L^{-1/3} \leq v_i \leq (\sqrt{\tau_i} + \sqrt{\eta_i})^2 - \ell L^{-2/3}. \qquad (6.68)$$

Moreover, assume that $|z_c - w_c|$ is bounded away from zero uniformly in L. Then, there exists ℓ large enough, such that

$$(W_1/W_2)|K(x_1, n_1, t_1; x_2, n_2, t_2)| \leq \frac{\mathrm{const}}{L \prod_{i=1}^2 \sqrt[4]{\eta_i \tau_i - \frac{1}{4}(\tau_i + \eta_i - v_i)^2}} \qquad (6.69)$$

uniformly in L for $L \geq L_0 \gg 1$.

Proof of Proposition 6.11. The proof follows the same argument as Lemma 6.8 for the variables which are close to the edge. For the one which is away from the edges, it is a consequence of the analysis of Proposition 6.9. □

When one or both positions are at the edge, we need a different bound.

Proposition 6.12. *Consider the setting of Proposition 6.9, but now with v_2 at the edge or in the facet, i.e.,*

$$v_2 \geq (\sqrt{\tau_2} + \sqrt{\eta_2})^2 - \ell L^{-2/3} \tag{6.70}$$

for any fixed ℓ. Assume $|z_c + \sqrt{\eta_2/\tau_2}|$ is bounded away from zero uniformly in L. Then,

$$(W_1/W_2)|K(x_1, n_1, t_1; x_2, n_2, t_2)| \leq \frac{\text{const}}{\sqrt{L}\sqrt[4]{\eta_1\tau_1 - \frac{1}{4}(\tau_1 + \eta_1 - v_1)^2}}$$
$$\times \frac{1}{L^{1/3}} \exp\left(-\frac{x_2 - (\sqrt{\tau_2} + \sqrt{\eta_2})^2 L}{(\tau_2 L)^{1/3}}\right), \tag{6.71}$$

uniformly in L for $L \geq L_0 \gg 1$.

Proof of Proposition 6.12. The proof is obtained along the same lines as Lemmas 6.1 and 6.2. With respect to those cases, the integral has however an extra factor $1/(w - z)$. Since we need just a bound, it can simply be replaced by $1/(w_c - z_c)$ as follows. In Lemma 6.1 w_c is replaced by $-\sqrt{\eta_2/\tau_2}$, while in Lemma 6.2, we need to replace w_c by ρ as given in (6.15). Notice that in the last case we can take $|w_c + \sqrt{\eta_2/\tau_2}|$ as small as desired. The assumption $|z_c + \sqrt{\eta_2/\tau_2}| > 0$ uniformly in L ensures then that $1/(w_c - z_c)$ remains bounded as $L \to \infty$. □

The last case to consider is when both v_1 and v_2 are at the upper edge.

Proposition 6.13. *Consider the setting of Proposition 6.9, but now with v_1 and v_2 at the edge or in the facet, i.e., with*

$$v_i \geq (\sqrt{\tau_i} + \sqrt{\eta_i})^2 - \ell L^{-2/3}, \quad i = 1, 2, \tag{6.72}$$

for any fixed ℓ. Assume that $|\sqrt{\eta_2/\tau_2} - \sqrt{\eta_1/\tau_1}|$ is bounded away from zero uniformly in L. Then,

$$(W_1/W_2)|K(x_1, n_1, t_1; x_2, n_2, t_2)| \leq \text{const}$$
$$\times \frac{1}{L^{2/3}} \exp\left(-\frac{x_2 - (\sqrt{\tau_2} + \sqrt{\eta_2})^2 L}{(\tau_2 L)^{1/3}}\right)\exp\left(-\frac{x_1 - (\sqrt{\tau_1} + \sqrt{\eta_1})^2 L}{(\tau_1 L)^{1/3}}\right), \tag{6.73}$$

uniformly in L for $L \geq L_0 \gg 1$.

Proof of Proposition 6.13. The proof is like Proposition 6.12. We will have $|w_c + \sqrt{\eta_2/\tau_2}|$ and $|z_c + \sqrt{\eta_1/\tau_1}|$ as small as desired. The assumption $|\sqrt{\eta_2/\tau_2} - \sqrt{\eta_1/\tau_1}| > 0$ uniformly in L allows us to easily bound uniformly in L the term $1/(w_c - z_c)$. □

A. Determinantal Structure of the Correlation Functions

Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_N$ be finite sets and $c(1), \ldots, c(N)$ be arbitrary nonnegative integers. Consider the set

$$\mathfrak{X} = (\mathfrak{X}_1 \sqcup \cdots \sqcup \mathfrak{X}_1) \sqcup \cdots \sqcup (\mathfrak{X}_N \sqcup \cdots \sqcup \mathfrak{X}_N) \qquad (A.1)$$

with $c(n) + 1$ copies of each $\mathfrak{X}_n$. We want to consider a particular form of weight $W(X)$ for any subset $X \subset \mathfrak{X}$, which turns out to have determinantal correlations.

To define the weight we need a bit of notations. Let

$$\begin{aligned}
\phi_n(\,\cdot\,, \,\cdot\,) : \mathfrak{X}_{n-1} \times \mathfrak{X}_n \to \mathbb{C}, & \quad n = 2, \ldots, N, \\
\phi_n(\text{virt}, \,\cdot\,) : \mathfrak{X}_n \to \mathbb{C}, & \quad n = 1, \ldots, N, \\
\Psi_j^N(\,\cdot\,) : \mathfrak{X}_N \to \mathbb{C}, & \quad j = 0, \ldots, N - 1,
\end{aligned} \qquad (A.2)$$

be arbitrary functions on the corresponding sets. Here the symbol virt stands for a "virtual" variable, which is convenient to introduce for notational purposes. In applications virt can sometimes be replaced by $+\infty$ or $-\infty$. The ϕ_n represents the transitions from $\mathfrak{X}_{n-1}$ to $\mathfrak{X}_n$.

Also, let

$$t_0^N \leq \cdots \leq t_{c(N)}^N = t_0^{N-1} \leq \cdots \leq t_{c(N-1)}^{N-1} = t_0^{N-2} \leq \cdots \leq t_{c(2)}^2 = t_0^1 \leq \cdots \leq t_{c(1)}^1 \qquad (A.3)$$

be real numbers. In applications, these numbers refer to time moments. Finally, let

$$\mathcal{T}_{t_a^n, t_{a-1}^n}(\,\cdot\,, \,\cdot\,) : \mathfrak{X}_n \times \mathfrak{X}_n \to \mathbb{C}, \qquad n = 1, \ldots, N, \quad a = 1, \ldots, c(n), \qquad (A.4)$$

be arbitrary functions. The $\mathcal{T}_{t_a^n, t_{a-1}^n}$ represents the transition between two copies of $\mathfrak{X}_n$ associated to "times" t_{a-1}^n and t_a^n.

Then, to any subset $X \subset \mathfrak{X}$ assign its weight $W(X)$ as follows. $W(X)$ is zero unless X has exactly n points in each copy of $\mathfrak{X}_n$, $n = 1, \ldots, N$. In the latter case, denote the points of X in the mth copy of $\mathfrak{X}_n$ by $x_k^n(t_m^n)$, $k = 1, \ldots, n$, $m = 0, \ldots, c(n)$. Thus,

$$X = \{x_k^n(t_m^n) \mid k = 1, \ldots, n; \ m = 0, \ldots, c(n); \ n = 1, \ldots, N\}. \qquad (A.5)$$

Set

$$\begin{aligned}
W(X) = \prod_{n=1}^N \Bigg[& \det \left[\phi_n(x_k^{n-1}(t_0^{n-1}), x_l^n(t_{c(n)}^n)) \right]_{1 \leq k, l \leq n} \\
& \times \prod_{a=1}^{c(n)} \det \left[\mathcal{T}_{t_a^n, t_{a-1}^n}(x_k^n(t_a^n), x_l^n(t_{a-1}^n)) \right]_{1 \leq k, l \leq n} \Bigg] \det \left[\Psi_{N-l}^N(x_k^N(t_0^N)) \right]_{1 \leq k, l \leq N},
\end{aligned} \qquad (A.6)$$

where $x_n^{n-1}(\,\cdot\,) = \text{virt}$ for all $n = 1, \ldots, N$.

In what follows we assume that the partition function of our weights does not vanish:

$$Z := \sum_{X \subset \mathfrak{X}} W(X) \neq 0. \qquad (A.7)$$

Under this assumption, the normalized weights $\widetilde{W}(X) = W(X)/Z$ define a (generally speaking, complex valued) measure on $2^{\mathfrak{X}}$ of total mass 1. One can say that we have a (complex valued) random point process on $\mathfrak{X}$, and its correlation functions are defined accordingly, see e.g. [26]. We are interested in computing these correlation functions.

Let us introduce the compact notation for the convolution of several transitions. For any $n = 1, \ldots, N$ and two time moments $t_a^n > t_b^n$ we define

$$\mathcal{T}_{t_a^n, t_b^n} = \mathcal{T}_{t_a^n, t_{a-1}^n} * \mathcal{T}_{t_{a-1}^n, t_{a-2}^n} * \cdots * \mathcal{T}_{t_{b+1}^n, t_b^n}, \qquad \mathcal{T}^n = \mathcal{T}_{t_{c(n)}^n, t_0^n}, \tag{A.8}$$

where we use the notation $(f * g)(x, y) := \sum_z f(x, z) g(z, y)$. For any time moments $t_{a_1}^{n_1} \geq t_{a_2}^{n_2}$ with $(a_1, n_1) \neq (a_2, n_2)$, we denote the convolution over all the transitions between them by $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}$:

$$\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = \mathcal{T}_{t_{a_1}^{n_1}, t_0^{n_1}} * \phi_{n_1+1} * \mathcal{T}^{n_1+1} * \cdots * \phi_{n_2} * \mathcal{T}_{t_{c(n_2)}^{n_2}, t_{a_2}^{n_2}}. \tag{A.9}$$

If there are no such transitions, i.e., if $t_{a_1}^{n_1} < t_{a_2}^{n_2}$ or $(a_1, n_1) = (a_2, n_2)$, we set $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = 0$.

Furthermore, define the matrix $M = \|M_{k,l}\|_{k,l=1}^N$ by

$$M_{k,l} = \left(\phi_k * \mathcal{T}^k * \cdots * \phi_N * \mathcal{T}^N * \Psi_{N-l}^N\right)(\mathrm{virt}) \tag{A.10}$$

and the vector

$$\Psi_{n-l}^{n, t_a^n} = \phi^{(t_a^n, t_0^N)} * \Psi_{N-l}^N, \qquad l = 1, \ldots, N. \tag{A.11}$$

The following statement describing the correlation kernel is a part of Theorem 4.2 of [13].

Theorem 6.14. *Assume that the matrix M is invertible. Then $Z = \det M \neq 0$, and the (complex valued) random point process on $\mathfrak{X}$ defined by the weights $\widetilde{W}(X)$ is determinantal. Its correlation kernel can be written in the form*

$$K(t_{a_1}^{n_1}, x_1; t_{a_2}^{n_2}, x_2) = -\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}(x_1, x_2)$$
$$+ \sum_{k=1}^N \sum_{l=1}^{n_2} \Psi_{n_1-k}^{n_1, t_{a_1}^{n_1}}(x_1) [M^{-1}]_{k,l} (\phi_l * \phi^{(t_{c(l)}^l, t_{a_2}^{n_2})})(\mathrm{virt}, x_2). \tag{A.12}$$

The proof of Theorem 6.14 given in [13] is based on the algebraic formalism of [26]. Another proof can be found in Sect. 4.4 of [44]. Although we stated Theorem 6.14 for the case when all sets $\mathfrak{X}_n$ are finite, one easily extends it to a more general setting. Indeed, the determinantal formula for the correlation functions is an algebraic identity, and the limit transition to the case when $\mathfrak{X}_n$'s are allowed to be countably infinite is immediate, under the assumption that all the sums needed to define the $*$-operations above are absolutely convergent. Another easy extension (which we do not need in this paper) is the case when the spaces $\mathfrak{X}_j$ become continuous, and the sums have to be replaced by the corresponding integrals over these spaces.

B. Further Developments

Here is an overview of the developments related to this paper since the appearance of the preprint version.

Growth models and random matrices. This paper provided the first example of a completely analyzed two-dimensional growth model in the so-called Kardar-Parisi-Zhang (KPZ) universality class in both the mathematical and physical literature. While some results were anticipated by physicists (logarithmic fluctuations of the height function at a fixed space location), others were not (Gaussian Free Field fluctuations with respect to a hidden conformal structure on the limiting surface). Results were reported in a physics publication [12].

This paper also delivered an overarching probabilistic structure for a number of results in (1+1)-dimensional surface growth [13, 15–18] that provided a conceptual probabilistic explanation for the appearance of random matrix statistics in the large time limit. The growing interface and the random matrix evolution were realized as two marginals of the (2+1)-dimensional object. This was the starting point of two further works relating minor processes of (perturbed) GUE random matrices, interacting particle systems and diffusion processes [40, 41].

Combinatorics and exact sampling. The sole previously known member of the mentioned class of growth processes was the celebrated domino shuffling algorithm (1992) that gives rise to uniformly distributed domino tilings of the Aztec diamond. The general construction of this paper was used in [19] to construct lozenge shuffling – a long sought analog of the domino shuffling on the hexagonal lattice. This lead to an efficient exact sampling algorithm for uniformly distributed lozenge tilings of hexagons (a.k.a. boxed plane partitions). This was further extended to more generally weighted plane partitions related to classical hypergeometric polynomials from the Askey scheme [21] and bi-orthogonal elliptic special functions [6, 21].

Another development of these ideas lead to a general construction of Markov chains preserving the class of the Schur processes, in particular, to an exact sampling algorithm for plane partitions with arbitrary back wall and for random Gelfand-Tsetlin patters related to irreducible characters of the infinite-dimensional unitary group [8].

Two-dimensional Gaussian Free Field (GFF). The 2d GFF may be viewed as a two-dimensional analog of the classical 1d Brownian Motion. This paper developed a new approach of proving the convergence to GFF. It applies to domains with facets, and one such domain was analyzed with complete details. Later the approach was successfully extended to growth models with reflecting wall in [57], to growth models with non-smooth limit shapes [36], to random surfaces described by Pfaffian (rather than determinantal) point processes [75], and to random surfaces arising from lozenge tilings of certain polygons drawn on the hexagonal lattice [64, 65]. The simplest result of the latter work covers lozenge tiling of the hexagon, thus proving a basic conjecture advertised a few years years ago by Kenyon [51].

Representation theory and infinite-dimensional Markov processes. One property of the construction of Markov dynamics presented in this paper is the fact that it preserves a certain class of *Gibbs* distributions. In a special case described in the paper, these Gibbs distributions are in one-to-one correspondence with characters (finite central traces) on the infinite-dimensional unitary group. One thus obtains a dynamics on the space of such characters. This dynamics may be deterministic or stochastic, and in the latter case one obtains a Markov process with an infinite-dimensional state space. In [24] and [20] properties of such Markov processes were investigated in detail. These processes remain the only proven examples of Feller processes that preserve infinite-particle random point processes similar to those that arise in random matrix theory.

Other root systems. From the viewpoint of representations of simple Lie groups, this paper dealt with the case of root systems of type A (unitary groups). The constructions have been partially extended to the cases of orthogonal and symplectic groups in [23,32,33,56].

Random polymers in random media. In a very recent development, the abstract formalism of the discussed paper was carried over to a new ground of the so-called *Macdonald processes* and applied to difference operators arising from the (multivariate) Macdonald polynomials [10]. This lead to a conceptual new understanding of the asymptotic behavior of (1+1)-dimensional random polymers in random media and finding explicit solutions of the (nonlinear stochastic partial differential) KPZ equation [10,11]. The development of the construction from this paper provided an alternative, analytically more powerful approach to earlier results of [1,29,60,71].

References

1. Amir, G., Corwin, I., Quastel, J.: Probability Distribution of the Free Energy of the Continuum Directed Random Polymer in 1 + 1 dimensions. Comm. Pure Appl. Math. **64**, 466–537 (2011)
2. Balázs, M., Komjáthy, J.: Order of current variance and diffusivity in the rate one totally asymmetric zero range process. J. Stat. Phys. **133**, 59–78 (2008)
3. Balázs, M., Seppäläinen, T.: Order of current variance and diffusivity in the asymmetric simple exclusion process. Ann. Math. **171**, 1237–1265 (2010)
4. Barabási, A.L., Stanley, H.E.: *Fractal concepts in surface growth*. Cambridge: Cambridge University Press, 1995
5. van Beijeren, H., Kutner, R., Spohn, H.: Excess noise for driven diffusive systems. Phys. Rev. Lett. **54**, 2026–2029 (1985)
6. Betea, D.: *Elliptically distributed lozenge tilings of a hexagon*. http://arXiv.org/abs/1110.4176v1 [math.ph], 2011
7. Biane, P.: Approximate factorization and concentration for characters of symmetric groups. Int. Math. Res. Not. **4**, 179–192 (2001)
8. Borodin, A.: Schur dynamics of the Schur processes. Adv. Math. **228**, 2268–2291 (2011)
9. Borodin, A.: Loop-free markov chains as determinantal point processes. Ann. Inst. H. Poincaré Probab. Statist. **44**, 19–28 (2008)
10. Borodin, A., Corwin, I.: *Macdonald processes*, to appear in Prob. Theory Rel. Fields, http://arXiv.org/abs/1111.4408v1 [math.PR], 2013
11. Borodin, A., Corwin, I., Ferrari, P.L.: *Free energy fluctuations for directed polymers in random media in 1+1 dimension*. http://arXiv.org.abs/1204.1024v2 [math.PR], 2013
12. Borodin, A., Ferrari, P.L.: Anisotropic growth of random surfaces in 2+1 dimensions: fluctuations and covariance structure. J. Stat. Mech. P02009 (2009)
13. Borodin, A.P.L. Ferrari: Large time asymptotics of growth models on space-like paths I: PushASEP. Electron. J. Probab. **13**, 1380–1418 (2008)
14. Borodin, A., Ferrari, P.L., Prähofer, M.: Fluctuations in the discrete TASEP with periodic initial configurations and the Airy$_1$ process. Int. Math. Res. Papers **2007**, rpm002 (2007)
15. Borodin, A., Ferrari, P.L., Prähofer, M., Sasamoto, T.: Fluctuation properties of the TASEP with periodic initial configuration. J. Stat. Phys. **129**, 1055–1080 (2007)
16. Borodin, A., Ferrari, P.L., Sasamoto, T.: Large time asymptotics of growth models on space-like paths II: PNG and parallel TASEP. Commun. Math. Phys. **283**, 417–449 (2008)
17. Borodin, A., Ferrari, P.L., Sasamoto, T.: Transition between Airy$_1$ and Airy$_2$ processes and TASEP fluctuations. Comm. Pure Appl. Math. **61**, 1603–1629 (2008)
18. Borodin, A., Ferrari, P.L., Sasamoto, T.: Two-speed TASEP. J. Stat. Phys. **137**, 936–977 (2009)
19. Borodin, A., Gorin, V.: Shuffling algorithm for boxed plane partitions. Adv. Math. **220**, 1739–1770 (2009)
20. Borodin, A., Gorin, V.: Markov processes of infinitely many nonintersecting random walks. Probab. Theory Relat. Fields **105**(3-4), 935–997 (2013)
21. Borodin, A., Gorin, V., Rains, E.M.: q-Distributions on boxed plane partitions. Selecta Math. (N.S.) **16**, 731–789 (2010)
22. Borodin, A., Kuan, J.: Asymptotics of Plancherel measures for the infinite-dimensional unitary group. Adv. Math. **219**, 894–931 (2008)

23. Borodin, A., Kuan, J.: Random surface growth with a wall and Plancherel measures for $O(\infty)$. Comm. Pure Appl. Math. **63**, 831–894 (2010)
24. Borodin, A., Olshanski, G.: Markov processes on the path space of the Gelfand-Tsetlin graph and on its boundary. Jour. Func. Anal. **263**, 248–303 (2012)
25. Borodin, A., Okounkov, A., Olshanski, G.: On asymptotics of Plancherel measures for symmetric groups. J. Amer. Math. Soc. **13**, 481–515 (2000)
26. Borodin, A., Rains, E.M.: Eynard-Mehta theorem, Schur process, and their Pfaffian analogs. J. Stat. Phys. **121**, 291–317 (2006)
27. Borodin, A., Shlosman, S.: Gibbs ensembles of nonintersecting paths. Commun. Math. Phys. **293**, 145–170 (2010)
28. Boutillier, C.: *Modèles de dimères: comportements limites*. Ph.D. thesis, Université de Paris-Sud, 2005
29. Corwin, I., O'Connell, N., Seppäläinen, T., Zygouras, N.: *Tropical combinatorics and Whittaker functions*. http://arXiv.org/abs/1110.3489v1 [math.PR], 2011
30. Corwin, I., Ferrari, P.L., Péché, S.: Universality of slow decorrelation in KPZ models. Ann. Inst. H. Poincaré Probab. Statist. **48**, 134–150 (2012)
31. Costin, O., Lebowitz, J.L.: Gaussian Fluctuation in Random Matrices. Phys. Rev. Lett. **75**, 69–72 (1995)
32. Defosseux, M.: An interacting particle model and a Pieri-type formula for the orthogonal group. J. Theor. Probab. **26**, 568–588 (2013)
33. Defosseux, M.: Interacting particle models and the Pieri-type formulas: the symplectic case with non equal weights. Electron. Commun. Probab. **17**(32), 1–12 (2012)
34. Diaconis, P., Fill, J.A.: Strong Stationary Times Via a New Form of Duality. Ann. Probab. **18**, 1483–1522 (1990)
35. Dieker, A.B., Warren, J.: Determinantal transition kernels for some interacting particles on the line. Ann. Inst. H. Poincaré Probab. Statist. **44**, 1162–1172 (2008)
36. Duits, M.: The Gaussian free field in an interlacing particle system with two jump rates. Comm. Pure Appl. Math. **66**, 600–643 (2013)
37. Elkies, N., Kuperbert, G., Larsen, M., Propp, J.: *Alternating-Sign Matrices and Domino Tilings I and II*. J. Alge. Comb. **1**, 111–132 and 219–234, (1992)
38. Ferrari, P.L.: *Java animation of a growth model in the anisotropic KPZ class in* 2 + 1 *dimensions*. http://www-wt.iam.uni-bonn.de/ferrari/research/#animations
39. Ferrari, P.L.: Slow decorrelations in KPZ growth. J. Stat. Mech., P07022 (2008)
40. Ferrari, P.L., Frings, F.: *Perturbed GUE minor process and Warren's process with drifts*. http://arXiv.org/abs/1212.5534v2 [math.ph], 2013
41. Ferrari, P.L., Frings, F.: On the partial connection between random matrices and interacting particle systems. J. Stat. Phys. **141**, 613–637 (2010)
42. Ferrari, P.L., Spohn, H.: Step fluctuations for a faceted crystal. J. Stat. Phys. **113**, 1–46 (2003)
43. Ferrari, P.L., Spohn, H.: Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process. Commun. Math. Phys. **265**, 1–44 (2006)
44. Forrester Peter, J., Nordenstam, Eric: The Anti-Symmetric GUE Minor Process. Mosc. Math. J. **9**, 749–774 (2009)
45. Halpin-Healy, T., Assdah, A.: On the kinetic roughening of vicinal surfaces. Phys. Rev. A **46**, 3527–3530 (1992)
46. Heong, H., Kahng, B., Kim, D.: Dynamics of a Toom Interface in Three Dimensions. Phys. Rev. Lett. **71**, 747–749 (1993)
47. Imamura, T., Sasamoto, T.: Dynamical properties of a tagged particle in the totally asymmetric simple exclusion process with the step-type initial condition. J. Stat. Phys. **128**, 799–846 (2007)
48. Johansson, K.: Shape fluctuations and random matrices. Commun. Math. Phys. **209**, 437–476 (2000)
49. Johansson, K.: Non-intersecting paths, random tilings and random matrices. Probab. Th. Rel. Fields **123**, 225–280 (2002)
50. Kenyon, R.: *Lectures on dimers*. available via http://www.math.brown.edu/rkenyon/papers/dimerlecture notes.pdf
51. Kenyon, R.: Height fluctuations in the honeycomb dimer model. Commun. Math. Phys. **281**, 675–709 (2008)
52. Kenyon, R., Okounkov, A.: Limit shapes and the complex Burgers equation. Acta Math. **199**, 263–302 (2007)
53. Kenyon, R., Okounkov, A., Sheffield, S.: Dimers and amoebae. Ann. of Math. **163**, 1019–1056 (2006)
54. Kim, H.-J., Kim, I., Kim, J.M.: Hybridized discrete model for the anisotropic Kardar-Parisi-Zhang equation. Phys. Rev. E **58**, 1144–1147 (1998)
55. König, W., O'Connel, N., Roch, S.: Non-colliding Random Walks, Tandem Queues and Discrete Orthogonal Polynomial Ensembles. Electron. J. Probab. **7**, 1–24 (2002)
56. Kuan, J.: Asymptotics of a Discrete-Time Particle System Near a Reflecting Boundary. J. Stat. Phys. **150**, 398–411 (2013)

57. Kuan, J.: *The Gaussian free field in interlacing particle systems*. http://arXiv.org/abs/1109.4444v1 [math.ph], 2011
58. Nagao, T., Sasamoto, T.: Asymmetric simple exclusion process and modified random matrix ensembles. Nucl. Phys. B **699**, 487–502 (2004)
59. Nordenstam, E.: On the shuffling algorithm for domino tilings. Electron. J. Probab. **15**, 75–95 (2010)
60. O'Connell, N.: Directed polymers and the quantum Toda lattice. Ann. Probab. **40**, 437–458 (2012)
61. O'Connell, N.: Conditioned random walks and the RSK correspondence. Random Matrix Theory. J. Phys. A **36**, 3049–3066 (2003)
62. Okounkov, A.: *Symmetric functions and random partitions*. http://arXiv.org/absmath/0309074v1 [math.co], 2003, Lecture notes for the NATO ASI "Symmetric Functions 2001", Isaac Newton Institute (Cambridge) (2003)
63. Okounkov, A., Reshetikhin, N.: Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram. J. Amer. Math. Soc. **16**, 581–603 (2003)
64. Petrov, L.: *Asymptotics of Random Lozenge Tilings via Gelfand-Tsetlin Schemes*. http://arXiv.org/abs/1202.3901v1 [math.PR], 2012
65. Petrov, L.: *Asymptotics of Uniformly Random Lozenge Tilings of Polygons. Gaussian Free Field*, to appear in Ann. Probab., http://arXiv.org/abs/1206.5123v2 [math.PR], 2012
66. M. Prähofer, *Stochastic surface growth*. Ph.D. thesis, Ludwig-Maximilians-Universität, München, http://edoc.ub.uni-muenchen.de/archive/00001381, 2003
67. Prähofer, M., Spohn, H.: An Exactly Solved Model of Three Dimensional Surface Growth in the Anisotropic KPZ Regime. J. Stat. Phys. **88**, 999–1012 (1997)
68. Prähofer, M., Spohn, H.: *Current fluctuations for the totally asymmetric simple exclusion process*. In: In and out of equilibrium V. Sidoravicius, ed., Progress in Probability, Basel-Boston: Birkhäuser, 2002
69. Prähofer, M., Spohn, H.: Exact scaling function for one-dimensional stationary KPZ growth. J. Stat. Phys. **115**, 255–279 (2004)
70. Quastel, J., Valkó, B.: $t^{1/3}$ Superdiffusivity of Finite-Range Asymmetric Exclusion Processes on $\mathbb{Z}$. Commun. Math. Phys. **273**, 379–394 (2007)
71. Sasamoto, T., Spohn, H.: One-dimensional KPZ equation: an exact solution and its universality. Phys. Rev. Lett. **104**, 23–26 (2010)
72. Sasamoto, T.: Spatial correlations of the 1D KPZ surface on a flat substrate. J. Phys. A **38**, L549–L556 (2005)
73. Sheffield, S.: Gaussian Free Field for mathematicians. Probab. Th. Rel. Fields **139**, 521–541 (2007)
74. Soshnikov, A.B.: Gaussian Fluctuation for the Number of Particles in Airy, Bessel, Sine and Other Determinantal Random Point Fields. J. Stat. Phys. **100**, 491–522 (2004)
75. Vuletic, M.: In preparation
76. Warren, Jon: Dyson's Brownian motions, intertwining and interlacing. Electron. J. Probab. **12**, 573–590 (2007)
77. Wolf, D.E.: Kinetic roughening of vicinal surfaces. Phys. Rev. Lett. **67**, 1783–1786 (1991)