

## MIT Open Access Articles

*K-theory of endomorphisms via noncommutative motives*

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

**Citation:** Blumberg, Andrew J., David Gepner, and Gonalo Tabuada. “ $\mathcal{K}$ -Theory of Endomorphisms via Noncommutative Motives.” Transactions of the American Mathematical Society 368.2 (2015): 1435–1465.

**As Published:** <http://dx.doi.org/10.1090/tran/6507>

**Publisher:** American Mathematical Society (AMS)

**Persistent URL:** <http://hdl.handle.net/1721.1/105205>

**Version:** Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

**Terms of Use:** Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use.



## **$K$ -THEORY OF ENDOMORPHISMS VIA NONCOMMUTATIVE MOTIVES**

ANDREW J. BLUMBERG, DAVID GEPNER, AND GONALO TABUADA

**ABSTRACT.** We extend the  $K$ -theory of endomorphisms functor from ordinary rings to (stable)  $\infty$ -categories. We show that  $K\text{End}(-)$  descends to the category of noncommutative motives, where it is corepresented by the noncommutative motive associated to the tensor algebra  $\mathbb{S}[t]$  of the sphere spectrum  $\mathbb{S}$ . Using this corepresentability result, we classify all the natural transformations of  $K\text{End}(-)$  in terms of an integer plus a fraction between polynomials with constant term 1; this solves a problem raised by Almkvist in the seventies. Finally, making use of the multiplicative coalgebra structure of  $\mathbb{S}[t]$ , we explain how the (rational) Witt vectors can also be recovered from the symmetric monoidal category of noncommutative motives. Along the way we show that the  $K_0$ -theory of endomorphisms of a *connective* ring spectrum  $R$  equals the  $K_0$ -theory of endomorphisms of the underlying ordinary ring  $\pi_0 R$ .

### CONTENTS

1. Introduction	1435
2. Preliminaries	1438
3. $K$ -theory of endomorphisms	1443
4. Endomorphisms of projective modules	1449
5. Natural operations	1454
6. (Rational) Witt vectors	1457
Appendix A. Splitting coalgebras	1460
Acknowledgments	1463
References	1463

### 1. INTRODUCTION

The  $K$ -theory of endomorphisms was introduced in the seventies by Almkvist [1, 2] and Grayson [23, 24]. Given an ordinary ring  $A$ , let  $\mathbf{P}(A)$  be the category of finitely generated projective (right)  $A$ -modules and  $\text{End}(\mathbf{P}(A))$  the associated category of endomorphisms: its objects are pairs  $(M, \alpha)$ , with  $M \in \mathbf{P}(A)$  and  $\alpha$

---

Received by the editors February 20, 2013 and, in revised form, March 3, 2014 and June 19, 2014.

2010 *Mathematics Subject Classification.* Primary 19D10, 19D25, 19D55, 18D20, 55N15.

*Key words and phrases.*  $K$ -theory of endomorphisms, noncommutative motives, stable  $\infty$ -categories, Witt vectors.

The first author was partially supported by the NSF grant DMS-0906105.

The third author was partially supported by the National Science Foundation CAREER Award #1350472 and by the Fundao para a Cincia e a Tecnologia (Portuguese Foundation for Science and Technology) through the project grant UID/MAT/00297/2013 (Centro de Matemtica e Aplicaes).

an endomorphism of  $M$ , and its morphisms  $(M, \alpha) \rightarrow (M', \alpha')$  are the  $A$ -linear homomorphisms  $f : M \rightarrow M'$  satisfying the equality  $f\alpha = \alpha'f$ . Note that this latter category inherits naturally from  $\mathbf{P}(A)$  an exact structure in the sense of Quillen [36]. The classical *K-theory of endomorphisms of  $A$*  was then defined as the homotopy groups of the (connective) algebraic  $K$ -theory spectrum  $\mathrm{KEnd}(\mathbf{P}(A))$  of the exact category  $\mathrm{End}(\mathbf{P}(A))$ .

Bloch [10], and later Stienstra [39, 40], related the  $K$ -theory of endomorphisms to crystalline cohomology. More recently, work of Betley-Schlichtkrull [7], Hesselholt [29], and Lindenstrauss-McCarthy [33], establishes precise connections between the  $K$ -theory of (parametrized) endomorphisms, Goodwillie calculus, and invariants arising from trace methods in algebraic  $K$ -theory. In this article, we study foundational aspects of the  $K$ -theory of endomorphisms using noncommutative motives.

**Noncommutative motives.** Let  $\mathrm{Cat}_\infty^{\mathrm{perf}}$  be the  $\infty$ -category of small idempotent-complete stable  $\infty$ -categories; see §2.2. Standard examples are the  $\infty$ -category  $\mathrm{Perf}_R$  of perfect modules over a ring spectrum  $R$  and the  $\infty$ -category  $\mathrm{Perf}_X$  of perfect complexes for a scheme  $X$ . Recall from [8, 6.1] that a functor  $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \rightarrow \mathcal{D}$ , with values in a stable presentable  $\infty$ -category  $\mathcal{D}$ , is called an *additive invariant* if it preserves filtered colimits and sends split exact sequences of stable  $\infty$ -categories to (necessarily split) cofiber sequences of spectra. Examples include algebraic  $K$ -theory (see §2.6) and topological Hochschild homology (THH). In [8, §6] we have constructed the *universal additive invariant*

$$(1.1) \quad \mathcal{U}_{\mathrm{add}} : \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{M}_{\mathrm{add}}.$$

Given any stable presentable  $\infty$ -category  $\mathcal{D}$ , there is an induced equivalence

$$(1.2) \quad (\mathcal{U}_{\mathrm{add}})^* : \mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathcal{D})$$

where the left-hand side denotes the  $\infty$ -category of colimit-preserving functors and the right-hand side the  $\infty$ -category of additive invariants. We refer to  $\mathcal{M}_{\mathrm{add}}$  as the category of *noncommutative spectral motives* or just *noncommutative motives*. As with any stable  $\infty$ -category,  $\mathcal{M}_{\mathrm{add}}$  carries a natural enrichment  $\mathrm{Map}(-, -)$  in spectra; see [8, §4.2]. In [8, §7.3] we proved that for every  $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$  there is a natural equivalence of spectra

$$(1.3) \quad \mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}}), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \simeq K(\mathcal{C}),$$

where  $\mathbb{S}$  denotes the sphere spectrum. That is, algebraic  $K$ -theory becomes corepresented by the noncommutative motive associated to  $\mathbb{S}$ .

**Statements of results.** Given an  $\infty$ -category  $\mathcal{C}$ , we start by defining the  $\infty$ -category  $\mathrm{End}(\mathcal{C})$  of endomorphisms in  $\mathcal{C}$  as the functor  $\infty$ -category  $\mathrm{End}(\mathcal{C}) := \mathrm{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C})$ ; see Definition 3.5. By first restricting this construction to  $\mathrm{Cat}_\infty^{\mathrm{perf}}$  and then applying the algebraic  $K$ -theory functor, we obtain a well-defined  $K$ -theory of endomorphisms functor

$$(1.4) \quad \mathrm{KEnd} : \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{S}_\infty$$

with values in the  $\infty$ -category of spectra. Our first main result characterizes this functor as follows.

**Theorem 1.5** (see Theorem 3.9). *The above functor (1.4) is an additive invariant. Moreover, for every  $\mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  there is a natural equivalence of spectra*

$$(1.6) \quad \text{Map}(\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]}), \mathcal{U}_{\text{add}}(\mathcal{C})) \simeq \text{KEnd}(\mathcal{C}),$$

where  $\mathbb{S}[t]$  stands for the tensor algebra of  $\mathbb{S}$ .

Theorem 1.5 shows that the functor (1.4) descends to  $\mathcal{M}_{\text{add}}$ , where it is corepresented by the noncommutative motive associated to  $\mathbb{S}[t]$ . By adding a “formal variable”  $t$  to  $\mathbb{S}$  one passes then from algebraic  $K$ -theory (1.3) to  $K$ -theory of endomorphisms (1.6). The following result justifies our definition of  $\text{KEnd}$ .

**Theorem 1.7** (see Theorem 3.12). *Given a regular ring  $A$ , the associated spectrum  $\text{KEnd}(\text{Perf}_{HA})$  (where  $HA$  denotes the Eilenberg-MacLane ring spectrum of  $A$ ) is naturally equivalent to the classical endomorphism  $K$ -theory  $\text{KEnd}(\mathbf{P}(A))$ .*

As explained by Almkvist in [1, page 339], a very interesting problem in the  $K$ -theory of endomorphisms is the classification of all the natural transformations of the functor  $A \mapsto \text{KEnd}(\mathbf{P}(A))$ . Classical examples include the Frobenius  $F_n$  and the Verschiebung  $V_n$  operations. This problem was studied in the particular case of the  $K_0$ -theory of endomorphisms by Hazewinkel [26], and many operations in the higher  $K$ -theory of endomorphisms were computed by Stienstra [39, 40]. In §5 we extend  $F_n$  and  $V_n$  to the  $\infty$ -categorical setting and (making use of Theorem 1.5) solve the problem raised by Almkvist as follows: given a (commutative) ring  $A$ , let us write  $W_0(A)$  for the multiplicative (abelian) group of *rational Witt vectors*:

$$(1.8) \quad W_0(A) = \left\{ \frac{1 + a_1 r + \cdots + a_i r^i + \cdots + a_n r^n}{1 + b_1 r + \cdots + b_j r^j + \cdots + b_m r^m} \mid a_i, b_j \in A, n, m \geq 0 \right\}.$$

**Theorem 1.9** (see Theorem 5.7). *There is a canonical weak equivalence of spectra*

$$(1.10) \quad \text{Nat}(\text{KEnd}, \text{KEnd}) \simeq \text{KEnd}(\text{Perf}_{\mathbb{S}[t]}),$$

where  $\text{Nat}$  stands for the spectrum of natural transformations. Moreover, the group  $\pi_0 \text{Nat}(\text{KEnd}, \text{KEnd})$  of natural transformations up to homotopy is isomorphic to

$$(1.11) \quad \pi_0 \text{KEnd}(\text{Perf}_{\mathbb{S}[t]}) \simeq \mathbb{Z} \oplus W_0(\mathbb{Z}[t]).$$

Furthermore, under the above identifications, the Frobenius operations  $F_n$  correspond to the elements  $(1, 1 + r^n t)$  and the Verschiebung operations  $V_n$  to the elements  $(n, 1 + r t^n)$ .

Roughly speaking, Theorem 1.9 shows us that all the information concerning a natural transformation of the above functor (1.4) can be completely encoded in an integer plus a fraction between polynomials with constant term 1. It also shows us that the Frobenius (resp. the Verschiebung) operation is the “simplest one” with respect to the variable  $t$  (resp.  $r$ ). In order to prove isomorphism (1.11), we establish the following computational result:

**Theorem 1.12** (see Theorem 4.17). *For every connective ring spectrum  $R$ , one has an isomorphism  $\pi_0 \text{KEnd}(\text{Perf}_R) \simeq K_0(\text{End}(\mathbf{P}(\pi_0 R)))$  of abelian groups.*

One interesting feature of the  $K$ -theory of endomorphisms is its connection with Witt vectors. Given a commutative ring  $A$ , the Witt ring  $W(A)$  of  $A$  is the abelian group of all power series of the form  $1 + a_1 r + a_2 r^2 + \cdots$ , with  $a_i \in A$ , endowed with

the multiplication  $*$  determined by the equality  $(1 - a_1 r) * (1 - a_2 r) = (1 - a_1 a_2 r)$ . The *rational* Witt ring  $W_0(A) \subset W(A)$  of  $A$  consists of the elements of the form (1.8). As observed by Grayson [23],  $W_0(A)$  is a dense  $\lambda$ -subring of  $W(A)$ , and hence  $W(A)$  can be recovered from  $W_0(A)$  by a completion procedure.

The standard symmetric monoidal structure  $-\otimes^\vee -$  on  $\mathrm{Cat}_\infty^{\mathrm{perf}}$  can be extended to  $\mathcal{M}_{\mathrm{add}}$  in a universal way making  $\mathcal{U}_{\mathrm{add}}: \mathrm{Cat}_\infty^{\mathrm{perf}} \rightarrow \mathcal{M}_{\mathrm{add}}$  a symmetric monoidal functor [9, §4]. This additional structure allows us to recover the rational Witt ring (and hence the Witt ring itself) from the  $\infty$ -category of noncommutative motives as follows:

**Theorem 1.13** (see §6). *The ring maps*

$$\mathbb{S}[t] \xrightarrow{t \mapsto t \wedge t} \mathbb{S}[t] \wedge \mathbb{S}[t], \quad \mathbb{S}[t] \xrightarrow{t=1} \mathbb{S}$$

*induce a counital coassociative cocommutative coalgebra structure on  $\mathrm{Perf}_{\mathbb{S}[t]} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ . Moreover, the ring maps  $\mathbb{S} \rightarrow \mathbb{S}[t]$  and  $\mathbb{S}[t] \xrightarrow{t=0} \mathbb{S}$  give rise to a wedge sum decomposition*

$$\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}) \simeq \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}}) \vee \mathbb{W}_0$$

*of counital coassociative cocommutative coalgebras in  $\mathcal{M}_{\mathrm{add}}$ .*

Using Theorem 1.13, we then obtain a lax symmetric monoidal functor  $\mathrm{Map}(\mathbb{W}_0, -)$  from  $\mathcal{M}_{\mathrm{add}}$  to  $\mathcal{S}_\infty$  which we call the *rational Witt ring spectrum functor*. This terminology is justified by the following agreement result:

**Theorem 1.14** (see Theorem 6.3). *For every ordinary commutative ring  $A$  one has a ring isomorphism*

$$(1.15) \quad \pi_0 \mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{HA})) \simeq W_0(A).$$

Isomorphism (1.15) provides a conceptual characterization of the rational Witt ring. Roughly speaking, by keeping track of only the “formal variable”  $t$  of  $\mathbb{S}[t]$  one passes from  $K$ -theory of endomorphisms (1.6) to rational Witt vectors (1.15).

Finally, making use of Lurie’s resolution of Mandell’s conjecture (see [32, 8.1.2.6]), we obtain the following result:

**Corollary 1.16** (see Corollary 6.8). *Let  $R$  be an  $E_n$  ring spectrum. Then the associated rational Witt ring spectrum  $\mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_R))$  is an  $E_{n-1}$  ring spectrum.*

## 2. PRELIMINARIES

In this section, we briefly review background about  $\infty$ -categories, spectral categories, and Waldhausen categories. We also prove some basic results about the algebraic  $K$ -theory of  $\infty$ -categories with cofibrations that we use later on.

**2.1. Notation.** Given an ordinary ring  $A$ , we will denote by  $\mathrm{Ch}(A)$  the category of chain complexes of (right)  $A$ -modules. We will assume that  $\mathrm{Ch}(A)$  is endowed with the projective model structure; see [30, §2.3]. The associated homotopy category (the derived category of  $A$ ) will be denoted by  $\mathcal{D}(A)$ . We will write  $\mathrm{perf}(A)$  for the category of *perfect* complexes of  $A$ -modules, i.e., the full subcategory of  $\mathrm{Ch}(A)$  consisting of those complexes that become compact in the derived category  $\mathcal{D}(A)$ . We will write  $\mathrm{Ch}^b(-)$  for the full subcategory  $\mathrm{Ch}(A)$  consisting of bounded complexes. Finally, we write  $\mathrm{Perf}_A$  for the  $\infty$ -category of perfect complexes over  $A$  (and

more generally the category of compact modules over a stable  $\infty$ -category); in other words,  $\mathrm{Perf}_A \simeq \mathrm{N}(\mathrm{perf}_A)[W^{-1}]$ , where  $W$  denotes the class of weak equivalences (i.e., the quasi-isomorphisms) in  $\mathrm{perf}_A$ .

**2.2.  $\infty$ -categories.** Throughout the article we will assume that the reader is familiar with the basics of the theory of  $\infty$ -categories. For technical convenience, we work in the setting of Joyal's quasi-categories, but nothing about our work depends on any particular model of  $\infty$ -categories. Standard references for quasi-categories material are [31, 32]. We briefly review a few aspects of the theory of particular note for our treatment.

There are a number of options for producing the “underlying”  $\infty$ -category of a category equipped with a notion of “weak equivalence”. The most structured setting is that of a simplicial model category  $\mathcal{C}$ , where the  $\infty$ -category can be obtained by restricting to the full simplicial subcategory  $\mathcal{C}^{\mathrm{cf}}$  of cofibrant-fibrant objects and then applying the simplicial nerve functor  $N$ . More generally, if  $\mathcal{C}$  is a category equipped with a subcategory of weak equivalences  $w\mathcal{C}$ , the Dwyer-Kan simplicial localization  $LC$  provides a corresponding simplicial category, and then  $N((LC)^{\mathrm{fib}})$ , where  $(-)^{\mathrm{fib}}$  denotes fibrant replacement in simplicial categories, yields an associated  $\infty$ -category. Lurie has given a version of this approach in [32, §1.3.3]: we associate to a (not necessarily simplicial) category  $\mathcal{C}$  with weak equivalences  $W$  an  $\infty$ -category  $N(\mathcal{C})[W^{-1}]$ ; when  $\mathcal{C}$  is a model category, for functoriality reasons it is usually convenient to restrict to the cofibrant objects  $\mathcal{C}^c$  and consider  $N(\mathcal{C}^c)[W^{-1}]$ .

We will primarily work with *stable*  $\infty$ -categories and *idempotent-complete* stable  $\infty$ -categories [32, §1]. An  $\infty$ -category is stable if it is pointed, admits finite colimits and limits, and the suspension functor induces an auto-equivalence. Let us denote by  $\mathrm{Cat}_\infty$  the  $\infty$ -category of small  $\infty$ -categories, by  $\mathrm{Cat}_\infty^{\mathrm{ex}}$  the  $\infty$ -category of small stable  $\infty$ -categories and exact functors, and by  $\mathrm{Cat}_\infty^{\mathrm{perf}}$  the  $\infty$ -category of small idempotent-complete stable  $\infty$ -categories. The inclusions of subcategories  $\mathrm{Cat}_\infty^{\mathrm{perf}} \subset \mathrm{Cat}_\infty^{\mathrm{ex}} \subset \mathrm{Cat}_\infty$  admit left adjoints

$$\mathrm{Stab}: \mathrm{Cat}_\infty \longrightarrow \mathrm{Cat}_\infty^{\mathrm{ex}}, \quad \mathrm{Idem}: \mathrm{Cat}_\infty^{\mathrm{ex}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{perf}}.$$

Note that the inclusion  $\mathrm{Cat}_\infty^{\mathrm{ex}} \subset \mathrm{Cat}_\infty^{\mathrm{perf}}$  is fully faithful, but  $\mathrm{Cat}_\infty^{\mathrm{ex}} \subset \mathrm{Cat}_\infty$  is not.

**2.3. Spectral categories.** Recall from [13, §2] and [38, Appendix A] that a small *spectral category* is a category enriched over the symmetric monoidal category  $\mathcal{S}$  of symmetric spectra. As explained in [8, §3] there is a close connection between spectral categories and  $\infty$ -categories. The category of small spectral categories  $\mathrm{Cat}_\mathcal{S}$  admits a Quillen model structure with weak equivalences the Morita equivalences [41], and the  $\infty$ -category  $\mathrm{Cat}_\infty^{\mathrm{perf}}$  is equivalent to the  $\infty$ -category associated to this model category; see [8, 3.20-3.20]. Hence, in order to simplify the exposition, we will sometimes abuse notation and elide the distinction between a spectral category (for instance a ring spectrum) and the associated  $\infty$ -category.

**2.4. The homotopy theory of a Waldhausen category.** An important source of categories with weak equivalences is provided by Waldhausen categories. Recall from [43] that this consists of a category  $\mathcal{C}$  endowed with a subcategory of weak equivalences  $w\mathcal{C}$  and with a subcategory of cofibrations  $\mathrm{cof}(\mathcal{C})$  such that the pushouts along cofibrations exist, the cobase change of a cofibration is a cofibration, and pushouts along cofibrations are homotopy pushouts.

Following [12], we will impose further hypotheses that afford control on the underlying homotopy category of a Waldhausen category. These come in two forms:

- (1) We want to ensure that  $\mathcal{C}$  has a homotopy calculus of left fractions (HCLF) in the sense of Dwyer-Kan [15]. Categories with weak equivalences that have homotopy calculi of fractions admit concise and tractable models for the mapping spaces in the Dwyer-Kan simplicial localization  $LC$  [15, 16]. Specifically, we can use concise models of  $L^H\mathcal{C}$  to represent the homotopy types of the mapping spaces as the nerves of certain categories of zig-zags [16].

For this purpose, we impose factorization hypotheses. Recall from [12, 2.6] the notion of a *Waldhausen category with functorial mapping cylinders*. Such cylinders allow a functorial factorization  $A \rightarrow Tf \rightarrow B$  of every map  $f: A \rightarrow B$ . The map  $A \rightarrow Tf$  is a cofibration and  $Tf \rightarrow B$  comes equipped with a natural section  $B \rightarrow Tf$  (which is a weak equivalence). We will say that a Waldhausen category admits functorial factorization if every map  $f: A \rightarrow B$  admits a functorial factorization as a cofibration followed by a weak equivalence; note that this implies functorial mapping cylinders (by factoring the fold map  $A \amalg A \xrightarrow{\nabla} A$ ). It is possible to weaken our factorization hypotheses to remove the hypotheses of functoriality; see for instance [12, §A]. In fact, it is enough to require functorial mapping cylinders for *weak cofibrations*, i.e., maps that are equivalent via a zig-zag of weak equivalences to a cofibration; see [12, §2.1]. Given functorial mapping cylinders for weak cofibrations, the category  $\mathcal{C}$  has a homotopy calculus of left fractions; see [12, 5.5].

- (2) We want to ensure the weak equivalences are compatible with the homotopy category of  $\mathcal{C}$ . Recall from [18] that a category  $\mathcal{C}$  with weak equivalences is called *DHKS-saturated* if a map is a weak equivalence if and only if it is an isomorphism in the homotopy category [18]. In a Waldhausen category with functorial mapping cylinders for weak cofibrations, the property of being DHKS-saturated is equivalent to the weak equivalences satisfying the two out of six property [12, 6.4]. In fact, it suffices that the weak equivalences are closed under retracts (which is easier to check).

**2.5. Algebraic  $K$ -theory of exact and Waldhausen categories.** We assume the reader has some familiarity with Quillen's  $K$ -theory of exact categories [36, §2] and with Waldhausen's  $K$ -theory of categories with cofibrations and weak equivalences [43, §1]. For each of these classes of input data, we can associate a connective algebraic  $K$ -theory spectrum. Given an exact category, we can regard it as a Waldhausen category with weak equivalences the isomorphisms and cofibrations the admissible monomorphisms; see [43, §1.9] for the agreement between the two possible constructions of  $K$ -theory in this situation.

**2.6. Algebraic  $K$ -theory of  $\infty$ -categories.** In this section we quickly review the analogue of the definition of Waldhausen  $K$ -theory in the setting of  $\infty$ -categories.

**Definition 2.1.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category. We say that  $\mathcal{C}$  is an  $\infty$ -category with cofibrations if we have the additional data of a subcategory  $\text{cof}(\mathcal{C})$  such that

- (1) for any object  $x$  in  $\mathcal{C}$ , the unique map  $* \rightarrow x$  is a cofibration,
- (2) the subcategory  $\text{cof}(\mathcal{C})$  contains all the equivalences,

- (3) pushouts along cofibrations exist, and the cobase change of a cofibration is again a cofibration.

We refer to the maps in  $\text{cof}(\mathcal{C})$  as cofibrations.

Such an  $\infty$ -category with cofibrations is referred to as a Waldhausen  $\infty$ -category in [4, 20]. Specifying a subcategory of cofibrations is a way of specifying which maps have homotopy cofibers. In many natural examples, all maps are cofibrations. For instance, any pointed category with pushouts admits the structure of an  $\infty$ -category with cofibrations where all maps are cofibrations.

There is a close connection between cofibrations in Waldhausen  $\infty$ -categories and the weak cofibrations studied in [12] (and reviewed in §2.4 above). More precisely, we have the following consistency check:

**Lemma 2.2.** *Let  $\mathcal{C}$  be a Waldhausen category with weak equivalences  $w\mathcal{C}$  and cofibrations  $\text{cof}(\mathcal{C})$  that satisfy the hypothesis of factorization of weak cofibrations. Let us denote by  $\text{cof}(\text{NC}[W^{-1}]) \subseteq \text{NC}[W^{-1}]$  the subcategory of those arrows which are equivalent to the image of an arrow in  $\text{Ncof}(\mathcal{C}) \subseteq \text{NC}$ . Under these assumptions and notation,  $(\text{NC}[W^{-1}], \text{cof}(\text{NC}[W^{-1}]))$  is an  $\infty$ -category with cofibrations.*

*Proof.* The only property that is not immediate is the existence of pushouts. By [12, 6.2], the hypotheses imply that  $\mathcal{C}$  has the property that a square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where the map  $A \rightarrow B$  is a weak cofibration is homotopy cocartesian if for every object  $E$  the square

$$\begin{array}{ccc} L^H\mathcal{C}(D, E) & \longrightarrow & L^H\mathcal{C}(C, E) \\ \downarrow & & \downarrow \\ L^H\mathcal{C}(B, E) & \longrightarrow & L^H\mathcal{C}(A, E) \end{array}$$

is a homotopy cartesian square of simplicial sets. Since cocartesian  $\infty$ -categories are detected in this way on mapping spaces and the Dwyer-Kan mapping spaces in  $\mathcal{C}$  compute the mapping spaces in  $\text{NC}[W^{-1}]$ , homotopy cocartesian squares in  $\mathcal{C}$  compute cocartesian squares in  $\text{NC}[W^{-1}]$ .  $\square$

*Remark 2.3.* In particular, a Waldhausen category with factorization has an underlying  $\infty$ -category with cofibrations where all maps are cofibrations (and all pushouts exist).

Generalizing [32, 1.2.2.2], we have the following version of the  $S_\bullet$  construction:

**Definition 2.4.** Let  $(\mathcal{C}, \text{cof}(\mathcal{C}))$  be an  $\infty$ -category with cofibrations. Denote by  $\text{Gap}([n], \mathcal{C}, \text{cof}(\mathcal{C}))$  the full subcategory of  $\text{Fun}(\text{N}(\text{Ar}[n]), \mathcal{C})$  spanned by the functors  $\text{N}(\text{Ar}[n]) \rightarrow \mathcal{C}$  such that, for each  $i \in I$ ,  $F(i, i)$  is a zero object of  $\mathcal{C}$ ,  $F(i, j) \rightarrow F(i, k)$  is a cofibration for  $i \leq j \leq k$ , and for each  $i < j < k$ , the following square is



cocartesian:

$$\begin{array}{ccc} F(i, j) & \longrightarrow & F(i, k) \\ \downarrow & & \downarrow \\ F(j, j) & \longrightarrow & F(j, k). \end{array}$$

Following [32, 1.2.2.5], we define a simplicial  $\infty$ -category  $S_{\bullet}^{\infty}\mathcal{C}$  by the rule  $S_n^{\infty}\mathcal{C} = \text{Gap}([n], \mathcal{C}, \text{cof}(\mathcal{C}))$ . Applying passage to the largest Kan complex levelwise, we obtain a simplicial space  $(S_{\bullet}^{\infty}\mathcal{C})_{\text{iso}}$ . Then  $\Omega|(S_{\bullet}^{\infty}\mathcal{C})_{\text{iso}}|$  is the  $\infty$ -categorical version of Waldhausen's  $K$ -theory space. Furthermore, for each  $n$ ,  $\text{Gap}([n], \mathcal{C}, \text{cof}(\mathcal{C}))$  is itself equipped with the usual subcategory of cofibrations: we can iterate this procedure. Since  $\text{Gap}([0], \mathcal{C}, \text{cof}(\mathcal{C}))$  is contractible (with preferred basepoint given by the point in  $\mathcal{C}$ ) and  $\text{Gap}([1], \mathcal{C}, \text{cof}(\mathcal{C}))$  is equivalent to  $\mathcal{C}$ , there is a natural map

$$S^1 \wedge (\mathcal{C})_{\text{iso}} \longrightarrow |(S_{\bullet}^{\infty}\mathcal{C})_{\text{iso}}|$$

given by the inclusion into the 1-skeleton. Therefore, the spaces  $|((S_{\bullet}^{\infty})^n(\mathcal{C}))_{\text{iso}}|$  assemble to form a spectrum  $K(\mathcal{C})$ , which is evidently the  $\infty$ -categorical analogue of Waldhausen's  $K$ -theory spectrum. Furthermore, it is natural in functors of  $\infty$ -categories with cofibrations, i.e., functors which preserve zero objects, cofibrations, and pushouts along cofibrations.

The following result records the connection of this definition to the usual definition. This is a generalization of the comparison of [8, 7.12], which handles the case when all maps are cofibrations.

**Theorem 2.5.** *Let  $\mathcal{C}$  be a Waldhausen category with weak equivalences  $w\mathcal{C}$  and cofibrations  $\text{cof}(\mathcal{C})$  that satisfies the hypothesis of factorization of weak cofibrations. Then there is a natural zig-zag of equivalences connecting  $K(\mathcal{C})$  and  $K(\text{NC}[W^{-1}])$ .*

*Proof.* Using the construction  $\text{NC}[W^{-1}]$  to associate an  $\infty$ -category to  $\mathcal{C}$ , we can simplify the arguments leading up [8, 7.12] to avoid embeddings in simplicial model categories. Specifically, the result follows directly from the fact that the comparison of [8, 7.7] (between the  $S'_{\bullet}$  construction of the algebraic  $K$ -theory of  $\mathcal{C}$  [11, 2.7] and  $S_{\bullet}^{\infty}$ ) in the case where all maps are cofibrations extends to the present situation, using the argument for Lemma 2.2.  $\square$

**2.7. Grothendieck group of  $\infty$ -categories.** In this last subsection we give an explicit description of the Grothendieck group ( $K_0$ ) of an  $\infty$ -category  $\mathcal{C}$  with cofibrations. Observe that we can describe  $S_2^{\infty}\mathcal{C}$  as the full subcategory of the  $\infty$ -category of 2-simplices  $\sigma: \Delta^2 \rightarrow \mathcal{C}$  with the property that the composite

$$\Delta^{\{0,2\}} \longrightarrow \Delta^2 \longrightarrow \mathcal{C}$$

is equivalent to the zero map and the map specified by  $\Delta^{\{0,1\}}$  is a cofibration. Moreover, Definition 2.1 implies that the subcategory  $S_2^{\infty}(\mathcal{C})$  always contains the “split exact sequence”  $A \rightarrow A \oplus B \rightarrow B$  corresponding to each pair of objects  $A$  and  $B$  of  $\mathcal{C}$ . The usual argument analyzing  $K_0$  of a Waldhausen category yields the following result.

**Lemma 2.6.** *The abelian group  $K_0(\mathcal{C})$  can be described as the cokernel*

$$(2.7) \quad \bigoplus_{\pi_0 S_2^{\infty}(\mathcal{C})_{\text{iso}}} \mathbb{Z} \longrightarrow \bigoplus_{\pi_0 \mathcal{C}_{\text{iso}}} \mathbb{Z} \longrightarrow K_0(\mathcal{C})$$

of the map which, on the component corresponding to the equivalence class of the exact sequence  $[A \rightarrow B \rightarrow C] \in \pi_0 S_2(\mathcal{C})$ , sends  $1 \in \mathbb{Z}$  to the element  $[A \oplus C] - [B] \in \bigoplus_{\pi_0 \mathcal{C}} \mathbb{Z}$ . Here  $\pi_0 \mathcal{C}_{\text{iso}}$  denotes the set of equivalence classes of objects of the  $\infty$ -category  $\mathcal{C}$ , which is to say the set of connected components of the underlying  $\infty$ -groupoid  $\mathcal{C}_{\text{iso}}$  of  $\mathcal{C}$ .

**Example 2.8.** If  $\mathcal{C}$  is a stable  $\infty$ -category, viewed as an  $\infty$ -category with cofibrations by taking all arrows to be cofibrations, we see that  $S_2^\infty(\mathcal{C})$  is the full subcategory of  $\text{Fun}(\Delta^2, \mathcal{C})$  consisting of the (co)fiber sequences; that is, those  $\sigma: \Delta^2 \rightarrow \mathcal{C}$  which extend to a cocartesian square of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \searrow \sigma & \downarrow \\ 0 & \longrightarrow & C \end{array}$$

where 0 denotes a zero object of  $\mathcal{C}$ . Equivalently, these are the 2-simplices which give rise to distinguished triangles  $A \rightarrow B \rightarrow C$  in the underlying triangulated homotopy category.

### 3. K-THEORY OF ENDOMORPHISMS

In this section we extend the  $K$ -theory of endomorphisms from ordinary rings to stable  $\infty$ -categories.

**Definition 3.1.** Let us denote by  $\mathcal{D} \simeq \underline{\mathbb{N}}$  the category with one object and endomorphism monoid  $\mathbb{N}$ . The category  $\text{End}(\mathcal{C})$  of endomorphisms in a category  $\mathcal{C}$  is the functor category  $\text{Fun}(\mathcal{D}, \mathcal{C})$ ; that is, it has as objects the pairs  $(x, \alpha)$  where  $x \in \mathcal{C}$  and  $\alpha: x \rightarrow x$  an endomorphism, and morphisms  $(x, \alpha) \rightarrow (x', \alpha')$  the maps  $f: x \rightarrow x'$  such that  $f\alpha = \alpha'f$ . Note that the unique functor  $* \rightarrow \underline{\mathbb{N}}$  induces a forgetful functor  $\text{End}(\mathcal{C}) \rightarrow \mathcal{C}$ .

**3.1. Exact categories and Waldhausen categories.** Given an exact category  $\mathcal{C}$  in the sense of Quillen [36, §2], the category  $\text{End}(\mathcal{C})$  inherits an exact structure by declaring a sequence to be exact if its image under the forgetful functor  $\text{End}(\mathcal{C}) \rightarrow \mathcal{C}$  is exact. The endomorphism  $K$ -theory of  $\mathcal{C}$  is then defined as the connective algebraic  $K$ -theory spectrum  $\text{KEnd}(\mathcal{C})$  of the exact category  $\text{End}(\mathcal{C})$ . Clearly this construction is functorial in exact functors.

Given an ordinary ring  $A$ , let  $\mathbf{P}(A)$  denote the exact category of finitely generated projective (right)  $A$ -modules. Following Almkvist [1, 2] and Grayson [23], the  $K$ -theory of endomorphisms of  $A$  is defined as the  $K$ -groups associated to the spectrum  $\text{KEnd}(\mathbf{P}(A))$ .

We now extend the above construction to the setting of Waldhausen categories.

**Lemma 3.2.** *The category  $\text{End}(\mathcal{C})$  of endomorphisms in a Waldhausen category  $\mathcal{C}$  carries a canonical Waldhausen structure. A morphism is a cofibration (resp. a weak equivalence) in  $\text{End}(\mathcal{C})$  if its image under  $\text{End}(\mathcal{C}) \rightarrow \mathcal{C}$  is a cofibration (resp. a weak equivalence) in  $\mathcal{C}$ . Furthermore, if  $\mathcal{C}$  admits functorial factorization of weak cofibrations, then so does  $\text{End}(\mathcal{C})$ .*

*Proof.* Pushouts along cofibrations in  $\text{End}(\mathcal{C})$  are computed in  $\mathcal{C}$ , using the induced endomorphism on the pushout. Hence, pushouts along cofibrations exist in  $\text{End}(\mathcal{C})$  and the gluing axiom holds. The remaining properties are clear.  $\square$

Using Lemma 3.2, one can associate (as in the case of exact categories) to every Waldhausen category  $\mathcal{C}$  a well-defined algebraic  $K$ -theory spectrum  $\mathrm{KEnd}(\mathcal{C})$ . This construction is clearly functorial in exact functors of Waldhausen categories.

**Example 3.3.** A motivating example is the case in which  $\mathcal{C} = \mathrm{perf}(A)$  is the category of perfect complexes over an ordinary ring  $A$ . The weak equivalences are the quasi-isomorphisms and the cofibrations are the morphisms which admit retractions as morphisms of graded (right)  $A$ -modules (i.e., the degree-wise split monomorphisms). We have an equivalence of  $\infty$ -categories

$$\mathrm{N}(\mathrm{perf}(A))[W^{-1}] \simeq \mathrm{Perf}_{HA},$$

where  $HA$  is the associated ring spectrum. The category  $\mathrm{End}(\mathrm{perf}(A))$  admits an algebraic description. Specifically,  $\mathrm{End}(\mathrm{perf}(A))$  is the category of those complexes of  $A[t]$ -modules that are perfect as underlying complexes of  $A$ -modules. Note that  $\mathrm{End}(\mathrm{perf}(A)) \neq \mathrm{perf}(A[t])$  since perfect complexes of  $A[t]$ -modules tend not to be perfect as complexes of  $A$ -modules.

**Example 3.4.** Example 3.3 can be generalized to the case where  $\mathcal{C}$  is the category  $\mathrm{Perf}_R$  of perfect complexes over a ring spectrum  $R$ . When  $R$  is the Eilenberg-MacLane spectrum  $HA$  of an ordinary ring  $A$ , we recover Example 3.3. Note that in this case  $\mathrm{End}(\mathrm{Perf}_R)$  also admits an algebraic description. Specifically, an  $R$ -module  $M$  endowed with an endomorphism is precisely the same data as a module over  $R[\mathbb{N}] := R \wedge \Sigma_+^\infty \mathbb{N}$ . This follows from the fact that  $R \wedge \Sigma_+^\infty \mathbb{N}$  is the free  $R$ -algebra on one generator, so that a map to  $\mathrm{End}(M)$  is (by adjunction) just a map of  $R$ -modules  $M \rightarrow M$  (see [19, II.4.4] or [37, 3.10]). As a consequence, we denote by  $R[t]$  the tensor algebra  $R[\mathbb{N}]$ ; in particular,  $\mathbb{S}[t]$  is the tensor algebra on the sphere spectrum  $\mathbb{S}$ .

**3.2. Endomorphisms of  $\infty$ -categories and corepresentability.** The category  $\underline{\mathbb{N}}$  with one object and endomorphism monoid  $\mathbb{N}$  (under addition) is freely generated by a single nonidentity arrow. Sending the unique nonidentity arrow of  $\Delta^1$  to this generator, we obtain a well-defined map  $\Delta^1/\partial\Delta^1 \rightarrow \mathrm{N}(\underline{\mathbb{N}})$ , which is a weak equivalence of simplicial sets in the Joyal model structure.

**Definition 3.5.** Let  $\mathcal{C} \in \mathrm{Cat}_\infty$  be an  $\infty$ -category.

- (1) The  $\infty$ -category  $\mathrm{End}(\mathcal{C})$  of endomorphisms in  $\mathcal{C}$  is the functor  $\infty$ -category  $\mathrm{End}(\mathcal{C}) := \mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C})$ . Note that as colimits in functor  $\infty$ -categories are computed pointwise,  $\mathrm{End}(\mathcal{C})$  has finite colimits if and only if  $\mathcal{C}$  has finite colimits. Moreover, if  $\mathcal{C}$  is stable, then  $\mathrm{End}(\mathcal{C})$  is also stable [32, 1.1.3.1].
- (2) If  $\mathcal{C}$  is a Waldhausen  $\infty$ -category, then the  $K$ -theory of endomorphisms of  $\mathcal{C}$  is defined as the (connective) spectrum  $\mathrm{KEnd}(\mathcal{C})$ ; see §2.6.

Clearly, Definition 3.5(ii) is functorial in exact functors  $\mathcal{C} \rightarrow \mathcal{D}$ . Therefore, we obtain a well-defined  $K$ -theory of endomorphisms functor

$$(3.6) \quad \mathrm{KEnd}: \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{S}_\infty$$

with values in the  $\infty$ -category of spectra. In this section we prove that (3.6) descends to the category  $\mathcal{M}_{\mathrm{add}}$  and becomes corepresentable there; see Theorem 3.9 below.

**Lemma 3.7.** *The  $\infty$ -category  $\mathrm{Perf}_{\mathbb{S}[t]} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$  is compact.*

*Proof.* Recall that there is an equivalence  $\mathrm{Cat}_\infty^{\mathrm{perf}} \simeq \mathrm{Pr}_{\mathrm{St},\omega}^{\mathrm{L}}$  induced by passage to the Ind-category [31, §5.5.7], where  $\mathrm{Pr}_{\mathrm{St},\omega}^{\mathrm{L}}$  is the  $\infty$ -category of compactly generated stable  $\infty$ -categories. Thus, it suffices to show that the  $\infty$ -category of  $\mathbb{S}[t]$ -modules is compact as a compactly generated stable  $\infty$ -category, i.e., as an object of  $\mathrm{Pr}_{\mathrm{St},\omega}^{\mathrm{L}}$ . By Proposition 3.5 of [3], it suffices to show that  $\mathbb{S}[t]$  is compact as an  $\mathbb{S}$ -algebra, which is clear because it is free on one generator.  $\square$

**Proposition 3.8.** *Let  $\mathcal{A}$  be an object in  $\mathrm{Cat}_\infty^{\mathrm{perf}}$ . Then there is a natural equivalence*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{A}) \simeq \mathrm{End}(\mathcal{A}).$$

*Proof.* By the work of [8, §4], we know that  $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{A})$  can be described as the  $\infty$ -category associated to the spectral category  $\mathrm{rep}(\mathbb{S}[t], \tilde{\mathcal{A}})$  of right compact  $\mathbb{S}[t]$ - $\tilde{\mathcal{A}}$ -bimodules, where  $\tilde{\mathcal{A}}$  is a spectral lift of  $\mathcal{A}$ . An  $\mathbb{S}[t]$ - $\tilde{\mathcal{A}}$ -bimodule is the same thing as an  $\tilde{\mathcal{A}}$ -module with an endomorphism. Next, the condition of being right compact means that these are precisely the  $\tilde{\mathcal{A}}$ -modules with endomorphisms that are compact as  $\tilde{\mathcal{A}}$ -modules; i.e., this is the category  $\mathrm{End}(\mathrm{perf}(\tilde{\mathcal{A}}))$ .  $\square$

**Theorem 3.9.** *The above functor (3.6) is an additive invariant. Moreover, for every  $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$  there is a natural weak equivalence of spectra*

$$\mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \simeq \mathrm{KEnd}(\mathcal{C}).$$

*Proof.* The first claim follows from the second. We can verify the second claim as follows. Since by Lemma 3.7 the  $\infty$ -category  $\mathrm{Perf}_{\mathbb{S}[t]}$  is compact, it follows from [8, 7.13] that we have a natural equivalence of spectra

$$\mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}), \mathcal{U}_{\mathrm{add}}(\mathcal{A})) \simeq K\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{A}).$$

Now we apply Proposition 3.8. To complete the proof, we need to compare  $K(\mathrm{rep}(\mathbb{S}[t], \tilde{\mathcal{A}}))$  and  $\mathrm{KEnd}(\mathrm{perf}(\tilde{\mathcal{A}}))$ . Proposition 3.10 below allows us to use the Waldhausen models for the  $K$ -theory spectra. The Waldhausen structure on  $\mathrm{End}(\tilde{\mathcal{A}})$  is inherited from the forgetful functor  $\mathrm{End}(\mathrm{perf}(\tilde{\mathcal{A}})) \rightarrow \mathrm{perf}(\tilde{\mathcal{A}})$ ; cofibrations are maps which are cofibrations in  $\mathrm{perf}(\tilde{\mathcal{A}})$ . On the other hand, the Waldhausen structure on  $\mathrm{rep}(\mathbb{S}[t], \tilde{\mathcal{A}})$  is given by maps which are cofibrations of bimodules.

The identity functor  $\mathrm{rep}(\mathbb{S}[t], \tilde{\mathcal{A}}) \rightarrow \mathrm{End}(\mathrm{perf}(\tilde{\mathcal{A}}))$  is exact and evidently induces an equivalence on homotopy categories. Since in both Waldhausen structures all maps are weak cofibrations and we have functorial factorization, the generalized version of the approximation theorem (e.g., see [12, 1.1]) implies that this exact functor induces an equivalence on  $K$ -theory spectra.  $\square$

**3.3. Agreement properties.** We start by showing that Definition 3.5 subsumes the definition of  $K$ -theory of endomorphisms in the setting of Waldhausen categories.

**Proposition 3.10.** *Let  $\mathcal{C}$  be a DHKS-saturated Waldhausen category that admits functorial factorization and  $\mathrm{NC}[W^{-1}]$  the associated  $\infty$ -category. Then there is a Waldhausen category  $\mathcal{C}'$  that is DK-equivalent to  $\mathcal{C}$  and a natural zig-zag of weak equivalences of spectra between  $\mathrm{KEnd}(\mathcal{C}')$  and  $\mathrm{KEnd}(\mathrm{NC}'[W^{-1}])$ .*

*Proof.* First, assume that  $\mathcal{C}$  is a full subcategory of the cofibrant objects in a combinatorial pointed model category  $\mathcal{A}$ . Then [32, 1.3.4.25] implies that there is an equivalence

$$\mathrm{N}(\mathrm{Fun}(\mathbb{N}, \mathcal{C}))[W^{-1}] \simeq \mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathrm{NC}[W^{-1}])$$

and therefore an equivalence of spectra

$$(3.11) \quad K(\mathbf{N}(\mathrm{Fun}(\mathbb{N}, \mathcal{C})[W^{-1}])) \simeq K \mathrm{End}(\mathbf{N}\mathcal{C}[W^{-1}]).$$

Although the Waldhausen structure on  $\mathrm{End}(\mathcal{C})$  need not arise as induced from a model structure by restriction to the cofibrant objects, since  $\mathrm{End}(\mathcal{C})$  inherits functorial factorization from  $\mathcal{C}$  we can reduce to this case [8, 7.7]. Next, using [8, 7.10], which compares the  $K$ -theory of a Waldhausen category to the  $K$ -theory of the underlying  $\infty$ -category, we find that the left-hand side of (3.11) is equivalent to the Waldhausen  $K$ -theory  $K\mathrm{End}(\mathcal{C})$ . When  $\mathcal{C}$  is an arbitrary Waldhausen category that admits functorial factorization and is DHKS-saturated, we can again use [8, 7.10] to reduce to the case of a full subcategory of the cofibrant objects in a pointed model category.  $\square$

The following result relates the classical definition of the  $K$ -theory of endomorphisms of an ordinary ring  $A$  with the definition given herein for the associated Eilenberg-Mac Lane spectrum  $HA$ .

**Theorem 3.12.** *Let  $A$  be a regular ring. There exists a canonical zig-zag of weak equivalences of spectra between the  $K$ -theory spectrum  $K\mathrm{End}(\mathbf{P}(A))$  of the exact category  $\mathrm{End}(\mathbf{P}(A))$  and the  $K$ -theory spectrum  $K\mathrm{End}(\mathrm{Perf}_{HA})$  of the  $\infty$ -category  $\mathrm{End}(\mathrm{Perf}_{HA})$ .*

*Proof.* We apply Barwick's theory of the  $K$ -theory of exact  $\infty$ -categories [5]. To explain our argument, recall that a stable  $\infty$ -category  $\mathcal{C}$  admits a  $t$ -structure if  $\mathrm{Ho}(\mathcal{C})$  admits a  $t$ -structure as a triangulated category. The stable  $\infty$ -category has a bounded  $t$ -structure when the  $t$ -structure on  $\mathrm{Ho}(\mathcal{C})$  is bounded. A full subcategory  $\mathcal{C}' \subset \mathcal{C}$  of a stable  $\infty$ -category is thick if  $\mathrm{Ho}(\mathcal{C}')$  is a thick subcategory of  $\mathrm{Ho}(\mathcal{C})$ . We will write that a thick subcategory  $\mathcal{C}' \in \mathcal{C}$  of a stable  $\infty$ -category with a  $t$ -structure is compatible with the  $t$ -structure if for any object  $x \in \mathcal{C}'$ , the truncations  $\tau_{\leq 0}x$  and  $\tau_{\geq 0}x$  are both objects of  $\mathcal{C}'$ . Barwick's result [5, 5.6.1] implies that if  $\mathcal{C}'$  is a thick subcategory of a small stable idempotent-complete  $\infty$ -category with a bounded  $t$ -structure such that  $\mathcal{C}'$  is compatible with the  $t$ -structure, then the algebraic  $K$ -theory of  $\mathcal{C}'$  is equivalent to the algebraic  $K$ -theory of the intersection of  $\mathcal{C}'$  and the heart of the  $t$ -structure on  $\mathcal{C}$  (regarded as an exact  $\infty$ -category).

Since Lemma 3.13 below implies that  $\mathrm{End}(\mathrm{Perf}_{HA})$  is a thick subcategory of a stable  $\infty$ -category with a bounded  $t$ -structure (and that  $\mathrm{End}(\mathrm{Perf}_{HA})$  is compatible with the  $t$ -structure), we can apply [5, 5.6.1] to compare  $K(\mathrm{End}(\mathrm{Perf}_{HA}))$  to the Quillen  $K$ -theory of the intersection of  $\mathrm{End}(\mathrm{Perf}_{HA})$  with the heart of the  $t$ -structure regarded as an exact  $\infty$ -category. As the intersection is the exact  $\infty$ -category associated to  $\mathrm{End}(\mathbf{M}(A))$  by Lemma 3.14, the comparison of [6, 4.6] and Quillen's resolution theorem completes the argument.  $\square$

We now establish that  $\mathrm{End}(\mathrm{Perf}_{HA})$  is a compatible thick subcategory of a small stable idempotent complete  $\infty$ -category with a bounded  $t$ -structure. Recall that a  $t$ -structure on a triangulated category  $\mathcal{C}$  is determined by a pair of full subcategories  $\mathcal{C}_{\leq 0}$  and  $\mathcal{C}_{\geq 0}$  such that:

- (1) For objects  $X, Y$  in  $\mathcal{C}$  such that  $X \in \mathcal{C}_{\geq 0}$  and  $Y \in \mathcal{C}_{\leq 0}$ ,  $\mathrm{Hom}_{\mathcal{C}}(X, Y[-1]) = 0$ .
- (2) There are inclusions  $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$  and  $\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$ .
- (3) For any object  $X$  in  $\mathcal{C}$ , there exists a distinguished triangle  $X' \rightarrow X \rightarrow X''$  such that  $X' \in \mathcal{C}_{\geq 0}$  and  $X'' \in \mathcal{C}_{\leq 0}[-1]$ .

A standard example of a  $t$ -structure is given by considering  $\mathrm{Ho}(\mathrm{Ch}^b(\mathcal{A}))$  for an abelian category  $\mathcal{A}$  (the derived category of bounded complexes) and defining  $\mathcal{C}_{\geq 0}$  and  $\mathcal{C}_{\leq 0}$  to be the complexes with nonnegative homology and nonpositive homology, respectively.

We write  $\mathcal{C}_{\leq n}$  for  $\mathcal{C}_{\leq 0}[n]$  and  $\mathcal{C}_{\geq m}$  for  $\mathcal{C}_{\geq 0}[m]$ . A  $t$ -structure is bounded if all objects in  $\mathcal{C}$  are contained in  $\mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq m}$  for some  $n$  and  $m$ . When  $\mathcal{A}$  is an abelian category, the standard  $t$ -structure on  $\mathrm{Ho}(\mathrm{Ch}^b(\mathcal{A}))$  is evidently bounded.

The  $\infty$ -category  $\mathrm{Mod}_{HA}$  is equivalent to the underlying  $\infty$ -category of the model category of  $HA$ -modules. Therefore,  $\mathrm{Mod}_{HA}$  is stable, idempotent-complete, and  $\mathrm{Ho}(\mathrm{Mod}_{HA}) \simeq \mathrm{Ho}(\mathrm{Ch}(A))$ . In the following lemma, let  $\mathrm{Mod}_{HA}^b$  denote the full subcategory of  $\mathrm{Mod}_{HA}$  on those objects that are isomorphic in  $\mathrm{Ho}(\mathrm{Ch}(A))$  to objects of  $\mathrm{Ho}(\mathrm{Ch}^b(A))$ . Observe that  $\mathrm{Mod}_{HA}^b$  is stable and idempotent-complete. We can regard it as small by imposing a cardinality bound on the spaces in the spectra, as in [11, 1.7].

**Lemma 3.13.** *Let  $A$  be an ordinary ring. Then the  $\infty$ -category  $\mathrm{Mod}_{HA}^b$  admits a bounded  $t$ -structure. When  $A$  is regular, the  $\infty$ -category  $\mathrm{End}(\mathrm{Perf}_{HA})$  is a stable subcategory of  $\mathrm{End}(\mathrm{Mod}_{HA}^b)$  that is compatible with the  $t$ -structure.*

*Proof.* Essentially by definition,  $\mathrm{Mod}_{HA}^b$  has a bounded  $t$ -structure arising from the equivalence

$$\mathrm{Ho}(\mathrm{Mod}_{HA}^b) \simeq \mathrm{Ho}(\mathrm{Ch}^b(A)).$$

The  $t$ -structure on  $\mathrm{Ho}(\mathrm{Mod}_{HA}^b)$  induces one on  $\mathrm{Ho}(\mathrm{End}(\mathrm{Mod}_{HA}^b))$  using the full subcategories  $\mathrm{Ho}(\mathrm{End}(\mathrm{Mod}_{HA}^b))_{\geq 0}$  and  $\mathrm{Ho}(\mathrm{End}(\mathrm{Mod}_{HA}^b))_{\leq 0}$  determined by the forgetful functor. The only nontrivial condition to check is that any  $X$  can be fit into a triangle  $X' \rightarrow X \rightarrow X''$  where  $X'$  has nonnegative homology and  $X''$  has nonpositive homology. For any connective ring spectrum  $R$ , there is a functorial construction of the connective cover  $C$  on the category of  $R$ -modules, such that we have a natural transformation  $C \rightarrow \mathrm{id}$  [34, 4.2]. Since it is functorial, this construction passes to  $\mathrm{End}(\mathrm{Mod}_{HA})$ , and the associated cofiber sequence gives the required triangle.

Finally, it is clear that the induced  $t$ -structure on  $\mathrm{Ho}(\mathrm{End}(\mathrm{Mod}_{HA}^b))$  is bounded since the one on  $\mathrm{Ho}(\mathrm{Mod}_{HA}^b)$  is, and moreover that  $\mathrm{End}(\mathrm{Perf}_{HA})$  is a compatible subcategory of  $\mathrm{End}(\mathrm{Mod}_{HA}^b)$  since  $A$  is regular.  $\square$

Finally, we need to compute the intersection of the heart of the  $t$ -structure on  $\mathrm{End}(\mathrm{Mod}_{HA}^b)$  with  $\mathrm{End}(\mathrm{Perf}_{HA})$ . Recall that the intersection of two full subcategories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  of an  $\infty$ -category  $\mathcal{C}$  is simply the subcategory of  $\mathcal{C}$  spanned by the objects that are in the intersection of  $\mathrm{Ho}(\mathcal{C}_1)$  and  $\mathrm{Ho}(\mathcal{C}_2)$ .

**Lemma 3.14.** *The intersection of the heart of the  $t$ -structure on  $\mathrm{End}(\mathrm{Mod}_{HA}^b)$  with  $\mathrm{End}(\mathrm{Perf}_{HA})$  is the full subcategory of  $\mathrm{End}(\mathrm{Mod}(A))$  of compact objects that are concentrated in degree 0, which is equivalent to the exact  $\infty$ -category underlying  $\mathrm{End}(\mathbf{M}(A))$ , where  $\mathbf{M}(A)$  denotes the category of finitely presented modules.*

*Proof.* The first part of the claim is clear from the definitions. For the comparison to the exact  $\infty$ -category associated to  $\mathrm{End}(\mathbf{M}(A))$ , first observe that a compact object of  $\mathrm{End}(\mathrm{Mod}(A))$  which is concentrated in degree 0 is equivalent to an object in the image of the natural inclusion  $\mathrm{N}(\mathrm{End}(\mathbf{M})(A))[W^{-1}] \rightarrow \mathrm{End}(\mathrm{Mod}(A))$ . The identification with the exact category underlying  $\mathrm{End}(\mathbf{M}(A))$  now follows from the discussion in [6, 1.4].  $\square$

Another approach to Theorem 3.12 would involve analysis of the exact inclusion functor

$$\iota: \operatorname{End}(\operatorname{Ch}^b(\mathbf{P}(A))) \longrightarrow \operatorname{End}(\operatorname{perf}(A)).$$

We conclude the section with a technical lemma which gives a partial analysis of the homotopy category of  $\operatorname{End}(\operatorname{Ch}^b(\mathbf{P}(A)))$ .

**Lemma 3.15.** *The induced functor*

$$(3.16) \quad \iota: \operatorname{End}(\operatorname{Ch}^b(\mathbf{P}(A))) \longrightarrow \operatorname{End}(\operatorname{perf}(A))$$

*is homotopically essentially surjective, where we equip each side with the weak equivalences given by the underlying quasi-isomorphisms.*

*Proof.* We will decorate quasi-isomorphisms with the symbol  $\sim$ . Recall that we have fully-faithful inclusions  $\operatorname{Ch}^b(\mathbf{P}(A)) \hookrightarrow \operatorname{perf}(A) \hookrightarrow \operatorname{Ch}(A)$  and that  $\operatorname{Ch}(A)$  carries a projective Quillen model structure in which every object is fibrant [30, 2.3.11]. The associated homotopy category is the derived category  $\mathcal{D}(A)$  of  $A$ .

Let  $(M^\bullet, \alpha)$  be an object of  $\operatorname{End}(\operatorname{perf}(A))$ , i.e., a complex  $M^\bullet \in \operatorname{perf}(A)$  and an endomorphism  $\alpha$  of  $M^\bullet$ . Since  $M^\bullet \in \operatorname{perf}(A)$ ,  $M^\bullet$  has the homotopy type of a wedge summand of a finite cell complex of  $A$ -modules; that is, there exists an isomorphism in  $\mathcal{D}(A)$  between  $M^\bullet$  and a complex  $P^\bullet \in \operatorname{Ch}^b(\mathbf{P}(A))$ . All the objects of  $\operatorname{Ch}^b(\mathbf{P}(A))$  are cofibrant [30, 2.3.6], and so this isomorphism lifts to a quasi-isomorphism  $\theta: P^\bullet \xrightarrow{\sim} M^\bullet$ . Associated to  $\alpha$  we obtain then a well-defined endomorphism of  $P^\bullet$  in the derived category  $\mathcal{D}(A)$ . Since  $P^\bullet$  is cofibrant we can choose a representative  $\bar{\alpha}: P^\bullet \rightarrow P^\bullet$  of this endomorphism. We obtain then a square

$$(3.17) \quad \begin{array}{ccc} P^\bullet & \xrightarrow[\sim]{\theta} & M^\bullet \\ \bar{\alpha} \downarrow & & \downarrow \alpha \\ P^\bullet & \xrightarrow[\theta]{\sim} & M^\bullet \end{array}$$

which is commutative only in the derived category  $\mathcal{D}(A)$ . The proof will consist now of replacing the quasi-isomorphism  $\theta$  in (3.17) by a zig-zag of strictly commutative squares relating  $\bar{\alpha}$  to  $\alpha$ . Let

$$P^\bullet \xrightarrow{\sim} X^\bullet \xrightarrow{\sim} M^\bullet$$

be a factorization of  $\theta$  provided by the projective model structure. Note that  $X^\bullet$  is cofibrant since this is the case for  $P^\bullet$ . Moreover, it belongs to  $\operatorname{perf}(A)$  since it is quasi-isomorphic to  $M^\bullet$ . The lifting properties of the projective model structure furnish us morphisms  $\bar{\beta}$  and  $\beta$  making the following two diagrams commutative:

$$(3.18) \quad \begin{array}{ccc} P^\bullet & \xrightarrow{\sim} & X_\bullet \\ \bar{\alpha} \downarrow & & \downarrow \bar{\beta} \\ P^\bullet & \xrightarrow{\sim} & X^\bullet \end{array} \quad \begin{array}{ccc} X_\bullet & \xrightarrow{\sim} & M^\bullet \\ \beta \downarrow & & \downarrow \alpha \\ X^\bullet & \xrightarrow{\sim} & M^\bullet \end{array}$$

By combining the squares (3.17)-(3.18), we conclude then that the endomorphisms  $\bar{\beta}$  and  $\beta$  of  $X^\bullet$  agree in the derived category  $\mathcal{D}(A)$ . Since  $X^\bullet$  is a cofibrant object

there exists a cylinder object

$$X^\bullet \oplus X^\bullet \xrightarrow{[i_0 \ i_1]} \mathrm{Cyl}(X^\bullet) \xrightarrow{\sim} X^\bullet$$

and a morphism  $H$  making the following diagram commute:

$$\begin{array}{ccccc} X^\bullet & \xrightarrow{i_0} & \mathrm{Cyl}(X^\bullet) & \xleftarrow{i_1} & X^\bullet \\ & \searrow \bar{\beta} & \downarrow H & \swarrow \beta & \\ & & X^\bullet & & \end{array}$$

Note that since  $\mathrm{Cyl}(X^\bullet)$  is quasi-isomorphic to  $X$ , it also belongs to  $\mathrm{perf}(A)$ . Consider the following commutative solid diagram:

$$(3.19) \quad \begin{array}{ccccc} X^\bullet \oplus X^\bullet & \xrightarrow{\bar{\beta} \oplus \beta} & X^\bullet \oplus X^\bullet & \xrightarrow{[i_0 \ i_1]} & \mathrm{Cyl}(X^\bullet) \\ \downarrow [i_0 \ i_1] & & \searrow \tilde{H} & \nearrow & \downarrow \sim \\ \mathrm{Cyl}(X^\bullet) & \xrightarrow{H} & & & X^\bullet \end{array}$$

By the lifting properties of the projective model structure there exists a well-defined morphism  $\tilde{H}$  as above making both triangles of the diagram commute. Now, consider the following commutative diagram:

$$(3.20) \quad \begin{array}{ccccccc} P^\bullet & \xrightarrow{\sim} & X^\bullet & \xrightarrow{i_0} & \mathrm{Cyl}(X) & \xleftarrow{i_1} & X^\bullet \xrightarrow{\sim} M^\bullet \\ \downarrow \bar{\alpha} & & \downarrow \bar{\beta} & & \downarrow \tilde{H} & & \downarrow \beta \\ P^\bullet & \xrightarrow{\sim} & X^\bullet & \xrightarrow{i_0} & \mathrm{Cyl}(X) & \xleftarrow{i_1} & X^\bullet \xrightarrow{\sim} M^\bullet \end{array}$$

Note that the commutativity of the two interior squares is equivalent to the commutativity of the upper triangle in (3.19). The diagram (3.20) can then be interpreted as a zig-zag of quasi-isomorphisms in the category  $\mathrm{End}(\mathrm{perf}(A))$  relating  $(P^\bullet, \bar{\alpha})$  with  $(M^\bullet, \alpha)$ . As a consequence,  $(P^\bullet, \bar{\alpha})$  and  $(M^\bullet, \alpha)$  become isomorphic in the homotopy category  $\mathrm{Ho}(\mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A))))$ . Since  $(P^\bullet, \alpha)$  belongs to  $\mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A)))$  the proof is then finished.  $\square$

#### 4. ENDOMORPHISMS OF PROJECTIVE MODULES

In this section, we identify  $\pi_0 \mathrm{KEnd}(\mathrm{Perf}_R)$  with  $\pi_0 \mathrm{KEnd}(\mathrm{Perf}_{\pi_0 R})$  for a connective  $A_\infty$  ring spectrum  $R$ ; see Theorem 4.17. To do this, we apply Bondarko's formalism of weight structures [14], which axiomatize the "CW cell structures" that exist on the triangulated categories of modules over a connective ring spectrum.

We begin by rapidly reviewing the theory of free and projective modules over connective ring spectra; see [32, 8.2.2.4] for further details or [3, §2] for an exposition of the relevant results.

**Definition 4.1.** An  $R$ -module  $M$  is (*finite*) *free* if there exists a (finite) set  $I$  and an equivalence of  $R$ -modules  $R^{\oplus I} \simeq M$ . An  $R$ -module is *projective* if it is projective as an object of the  $\infty$ -category  $\mathrm{Mod}_R^{\geq 0}$  of connective  $R$ -modules; i.e., the functor

$$\mathrm{map}_R(P, -): \mathrm{Mod}_R^{\geq 0} \longrightarrow \mathcal{T}$$

commutes with geometric realizations of connective  $R$ -modules.



*Remark 4.2.* Definition 4.1 really only makes sense under connectivity hypotheses on  $R$ . The notion of shifted free module is only sensible if  $R$  is not periodic, that is, if the only integer  $n$  for which there exists an equivalence  $\Sigma^n R \rightarrow R$  is zero (of course, this property always holds for connective ring spectra, except when  $R \simeq 0$ ). Similarly, for an arbitrary ring spectrum  $R$ , there are no nontrivial projective objects of the  $\infty$ -category  $\mathrm{Mod}_R$  of all right  $R$ -modules (and of course the same is true in the  $\infty$ -category  $\mathrm{Mod}_{R^{\mathrm{op}}}$  of left  $R$ -modules). See the argument immediately following [3, 2.5] for details.

The following result (proved in [32, §8]) relates Definition 4.1 to other possible notions of projective  $R$ -modules.

**Proposition 4.3.** *Suppose that  $P$  is a connective  $R$ -module. Then the following are equivalent:*

- (1) *The  $R$ -module  $P$  is projective.*
- (2) *The  $R$ -module  $P$  is a retract of a free  $R$ -module.*
- (3) *The functor*

$$\mathrm{map}_R(P, -): \mathrm{Mod}_R^{\geq 0} \longrightarrow \mathcal{T}$$

*preserves surjections (i.e., morphisms which are surjective on  $\pi_0$ ).*

- (4) *Given a surjection (on  $\pi_0$ ) of (not necessarily connective)  $R$ -modules  $N \rightarrow M$  and any map  $P \rightarrow M$ , there exists a map  $g: P \rightarrow N$  such that the resulting diagram*

$$\begin{array}{ccc} & P & \\ \swarrow & & \searrow \\ N & \xrightarrow{\quad} & M \end{array}$$

*commutes in  $\mathrm{Mod}_R$ . (We may assume without loss of generality that  $M$  and  $N$  are connective, as  $\pi_0 \mathrm{map}(P, M) \cong \pi_0 \mathrm{map}(P, \tau_{\geq 0} M)$ .)*

Proposition 4.3 shows that projective modules over connective ring spectra behave in much the same way as projective modules over ordinary rings.

Note that finite projective modules are perfect, since finite free modules are perfect and retracts of compact objects are still compact. We write  $\mathrm{Proj}_R \subset \mathrm{Perf}_R$  for the full subcategory of  $\mathrm{Perf}_R$  consisting of the finite projective  $R$ -modules, and we write  $\mathrm{Free}_R \subset \mathrm{Perf}_R$  for the full subcategory of  $\mathrm{Perf}_R$  consisting of the finite free  $R$ -modules.

The following standard comparison provides a description of  $\mathrm{Proj}_R$  in terms of the discrete ring  $\pi_0 R$  (see, e.g., [3, 2.12] for a proof).

**Proposition 4.4.** *Let  $R$  be a connective ring spectrum. Then*

$$\pi_0: \mathrm{Ho}(\mathrm{Proj}_R) \longrightarrow \mathrm{Proj}_{\pi_0 R}$$

*is an equivalence of categories.*

We also have the following easy observations about the relationships of  $\mathrm{Free}_R$ ,  $\mathrm{Proj}_R$ , and  $\mathrm{Perf}_R$ .

**Lemma 4.5.** *The category  $\mathrm{Ho}(\mathrm{Perf}_R)$  is triangulated, with additive subcategories  $\mathrm{Ho}(\mathrm{Proj}_R)$  and  $\mathrm{Ho}(\mathrm{Free}_R)$ . The  $\infty$ -category  $\mathrm{Proj}_R$  is the idempotent completion of  $\mathrm{Free}_R$ , and similarly  $\mathrm{Ho}(\mathrm{Proj}_R)$  is the idempotent completion of  $\mathrm{Ho}(\mathrm{Free}_R)$ .*

Moreover, we have the following relationship between  $\mathrm{Free}_R$  and  $\mathrm{Perf}_R$ .

**Lemma 4.6.** *The  $\infty$ -category  $\mathrm{Perf}_R$  is the smallest idempotent-complete stable  $\infty$ -subcategory of  $\mathrm{Mod}_R$  containing  $\mathrm{Free}_R$ . Equivalently,  $\mathrm{Ho}(\mathrm{Perf}_R)$  is the smallest thick triangulated subcategory of  $\mathrm{Ho}(\mathrm{Mod}_R)$  containing  $\mathrm{Ho}(\mathrm{Free}_R)$ .*

*Proof.* Since  $\mathrm{Mod}_R$  is equivalent to the  $\infty$ -category underlying the model category of  $R$ -modules, we can deduce the result from the fact that  $\mathrm{Perf}_R$  is equivalent to the underlying  $\infty$ -category of the retracts of finite CW  $R$ -modules [19, III.7.9].  $\square$

Next, observe that since  $R$  is a connective ring spectrum,  $\mathrm{Free}_R$  is connective in the following sense.

**Lemma 4.7.** *For any objects  $x$  and  $y$  in  $\mathrm{Free}_R$  and  $n > 0$ , the derived mapping space  $\mathrm{Map}_{\mathrm{Mod}_R}(x, \Sigma^n y)$  is trivial.*

Bondarko's theory of weight structures now yields the following result.

**Theorem 4.8.** *For any connective ring spectrum  $R$ , the map*

$$i: K_0(\mathrm{Proj}_R) \longrightarrow K_0(\mathrm{Perf}_R),$$

*induced by the inclusion  $\mathrm{Proj}_R \rightarrow \mathrm{Perf}_R$ , is an isomorphism.*

*Proof.* By [14, 4.3.2 II], Lemmas 4.6 and 4.7 imply that there exists a unique weight structure on  $\mathrm{Ho}(\mathrm{Perf}_R)$  with heart  $\mathrm{Ho}(\mathrm{Proj}_R)$ . Next, [14, 5.3.1] now implies that there is an isomorphism  $K_0(\mathrm{Ho}(\mathrm{Perf}_R)) \cong K_0(\mathrm{Ho}(\mathrm{Proj}_R))$ . By Proposition 4.4, we have the further equivalence  $K_0(\mathrm{Ho}(\mathrm{Proj}_R)) \cong K_0(\pi_0(R))$ .  $\square$

We now make an analogous argument for  $\mathrm{End}(\mathrm{Perf}_R)$ , following the same general outline. First, recall that  $\mathrm{End}(\mathrm{Perf}_R)$  is stable since  $\mathrm{Perf}_R$  is stable [32, 1.1.3.1]. As a consequence,  $\mathrm{Ho}(\mathrm{End}(\mathrm{Perf}_R))$  is a triangulated category. Moreover,  $\mathrm{End}(\mathrm{Free}_R)$  and  $\mathrm{End}(\mathrm{Proj}_R)$  are additive subcategories of  $\mathrm{End}(\mathrm{Perf}_R)$  and correspondingly  $\mathrm{Ho}(\mathrm{End}(\mathrm{Free}_R))$  and  $\mathrm{Ho}(\mathrm{End}(\mathrm{Proj}_R))$  are additive subcategories of  $\mathrm{Ho}(\mathrm{End}(\mathrm{Perf}_R))$ . It is also clear that  $\mathrm{End}(\mathrm{Proj}_R)$  is the idempotent-completion of  $\mathrm{End}(\mathrm{Free}_R)$ .

We now establish the analogue of Lemma 4.6 in this context. The proof uses the notion of Tor-amplitude; see [25] or [42] for a discussion of Tor-amplitude in the setting of derived categories and [3, §2.4] for the analogous treatment in the setting of modules over a ring spectrum. Briefly, an  $R$ -module  $M$  has Tor-amplitude contained in  $[a, b]$  if for any  $\pi_0 R$ -module  $N$ ,  $H_i(M \wedge_{\pi_0 R} N) = 0$  except when  $i \in [a, b]$ .

**Proposition 4.9.** *The  $\infty$ -category  $\mathrm{End}(\mathrm{Perf}_R)$  is the smallest idempotent-complete stable  $\infty$ -subcategory of  $\mathrm{End}(\mathrm{Mod}_R)$  containing  $\mathrm{End}(\mathrm{Free}_R)$ . Equivalently, the triangulated category  $\mathrm{Ho}(\mathrm{End}(\mathrm{Perf}_R))$  is the smallest thick triangulated subcategory of  $\mathrm{Ho}(\mathrm{End}(\mathrm{Mod}_R))$  containing  $\mathrm{Ho}(\mathrm{End}(\mathrm{Free}_R))$ .*

*Proof.* It is enough to show that any object of  $\mathrm{End}(\mathrm{Perf}_R)$  can be built in finitely many steps from suspensions of objects of the form  $\mathrm{End}(\mathrm{Proj}_R)$ . The proof goes by induction on the length of the Tor-amplitude of a given object  $M$  of  $\mathrm{End}(\mathrm{Perf}_R)$ . If  $M$  has Tor-amplitude contained in  $[a, b]$  with  $a - b = 0$ , then  $M \simeq \Sigma^a P$  for some finite projective  $R$ -module  $P$ . Inductively, suppose that  $M$  has Tor-amplitude contained in an interval  $[a, b]$  of length not more than  $l$ . Let  $P$  be a finite projective and  $\Sigma^a P \rightarrow M \rightarrow N$  a cofiber sequence such that  $N$  has Tor-amplitude contained in  $[a + 1, b]$  ([3, 2.7]). Taking homotopy, the resulting exact sequence

$$\pi_0(P) \longrightarrow \pi_0(\Sigma^{-a} M) \longrightarrow \pi_0(\Sigma^{-a} N)$$

shows that  $P \rightarrow \Sigma^{-a}M$  is surjective on  $\pi_0$ , since  $\pi_0(\Sigma^{-a}N) \cong 0$  by Lemma 4.10 below. It follows from Proposition 4.3 that there exists an endomorphism  $e$  of  $P$  making the lower triangle in the diagram

$$\begin{array}{ccc} P & \longrightarrow & \Sigma^{-a}M \\ \downarrow & \searrow & \downarrow \\ P & \longrightarrow & \Sigma^{-a}M \end{array}$$

commute; that is, there is a map from  $\Sigma^a P \xrightarrow{\Sigma^a e} \Sigma^a P$  to  $M \xrightarrow{\alpha} M$  in  $\text{End}(\text{Perf}_R)$ . Writing  $\beta: N \rightarrow N$  for the cofiber of this map in  $\text{End}(\text{Perf}_R)$ , we see that  $N$  is a perfect  $R$ -module with Tor-amplitude in the interval  $[a+1, b]$ . Repeating this process gives a filtration

$$\begin{array}{ccccccc} M = M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_l \longrightarrow M_{l+1} \simeq 0 \\ \downarrow \alpha_0 & & \downarrow \alpha_1 & & & & \downarrow \alpha_l & \downarrow \alpha_{l+1} \\ M = M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_l \longrightarrow M_{l+1} \simeq 0 \end{array}$$

of  $M \xrightarrow{\alpha} M$  in  $\text{End}(\text{Perf}_R)$  in which  $\alpha_0 \simeq \alpha$  and  $\alpha_{l+1} \simeq 0$ , and the filtration quotients are of the form

$$\Sigma^{a+k+1} P_k \xrightarrow{\Sigma^{a+k} e_k} \Sigma^{a+k+1} P_k$$

for some  $P_k \xrightarrow{e_k} P_k$  in  $\text{End}(\text{Proj}_R)$ ,  $0 \leq k \leq l$ . □

**Lemma 4.10.** *Let  $R$  be a connective ring spectrum and let  $M$  be a perfect  $R$ -module with Tor-amplitude contained in  $[0, \infty]$ . Then  $M$  is connective.*

*Proof.* This follows from the convergent spectral sequence

$$\text{Tor}_p^{\pi_* R}(\pi_q M, \pi_0 R) \implies \pi_{p+q}(M \wedge_R H\pi_0 R);$$

in particular, if  $\pi_q M \neq 0$  for some  $q < 0$ , then  $\pi_q M \otimes_{\pi_* R} \pi_0 R \neq 0$  since  $R$  is connective, giving nonzero classes in  $\pi_q(M \wedge_R H\pi_0 R)$ . □

Next, we have the following analogue of Lemma 4.7.

**Lemma 4.11.** *For any objects  $x$  and  $y$  in  $\text{End}(\text{Free}_R)$  and  $n > 0$ , the derived mapping space  $\text{Map}_{\text{End}(\text{Mod}_R)}(x, \Sigma^n y)$  is trivial.*

*Proof.* This follows from the fact that the mapping spaces in  $\text{End}(\text{Mod}_R)$  can be computed as homotopy equalizers of associated mapping spaces in  $\text{Mod}_R$ . Since  $R$  is connective, these mapping spaces are all trivial by Lemma 4.7, and homotopy limits of contractible spaces are themselves contractible. □

The same argument as for Theorem 4.8 (using Proposition 4.9 and Lemma 4.11 in place of Lemmas 4.6 and 4.7, respectively) now can be applied to prove the following result. Here to define  $K_0(\text{End}(\text{Proj}_R)) = \pi_0 \text{KEnd}(\text{Proj}_R)$  we will specify the structure of an  $\infty$ -category with cofibrations on  $\text{End}(\text{Proj}_R)$ . First, we specify the cofibrations on  $\text{Proj}_R$  as the maps  $P \rightarrow P \amalg Q$  such that the cofiber (in  $\text{Perf}_R$ ) is  $Q$ . It is straightforward to check that this definition satisfies the conditions of Definition 2.1. Then we define a map to be a cofibration in  $\text{End}(\text{Proj}_R)$  if its image under the forgetful functor  $\text{End}(\text{Proj}_R) \rightarrow \text{Proj}_R$  is a cofibration.

**Theorem 4.12.** *For any connective ring spectrum  $R$ , the map*

$$i: K_0(\text{End}(\text{Proj}_R)) \longrightarrow K_0(\text{End}(\text{Perf}_R)),$$

*induced by the inclusion  $\text{Proj}_R \rightarrow \text{Perf}_R$ , is an isomorphism.*

Finally, we want to complete the computation by showing that  $K_0(\text{End}(\text{Proj}_R)) \cong K_0(\text{End}(\text{Proj}_{\pi_0 R}))$ . To do so, we require a technical lemma about rigidifying homotopy commutative triangles in an  $\infty$ -category  $\mathcal{C}$ .

**Lemma 4.13.** *Let  $\mathcal{C}$  be an  $\infty$ -category and let  $K$  be a 2-skeletal simplicial set. Then any diagram*

$$\sigma: K \longrightarrow \mathbf{N}(\text{Ho}(\mathcal{C}))$$

*lifts to a diagram  $\tau: K \rightarrow \mathcal{C}$  such that  $\eta \circ \tau = \sigma$ , where  $\eta: \mathcal{C} \rightarrow \mathbf{N}(\text{Ho}(\mathcal{C}))$  denotes the unit of the adjunction  $\text{Ho}: \text{Set}_\Delta \rightleftarrows \text{Cat}: \mathbf{N}$ .*

*Proof.* First suppose  $K = \Delta^2$  and that  $\mathcal{C} = \mathbf{N}(\mathcal{C}')$  for some fibrant simplicial category  $\mathcal{C}'$  such that  $\text{Ho}(\mathcal{C}') \cong \text{Ho}(\mathcal{C})$ . By adjunction, a 2-simplex of  $\mathbf{N}(\mathcal{C}')$  is a map  $\tau: \mathfrak{C}[\Delta^2] \rightarrow \mathcal{C}'$ , which is to say objects  $\tau_i$ ,  $0 \leq i \leq 2$ , maps  $\tau_{ji}: \tau_i \rightarrow \tau_j$ ,  $0 \leq i < j \leq 2$ , and a homotopy  $\tau_{210}: \tau_{20} \rightarrow \tau_{21} \circ \tau_{10}$  in  $\text{map}_{\mathcal{C}'}(\tau_0, \tau_2)$ . Since we're given a map  $\sigma: \mathfrak{C}[\Delta^2] \rightarrow \text{Ho}(\mathcal{C}')$ , we have homotopy classes of maps  $\sigma_{ji} \in \pi_0 \text{map}_{\mathcal{C}}(\sigma_i, \sigma_j)$  such that  $\sigma_{20} = \sigma_{21} \circ \sigma_{10}$ . Taking  $\tau_i = \sigma_i$ , we may choose representative  $\tau_{ji}: \tau_i \rightarrow \tau_j$  of  $\sigma_{ji}$ , and as the two resulting maps  $\tau_{20}$  and  $\tau_{21} \circ \tau_{10}$  from  $\tau_0$  to  $\tau_2$  are homotopic, we may also choose a 1-simplex  $\tau_{210}: \Delta^1 \rightarrow \text{map}_{\mathcal{C}'}(\tau_0, \tau_2)$  realizing this.

For the general case, note that there exists a categorical equivalence

$$f: \mathbf{N}(\mathfrak{C}[\mathcal{C}]^{\text{fib}}) \longrightarrow \mathcal{C}$$

which we may suppose is an isomorphism on homotopy categories. First lift the 1-skeleton  $\text{sk}_1 K$  to  $\mathbf{N}(\mathfrak{C}[\mathcal{C}]^{\text{fib}})$  by choosing a representative for homotopy classes of arrows in  $\mathcal{C}$ , and then extend this to the 2-skeleton by choosing lifts of each 2-simplex compatibly with the chosen lifts on the boundary. Composing with  $f$  then gives the desired lift to  $\mathcal{C}$ .  $\square$

**Corollary 4.14.** *Let  $\mathcal{C}$  be an  $\infty$ -category. Then the canonical map*

$$\text{Ho}(\text{End}(\mathcal{C})) \longrightarrow \text{End}(\text{Ho}(\mathcal{C}))$$

*is surjective on equivalence classes of arrows.*

*Proof.* Since  $\Delta^1 \times \Delta^1 / \partial \Delta^1$  is 2-skeletal, the map

$$\pi_0 \text{map}(\Delta^1 \times \Delta^1 / \partial \Delta^1, \mathcal{C}) \longrightarrow \pi_0 \text{map}(\Delta^1 \times \Delta^1 / \partial \Delta^1, \mathbf{N}(\text{Ho}(\mathcal{C})))$$

is surjective by Lemma 4.13. But, by adjunction, the source is isomorphic to the set of equivalence classes of arrows in  $\text{Ho}(\text{End}(\mathcal{C}))$ , and the target is isomorphic to the set of equivalence classes of arrows in  $\text{End}(\text{Ho}(\mathcal{C}))$ .  $\square$

**Proposition 4.15.** *If  $R$  is a connective ring spectrum, then the canonical functor*

$$i: K_0(\text{End}(\text{Proj}_R)) \longrightarrow K_0(\text{End}(\text{Ho}(\text{Proj}_R)))$$

*is an isomorphism.*

*Proof.* Using the presentation for  $K_0$  of (2.7), we first observe that  $i$  is surjective, as

$$\mathrm{End}(\mathrm{Proj}_R)^\simeq \longrightarrow \mathrm{N}(\mathrm{End}(\mathrm{Ho}(\mathrm{Proj}_R)))^\simeq$$

is surjective on  $\pi_0$ . To see that  $i$  is also injective, we must show that any exact sequence in  $\mathrm{End}(\mathrm{Ho}(\mathrm{Proj}_R))$  lifts to an exact sequence in  $\mathrm{End}(\mathrm{Proj}_R)$ . Since an exact sequence in  $\mathrm{End}(\mathrm{Proj}_R)$  is in particular a cofiber sequence, any exact sequence in  $S_2^\infty(\mathrm{End}(\mathrm{Proj}_R))$  is determined (up to contractible ambiguity) by a suitable arrow  $\Delta^1 \rightarrow \mathrm{End}(\mathrm{Proj}_R)$ . Thus, the vertical fibers in the commutative square

$$\begin{array}{ccc} S_2^\infty(\mathrm{End}(\mathrm{Proj}_R))_{\mathrm{iso}} & \longrightarrow & S_2^\infty(\mathrm{End}(\mathrm{N Ho}(\mathrm{Proj}_R)))_{\mathrm{iso}} \\ \downarrow & & \downarrow \\ \mathrm{map}(\Delta^1, \mathrm{End}(\mathrm{Proj}_R)) & \longrightarrow & \mathrm{map}(\Delta^1, \mathrm{N Ho}(\mathrm{Proj}_R)), \end{array}$$

in which the vertical maps are the restrictions along  $\Delta^{\{0,1\}} \rightarrow \Delta^2$ , are contractible, so the diagram is cartesian. But the bottom horizontal map is surjective on  $\pi_0$  by Corollary 4.14, so the top horizontal map must be surjective on  $\pi_0$  as well.  $\square$

**Corollary 4.16.** *The map*

$$\mathrm{End Ho}(\mathrm{Proj}_R) \longrightarrow \mathrm{End Proj}_{\pi_0 R}$$

*induced by the equivalence  $\pi_0: \mathrm{Ho}(\mathrm{Proj}_R) \simeq \mathrm{Proj}_{\pi_0 R}$  is a  $K_0$ -isomorphism.*

*Proof.* This is immediate from Proposition 4.15.  $\square$

Finally, assembling the comparisons of Corollary 4.16, Proposition 4.15, and Theorem 4.12, we obtain the following:

**Theorem 4.17.** *For every connective ring spectrum  $R$  one has an isomorphism*

$$K_0(\mathrm{End}(\mathrm{Perf}_R)) \cong K_0(\mathrm{End}(\mathbf{P}(\pi_0(R)))) = K_0(\mathrm{End}(\pi_0 R))$$

*of abelian groups.*

## 5. NATURAL OPERATIONS

In this section we classify all the functorial operations on  $\mathrm{KEnd}$ , i.e., the natural transformation  $\mathrm{KEnd}(-) \rightarrow \mathrm{KEnd}(-)$ ; see Theorem 5.7. The constructions of §3.1 are functorial on exact categories and hence (after the usual fixes associated to the fact that the passage from rings to exact categories of modules is only a pseudo-functor; e.g., see [12, 9.1]) give rise to a well-defined functor

$$(5.1) \quad \mathrm{KEnd}(\mathbf{P}(-)): \mathrm{Rings} \longrightarrow \mathcal{S}$$

from ordinary rings to symmetric spectra. In particular, we have the classical functor

$$(5.2) \quad K_0(\mathrm{End}(\mathbf{P}(-))): \mathrm{Rings} \longrightarrow \mathrm{Ab}$$

from ordinary rings to abelian groups. As explained by Almkvist in [1, page 339], an interesting problem is the computation of all the functorial operations of the above functors; see also [40], [39, §1]. This problem was solved by Hazewinkel [26] for  $K_0 \operatorname{End}(-)$ , and we explain the solution for  $\operatorname{KEnd}$  in this section.

Classical examples of such operations are given by the Frobenius operations  $F_n: [(M, \alpha)] \mapsto [(M, \alpha^n)]$ ,  $n \geq 0$ , and by the Verschiebung operations  $V_n: [(M, \alpha)] \mapsto [(M^{\oplus n}, V_n(\alpha))]$ , where

$$(5.3) \quad V_n(\alpha) := \begin{bmatrix} 0 & \cdots & \cdots & 0 & (-1)^{n+1}\alpha \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(n \times n)}$$

These natural operations can be generalized to the  $\infty$ -categorical setting as follows:

**Definition 5.4** (Frobenius). Let  $f_n$  be the endofunctor of  $\underline{\mathbb{N}}$  induced by the monoid map  $n: \mathbb{N} \rightarrow \mathbb{N}$  which sends  $m$  to  $nm$ . Using the equivalence

$$\operatorname{Fun}(\operatorname{N}(\underline{\mathbb{N}}), \mathcal{A}) \longrightarrow \operatorname{Fun}(\Delta^1/\partial\Delta^1, \mathcal{A})$$

induced by the categorical equivalence  $\Delta^1/\partial\Delta^1 \rightarrow \operatorname{N}(\underline{\mathbb{N}})$ , one obtains by precomposition with  $f_n$  an exact functor  $f_n^*: \operatorname{End}(\mathcal{A}) \rightarrow \operatorname{End}(\mathcal{A})$  and consequently a map of spectra  $\operatorname{KEnd}(\mathcal{A}) \rightarrow \operatorname{KEnd}(\mathcal{A})$ . This construction is natural on  $\mathcal{A}$  and hence gives rise to a natural transformation  $F_n: \operatorname{KEnd} \rightarrow \operatorname{KEnd}$  of the functor (3.6) that we call the  $n^{\text{th}}$ -Frobenius operation.

For an  $\infty$ -category  $\mathcal{C} \in \operatorname{Cat}_{\infty}$ , there is a natural functor  $\mathcal{C} \rightarrow \operatorname{End}(\mathcal{C})$ , induced by projection  $\Delta^1/\partial\Delta^1 \rightarrow \Delta^0$ , that takes  $x$  to  $\operatorname{id}: x \rightarrow x$ . By precomposing with the forgetful functor  $\operatorname{End}(\mathcal{C}) \rightarrow \mathcal{C}$ , we obtain the composite functor

$$(5.5) \quad \iota: \operatorname{End}(\mathcal{C}) \longrightarrow \mathcal{C} \longrightarrow \operatorname{End}(\mathcal{C})$$

which sends the endomorphism  $\alpha: x \rightarrow x$  to  $\operatorname{id}: x \rightarrow x$ . Moreover, given an  $\infty$ -category  $\mathcal{D}$  with finite coproducts and given functors  $f_i: \mathcal{C} \rightarrow \mathcal{D}$  we can construct a functor  $\coprod f_i: \mathcal{C} \rightarrow \mathcal{D}$  as the composite

$$\mathcal{C} \longrightarrow \prod_i \mathcal{C} \xrightarrow{\coprod f_i} \prod_i \mathcal{D} \longrightarrow \mathcal{D},$$

where the last map is a choice of functorial coproduct. Similarly, given  $\tau \in \Sigma_n$ , we can permute the factors of this coproduct by  $\tau$ . We now use these constructions to generalize the Verschiebung:

**Definition 5.6** (Verschiebung). For each  $\infty$ -category  $\mathcal{A} \in \operatorname{Cat}_{\infty}^{\operatorname{perf}}$  there is an endofunctor on  $\operatorname{End}(\mathcal{A})$  defined by applying the cyclic permutation to the coproduct of  $(n-1)$  copies of  $\iota$  and one copy of  $(-1)^{n+1} \operatorname{id}$ . This functor gives rise to a natural transformation  $\operatorname{End}(\mathcal{A}) \rightarrow \operatorname{End}(\mathcal{A})$  and hence a natural transformation  $V_n: \operatorname{KEnd} \rightarrow \operatorname{KEnd}$  that we call the  $n^{\text{th}}$ -Verschiebung operation.

We now explain our solution for  $\mathbf{KEnd}$  to the problem stated by Almkvist, generalizing the solution for  $K_0\mathbf{End}$  given by Hazewinkel [26]. In the following, let

$$W_0(\mathbb{Z}[t]) := \left\{ \frac{1 + p_1(t)r + \cdots + p_i(t)r^i + \cdots + p_n(t)r^n}{1 + q_1(t)r + \cdots + q_j(t)r^j + \cdots + q_m(t)r^m} \mid p_i(t), q_j(t) \in \mathbb{Z}[t] \right\}$$

denote the multiplicative group of fractions of polynomials in the variable  $r$  with coefficients in  $\mathbb{Z}[t]$  and constant term 1.

**Theorem 5.7.** *There is a canonical equivalence of spectra between the spectrum  $\mathrm{Nat}(\mathbf{KEnd}, \mathbf{KEnd})$  of natural transformations of the functor (3.6) and the spectrum  $\mathbf{KEnd}(\mathrm{Perf}_{\mathbb{S}[t]})$ . In particular, the abelian group  $\pi_0 \mathrm{Nat}(\mathbf{KEnd}, \mathbf{KEnd})$  of natural transformations up to homotopy is isomorphic to*

$$K_0\mathbf{End}(\mathrm{Perf}_{\mathbb{S}[t]}) \simeq K_0\mathbf{End}(\mathbf{P}(\pi_0\mathbb{S}[t])) \simeq K_0\mathbf{End}(\mathbf{P}(\mathbb{Z}[t])) \simeq \mathbb{Z} \oplus W_0(\mathbb{Z}[t]).$$

Moreover, under these isomorphisms, the Frobenius operations  $F_n$  are identified with the elements  $(1, 1 + r^n t)$  and the Verschiebung operations  $V_n$  with the elements  $(n, 1 + rt^n)$ .

*Proof.* The natural equivalence of spectra  $\mathrm{Nat}(\mathbf{KEnd}, \mathbf{KEnd}) \simeq \mathbf{KEnd}(\mathrm{Perf}_{\mathbb{S}[t]})$  follows from Lemma 5.8 below (with  $E$  the functor (3.6)). The isomorphisms follow from Theorem 4.17 applied to  $R = \mathbb{S}[t]$ , from the equality  $\pi_0\mathbb{S}[t] = \mathbb{Z}[t]$ , and from Almkvist's isomorphism (see [2])

$$K_0\mathbf{End}(\mathbf{P}(A)) \xrightarrow{\cong} K_0(A) \oplus W_0(A) \quad (M, \alpha) \mapsto ([M], \det(\mathrm{Id} + \alpha r))$$

applied to  $A = \mathbb{Z}[t]$ . The identifications of  $F_n$  and  $V_n$  as the elements in question also follow from the preceding computation and Theorem 3.12; on  $\pi_0$ , the operations of Definitions 5.4 and 5.6 give rise to the classical operations on the  $K$ -theory of endomorphisms. Specifically, on passage to  $K_0$ , it is clear that the operation of Definition 5.4 takes the class  $[M, \alpha]$  to  $[M, \alpha^n]$ . Moreover, since for any connective ring spectrum  $R$  and compact  $R$ -module  $M$ ,  $\mathrm{Map}_R(\vee_n M, \vee_n M) \simeq \prod_n \coprod_n \mathrm{Map}_R(M, M)$ , on passage to  $K_0$  the operation of Definition 5.6 gives rise to the matrix specified above in equation (5.3). Now Theorem 3.12 coupled with Almkvist's identification of these operations in terms of the isomorphism (5.8) establishes the desired comparison.  $\square$

**Lemma 5.8.** *For every additive invariant  $E: \mathrm{Cat}_{\infty}^{\mathrm{perf}} \rightarrow \mathcal{S}_{\infty}$  there is a natural equivalence of spectra*

$$(5.9) \quad \mathrm{Nat}(\mathbf{KEnd}, E) \longrightarrow E(\mathrm{Perf}_{\mathbb{S}[t]}).$$

*Proof.* As shown in Theorem 3.9(i), the functor  $\mathbf{KEnd}: \mathrm{Cat}_{\infty}^{\mathrm{perf}} \rightarrow \mathcal{S}_{\infty}$  is an additive invariant. Hence, by equivalence (1.2) one obtains well-defined colimit preserving functors

$$\overline{\mathbf{KEnd}}, \overline{E}: \mathcal{M}_{\mathrm{add}} \longrightarrow \mathcal{S}_{\infty}$$

satisfying  $\overline{\mathbf{KEnd}} \circ \mathcal{U}_{\mathrm{add}} \simeq \mathbf{KEnd}$  and  $\overline{E} \circ \mathcal{U}_{\mathrm{add}} \simeq E$ , as well as a natural equivalence of spectra

$$(5.10) \quad \mathrm{Nat}(\mathbf{KEnd}, E) \longrightarrow \mathrm{Nat}(\overline{\mathbf{KEnd}}, \overline{E}).$$

By Theorem 3.9(ii) the functor  $\overline{\mathbf{K}\mathbf{End}}$  is corepresented in  $\mathcal{M}_{\text{add}}$  by the noncommutative motive  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]})$ . Hence, the  $\infty$ -categorical version of the Yoneda lemma [31, §5.1.3] provides an equivalence of spectra

$$(5.11) \quad \text{Nat}(\overline{\mathbf{K}\mathbf{End}}, \overline{E}) \longrightarrow \overline{E}(\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]}) = E(\text{Perf}_{\mathbb{S}[t]}).$$

By combining (5.10) with (5.11) we obtain then the above equivalence (5.9).  $\square$

## 6. (RATIONAL) WITT VECTORS

Witt vectors were introduced in the thirties by E. Witt [44]. Given a commutative ring  $A$ , the *Witt ring*  $W(A)$  of  $A$  is the abelian group of all power series of the form  $1 + a_1r + a_2r^2 + \cdots$ , with  $a_i \in A$ , endowed with the multiplication  $*$  determined by the equality  $(1 - a_1r) * (1 - a_2r) = (1 - a_1a_2r)$ . The *rational* Witt ring  $W_0(A)$  of  $A$  consists of the elements which are quotients of polynomials, that is, those of the form

$$\left\{ \frac{1 + a_1r + \cdots + a_i r^i + \cdots + a_m r^m}{1 + b_1r + \cdots + b_j r^j + \cdots + b_n r^n} \mid a_i, b_j \in A \right\} \subset W(A);$$

consult [27] for further details.

Recall from [8, §2.3] that the category  $\text{Cat}_{\infty}^{\text{perf}}$  carries a symmetric monoidal structure in which the symmetric monoidal product  $- \otimes^{\vee} -$  is characterized by the property that functors out of  $\mathcal{A} \otimes^{\vee} \mathcal{B}$  are in correspondence with functors out of the product  $\mathcal{A} \times \mathcal{B}$  which preserve colimits in each variable. The  $\otimes^{\vee}$ -unit is the  $\infty$ -category  $\text{Perf}_{\mathbb{S}}$ . In this section, we will be working with coalgebra objects in symmetric monoidal  $\infty$ -categories; we define these simply to be algebras in the opposite  $\infty$ -category.

**Proposition 6.1.** *Let  $M$  be a monoid in the  $\infty$ -category of spaces. Then the  $\infty$ -category  $\text{Perf}_{\mathbb{S}[M]}$  of perfect modules for the monoid-ring  $\mathbb{S}[M]$  carries a canonical counital, coassociative, and cocommutative coalgebra structure in  $\text{Cat}_{\infty}^{\text{perf}}$ .*

*Proof.* First recall that if  $\mathcal{C}^{\otimes}$  is a symmetric monoidal  $\infty$ -category in which the symmetric monoidal product is the coproduct, then the projection  $\text{CAlg}(\mathcal{C}^{\otimes}) \rightarrow \mathcal{C}$  is an equivalence. Taking  $\mathcal{C}$  to be the opposite of the  $\infty$ -category of  $A_{\infty}$ -spaces, we see that  $M$  has a coassociative and cocommutative coalgebra structure, so that  $\mathbb{S}[M] \simeq \Sigma_+^{\infty} M$  is a coassociative and cocommutative object in  $A_{\infty}$ -spectra. Now, we note that the “one-object spectral category” functor  $\text{Alg}_{\mathbb{S}} \rightarrow \text{N}(\text{Cat}_{\mathbb{S}})[W^{-1}]$  extends to a symmetric monoidal functor  $\text{Alg}_{\mathbb{S}}^{\otimes} \rightarrow \text{N}(\text{Cat}_{\mathbb{S}})[W^{-1}]^{\otimes}$ , and according to [9, §3],  $\text{Cat}_{\infty}^{\text{perf}\otimes}$  is a symmetric monoidal localization of  $\text{N}(\text{Cat}_{\mathbb{S}})[W^{-1}]^{\otimes}$ . Thus  $\text{Perf}_{\mathbb{S}[M]}$  inherits a canonical coassociative and cocommutative coalgebra structure.  $\square$

Specializing to the case in which  $M = \mathbb{N}$ , the free  $A_{\infty}$ -monoid on one generator, this amounts to saying that the diagonal  $\Delta: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  and the projection  $\mathbb{N} \rightarrow *$  are monoid maps. Applying  $\Sigma_+^{\infty}$ , we obtain ring maps

$$\Delta: \mathbb{S}[t] \xrightarrow{t \mapsto t \wedge t} \mathbb{S}[t] \wedge \mathbb{S}[t], \quad \epsilon: \mathbb{S}[t] \xrightarrow{t=1} \mathbb{S}.$$



As proved in [9, §4], the category of noncommutative motives  $\mathcal{M}_{\text{add}}$  carries a symmetric monoidal structure making the universal additive invariant  $\mathcal{U}_{\text{add}}: \text{Cat}_{\infty}^{\text{perf}} \rightarrow \mathcal{M}_{\text{add}}$  symmetric monoidal. Hence, by Proposition 6.1, the noncommutative motive  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]})$  becomes a counital coassociative coalgebra in  $\mathcal{M}_{\text{add}}$ . As a consequence, we obtain the following result:

**Proposition 6.2.** *The functor*

$$\text{Map}(\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]}), -): \mathcal{M}_{\text{add}} \longrightarrow \mathcal{S}_{\infty}$$

*lifts canonically to a lax symmetric monoidal functor.*

*Proof.* Recall that if  $\mathcal{C}^{\otimes}$  is a symmetric monoidal  $\infty$ -category, then the Yoneda embedding  $\mathcal{C} \rightarrow \text{Pre}(\mathcal{C})$  extends to a symmetric monoidal functor  $\mathcal{C}^{\otimes} \rightarrow \text{Pre}(\mathcal{C})^{\otimes}$ , where the symmetric monoidal structure on the  $\infty$ -category of presheaves of spaces on  $\mathcal{C}$  is given by the Day convolution [32, 4.8.1.12]. If additionally  $\mathcal{C}$  is pointed, then passing to  $\infty$ -categories of pointed objects yields a symmetric monoidal pointed Yoneda embedding

$$\mathcal{C}^{\otimes} \simeq \mathcal{C}_{*}^{\otimes} \longrightarrow \text{Pre}(\mathcal{C})_{*}^{\otimes} \simeq \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{T}_{*})^{\otimes}$$

where the  $\infty$ -category of pointed presheaves of spaces is symmetric monoidal with respect to the smash product. When  $\mathcal{C}$  is stable, stabilizing by passing to  $\infty$ -categories of spectrum objects [32, §1.4.2] yields a symmetric monoidal spectral Yoneda embedding

$$\mathcal{C}^{\otimes} \simeq \text{Sp}(\mathcal{C}_{*})^{\otimes} \longrightarrow \text{Sp}(\text{Pre}(\mathcal{C})_{*})^{\otimes} \simeq \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}_{\infty})^{\otimes}$$

whose underlying functor is the spectral Yoneda embedding constructed in [8, 2.15]. Therefore, there is an induced functor

$$\text{CAlg}(\mathcal{C}^{\otimes}) \longrightarrow \text{CAlg}(\text{Pre}_{\mathcal{S}_{\infty}}(\mathcal{C})^{\otimes})$$

from commutative algebra objects in  $\mathcal{C}$  to commutative algebra objects in spectral presheaves on  $\mathcal{C}$ . Finally, since commutative algebra objects in the convolution symmetric monoidal structure are lax symmetric monoidal functors  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{S}_{\infty}$  by [21], a commutative algebra in  $\mathcal{C}^{\otimes}$  represents such a functor.

We now restrict to the case in which  $\mathcal{C}^{\otimes} = (\mathcal{M}_{\text{add}}^{\text{op}})^{\otimes}$ . (In fact, to avoid set-theoretic issues we implicitly restrict to the subcategory of  $\kappa$ -compact objects for a suitable cardinal  $\kappa$ .) Since  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]})$  is a counital, coassociative, and cocommutative coalgebra in  $\mathcal{M}_{\text{add}} \simeq \mathcal{C}^{\text{op}}$ , we conclude that the functor

$$\text{Map}(\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]}), -): \mathcal{M}_{\text{add}} \longrightarrow \mathcal{S}_{\infty}$$

corepresented by  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]})$  lifts canonically to a lax symmetric monoidal functor.  $\square$

**Theorem 6.3.** *The ring maps  $\mathbb{S} \xrightarrow{\iota} \mathbb{S}[t]$  and  $\mathbb{S}[t] \xrightarrow{t=0} \mathbb{S}$  give rise to a wedge sum decomposition  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]}) \simeq \mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}}) \vee \mathbb{W}_0$  of counital coassociative cocommutative coalgebras in  $\mathcal{M}_{\text{add}}$ . Moreover, for every ordinary commutative ring  $A$ , we have an isomorphism of commutative rings*

$$(6.4) \quad \pi_0 \text{Map}(\mathbb{W}_0, \mathcal{U}_{\text{add}}(\text{Perf}_{HA})) \simeq W_0(A).$$

*Proof.* Recall from Proposition 6.1 that  $\mathbb{S}[t] \simeq \mathbb{S}[\mathbb{N}]$  and  $\mathbb{S} \simeq \mathbb{S}[*]$  are counital, coassociative, and cocommutative coalgebras. Since the map  $* \rightarrow \mathbb{N}$  is a map of commutative monoids in sets, and hence in the  $\infty$ -category of spaces, applying  $\Sigma_{+}^{\infty}$  shows that  $\iota: \mathbb{S} \rightarrow \mathbb{S}[t]$  is a map of counital, coassociative, cocommutative

coalgebras in  $\text{Alg}_{\mathbb{S}}$ . The map  $(t = 0)$  is not of this form, however; rather, it is induced from the map  $\mathbb{N}_+ \rightarrow *_+$  of non-counital, coassociative, cocommutative coalgebras in pointed commutative monoids which sends  $n \in \mathbb{N}_+$  to the basepoint of  $*_+$  for  $n > 0$ , and  $0 \in \mathbb{N}$  to the nonbasepoint of  $*_+$ . Applying  $\Sigma^\infty$ , we deduce that  $(t = 0): \mathbb{S}[t] \rightarrow \mathbb{S}$  is a map of non-counital, coassociative, cocommutative coalgebras. Finally, the composite  $(t = 0) \circ \iota$  is equivalent to the identity of  $\mathbb{S}$ , as the composite  $*_+ \rightarrow \mathbb{N}_+ \rightarrow *_+$  is equal to the identity of  $*_+$ . It follows that the composition  $\iota \circ (t = 0)$  is an idempotent of  $\mathbb{S}[t]$  as a non-counital coalgebra; in particular,  $\mathbb{S}$  is a retract of  $\mathbb{S}[t]$ .

Applying the composite functor

$$(6.5) \quad \text{Cat}_\infty^{\text{ex}} \xrightarrow{(-)^{\text{perf}}} \text{Cat}_\infty^{\text{perf}} \xrightarrow{\mathcal{U}_{\text{add}}} \mathcal{M}_{\text{add}},$$

we obtain then a retraction of coalgebra objects

$$\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}}) \longrightarrow \mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]}) \longrightarrow \mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}}).$$

Since  $\mathcal{M}_{\text{add}}$  is idempotent complete, we may split an idempotent endomorphism  $e: M \rightarrow M$  in  $\mathcal{M}_{\text{add}}$  by taking the filtered colimit

$$M_0 \simeq \text{colim}\{M \xrightarrow{e} M \xrightarrow{e} M \xrightarrow{e} \dots\};$$

taking the fiber  $M_1 \rightarrow M \rightarrow M_0$ , we obtain a splitting  $M \simeq M_0 \vee M_1$  of  $M$ , where  $e$  restricts to the identity on  $M_0$  and zero on  $M_1$ . Lastly, we note that colimits in the  $\infty$ -category of coalgebra objects are computed in the underlying  $\infty$ -category. Putting all of this together, we see that we can decompose  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]})$  as a coproduct of  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}})$  together with the coassociative and cocommutative coalgebra object  $\mathbb{W}_0$  of  $\mathcal{M}_{\text{add}}$ . To see that  $\mathbb{W}_0$  is counital, we observe that by [32, §5.2.3] it suffices to produce a homotopy counit; the existence of such now follows from Proposition A.3 (more generally the arguments of the appendix provide a splitting on the level of the homotopy category).

The identification of  $\pi_0 \text{Map}(\mathbb{W}_0, \mathcal{U}_{\text{add}}(\text{Perf}_{HA})) \simeq W_0(A)$  follows from the same considerations as in the argument for Theorem 5.7. Specifically, we can identify  $\pi_0 \text{Map}(\text{Perf}_{\mathbb{S}[t]}, \mathcal{U}_{\text{add}}(\text{Perf}_{HA})) \cong K_0 \text{End}(\text{Perf}_{HA})$  as  $K_0(A) \oplus W_0(A)$ , and Almkvist's results [2, pages 2-3] imply that the maps that split off  $\mathbb{W}_0$  in the preceding argument split off the  $W_0(A)$  component on  $\pi_0$ . This splitting induces the stated commutative ring isomorphism since it is induced by the splitting  $\mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}[t]}) \simeq \mathcal{U}_{\text{add}}(\text{Perf}_{\mathbb{S}}) \vee \mathbb{W}_0$  of counital coassociative cocommutative coalgebras.  $\square$

Isomorphism (6.4) motivates the following definition:

**Definition 6.6.** The spectrum of rational Witt vectors of a small idempotent-complete stable  $\infty$ -category  $\mathcal{A} \in \text{Cat}_\infty^{\text{perf}}$  is defined as  $\text{Map}(\mathbb{W}_0, \mathcal{U}_{\text{add}}(\mathcal{A}))$ .

The argument for Proposition 6.2 and the fact that  $\mathcal{U}_{\text{add}}$  is symmetric monoidal [9, §4] yields the following corollary:

**Corollary 6.7.** *The mapping spectrum  $\text{Map}(\mathbb{W}_0, -)$  admits a canonical refinement to a lax symmetric monoidal functor  $\mathcal{M}_{\text{add}} \rightarrow \mathcal{S}_\infty$ . Therefore, if  $\mathcal{A}$  has the structure of an  $E_n$  object in  $\text{Cat}_\infty^{\text{perf}}$ , the associated spectrum of rational Witt vectors  $\text{Map}(\mathbb{W}_0, \mathcal{A})$  is an  $E_n$  ring spectrum.*

Specializing to the case of  $E_n$  ring spectra, we have the following further corollary.

**Corollary 6.8.** *Let  $R$  be an  $E_n$  ring spectrum. Then the associated rational Witt ring spectrum  $\mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_R))$  is an  $E_{n-1}$  ring spectrum.*

*Proof.* By Lurie’s resolution of Mandell’s conjecture (see [32, 8.1.2.6]), the  $\infty$ -category of modules for an  $E_n$  ring spectrum  $R$  is an  $E_{n-1}$  object in the  $\infty$ -category  $\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}$  of presentable stable  $\infty$ -categories. Since the category of  $R$ -modules is compactly generated and the symmetric monoidal structure on the  $\infty$ -category  $\mathrm{Pr}_{\mathrm{St}, \omega}^{\mathrm{L}}$  of compactly generated stable  $\infty$ -categories is induced by the symmetric monoidal structure on  $\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}$ , we can conclude from [31, §5.5.7] that the  $\infty$ -category of perfect modules for an  $E_n$  ring spectrum is an  $E_{n-1}$  object in  $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ . The result now follows from Corollary 6.7.  $\square$

## APPENDIX A. SPLITTING COALGEBRAS

In this appendix we verify some technical results about splitting of (point-set) coalgebras. Let  $(\mathcal{C}, \otimes, \mathbf{1})$  be a symmetric monoidal category. Recall that a *coalgebra*  $(A, \mu_A, \eta_A)$  in  $\mathcal{C}$  consists of an object  $A \in \mathcal{C}$  and two maps  $\mu_A: A \rightarrow A \otimes A$  (the comultiplication) and  $\eta_A: A \rightarrow \mathbf{1}$  (the counit). If the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mu_A} & A \otimes A & \xrightarrow{\mu_A \otimes \mathrm{Id}} & (A \otimes A) \otimes A \\ \mu_A \downarrow & & & & \downarrow \simeq \\ A \otimes A & \xrightarrow{\mathrm{Id} \otimes \mu_A} & A \otimes (A \otimes A) & & \end{array}$$

commutes we say that  $(A, \mu_A, \eta_A)$  is *coassociative*, and if the diagram

$$(A.1) \quad \begin{array}{ccccc} A & \xleftarrow{\eta_A \otimes \mathrm{Id}} & A \otimes A & \xrightarrow{\mathrm{Id} \otimes \eta_A} & A \\ & \searrow & \uparrow \mu_A & \nearrow & \\ & & A & & \end{array}$$

commutes we say that  $(A, \mu_A, \eta_A)$  is *counital*. Finally,  $(A, \mu_A, \eta_A)$  is *cocommutative* if in addition the diagram

$$(A.2) \quad \begin{array}{ccc} A & \xrightarrow{\mu_A} & A \otimes A \\ \mu_A \downarrow & \nearrow \tau_{A,A} & \\ A \otimes A & & \end{array}$$

commutes, where  $\tau_{A,A}$  stands for the symmetry constraint. A *coalgebra map*  $f: (A, \mu_A, \eta_A) \rightarrow (B, \mu_B, \eta_B)$  consists of a map  $f: A \rightarrow B$  in  $\mathcal{C}$  making the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ \mu_A \uparrow & & \uparrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

commute. When  $\eta_B \circ f = \eta_A$  we say that it is *counit preserving*.

Now, let us assume that the symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbf{1})$  is moreover additive. Note that given counital coassociative coalgebras  $(A, \mu_A, \eta_A)$  and

$(B, \mu_B, \eta_B)$ , the direct sum  $A \oplus B \in \mathcal{C}$  becomes a counital coassociative coalgebra. Its comultiplication is given by

$$\mu_{A \oplus B}: A \oplus B \xrightarrow{\mu_A \oplus \mu_B} (A \otimes A) \otimes (B \otimes B) \subset (A \oplus B) \otimes (A \oplus B),$$

and its counit is given by

$$\eta_{A \oplus B}: A \oplus B \xrightarrow{\eta_A \oplus \eta_B} \mathbf{1} \oplus \mathbf{1} \xrightarrow{\nabla} \mathbf{1}.$$

Moreover, if  $(A, \mu_A, \eta_A)$  and  $(B, \mu_B, \eta_B)$  are cocommutative the same holds for the direct sum  $A \oplus B$ . We now have all the ingredients needed for the following result:

**Proposition A.3.** *Consider the following diagram in  $\mathcal{C}$ :*

$$C \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} A \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{g} \end{array} B.$$

Assume that  $r \circ f = \text{Id}$ ,  $g \circ s = \text{Id}$  and  $f \circ r + s \circ g = \text{Id}$ . Assume also that  $A$  and  $B$  are counital coassociative coalgebras and that  $s$  and  $g$  are coalgebra maps with  $s$  counit preserving. Under these assumptions,  $C$  becomes a counital coassociative coalgebra and  $r$  and  $f$  coalgebra maps with  $f$  counit preserving. Moreover, if  $A$  and  $B$  are cocommutative the same holds for  $C$ . Furthermore, the induced isomorphism  $[r \ g]: A \xrightarrow{\sim} C \oplus B$  is a counit preserving coalgebra map.

*Proof.* Let us start by constructing the comultiplication on  $C$ . Note that it follows from our hypothesis that  $C$  identifies with the cokernel of  $s$ . Hence, consider the following commutative diagram:

$$\begin{array}{ccccccc} B \otimes B & \xrightarrow{s \otimes s} & A \otimes A & \longrightarrow & \text{cok}(s \otimes s) & \longrightarrow & \text{cok}(s) \otimes \text{cok}(s) \\ \uparrow \mu_B & & \uparrow \mu_A & & \uparrow & & \uparrow =: \mu_C \\ B & \xrightarrow{s} & A & \xrightarrow{r} & \text{cok}(s) & \xlongequal{\quad} & \text{cok}(s). \end{array}$$

The left-hand side square commutes since  $s$  is a coalgebra map; the middle one commutes since the dashed arrow is induced by the universal property of the cokernel; and the right-hand side one commutes since the composition

$$B \oplus B \xrightarrow{s \otimes s} A \otimes A \xrightarrow{r \otimes r} \text{cok}(s) \otimes \text{cok}(s)$$

is trivial. The comultiplication  $\mu_C$  on  $C$  is then given by the vertical arrow on the right-hand side. With this definition it is clear that  $r$  becomes a coalgebra map. Note also that since  $\mu_A$  and  $\mu_B$  are coassociative the same holds for  $\mu_C$ . Similarly, if by hypothesis  $\mu_A$  and  $\mu_B$  are cocommutative the same holds for  $\mu_C$ .

Let us now prove that  $f$  is also a coalgebra map. Consider the diagram:

$$(A.4) \quad \begin{array}{ccccc} C \otimes C & \xrightarrow{f \otimes f} & A \otimes A & \xrightarrow{g \otimes g} & B \otimes B \\ \uparrow \mu_C & & \uparrow \mu_A & & \uparrow \mu_B \\ C & \xrightarrow{f} & A & \xrightarrow{g} & B. \end{array}$$

One needs to show that the left-hand side square is commutative. By hypothesis  $g$  is a coalgebra map, and so the right-hand side square is commutative. Moreover,

since  $g \circ f = 0$ , the outer square is also commutative (since both maps from  $C$  to  $B \otimes B$  are trivial). This implies that the two maps

$$C \xrightarrow{\mu_C} C \otimes C \xrightarrow{f \otimes f} A \otimes A \xrightarrow{g \otimes g} B \otimes B, \quad C \xrightarrow{f} A \xrightarrow{\mu_A} A \otimes A \xrightarrow{g \otimes g} B \otimes B$$

agree. Since  $g \otimes g$  is surjective (note that it admits a section  $s \otimes s$ ) we then conclude that  $(f \otimes f) \circ \mu_C = \mu_A \circ f$ , i.e., that the left-hand side square in (A.4) commutes.

Let us now define the counit of  $C$  as the composition  $\mu_C: C \xrightarrow{f} A \xrightarrow{\eta_A} \mathbf{1}$ . Note that proving the commutativity of diagram (A.1) amounts to showing that both composites

$$C \xrightarrow{\mu_C} C \otimes C \xrightarrow{\text{Id} \otimes f} C \otimes A \xrightarrow{\text{Id} \otimes \eta_A} C, \quad C \xrightarrow{\mu_C} C \otimes C \xrightarrow{f \otimes \text{Id}} A \otimes C \xrightarrow{\eta_A \otimes \text{Id}} C$$

are the identity. The proof is similar, and so we restrict ourselves to the left-hand side case. Consider the following commutative diagram:

$$\begin{array}{ccccc} C & \xrightarrow{\mu_C} & C \otimes C & \xrightarrow{\text{Id} \otimes f} & C \otimes A & \xrightarrow{\text{Id} \otimes \eta_A} & C \\ \uparrow r & & \uparrow r \otimes r & & & & \parallel \\ A & \xrightarrow{\mu_A} & A \otimes A & \xrightarrow{r \otimes (\eta_A \circ f \circ r)} & & & C \\ \uparrow f & & \uparrow f \otimes f & & & & \\ C & \xrightarrow{\mu_C} & C \otimes C & & & & \end{array}$$

Since  $r \circ f = \text{Id}$  it suffices to show that the composite

$$(A.5) \quad C \xrightarrow{f} A \xrightarrow{\mu_A} A \otimes A \xrightarrow{r \otimes (\eta_A \circ f \circ r)} C$$

is the identity map. Since  $f \circ r = (\text{Id} - s \circ g)$ , we have  $r \otimes (\eta_A \circ f \circ r) = r \otimes \eta_A - r \otimes (\eta_A \circ s \circ g)$ , and hence, since  $(r \otimes \eta_A) \circ \mu_A = r$ , we obtain the equality

$$(r \otimes (\eta_A \circ f \circ r)) \circ \mu_A = r - (r \otimes (\eta_A \circ s \circ g)) \circ \mu_A.$$

Now, note that  $g \circ f = 0$ . Since  $s$  is a monomorphism it suffices to show that  $s \circ g \circ f = 0$ , which follows from the equalities

$$s \circ g \circ f = s \circ g \circ (f \circ r \circ f) = (s \circ g) \circ (\text{Id} - s \circ g) \circ f = (s \circ g - s \circ g) \circ f = 0.$$

The equalities  $g \circ f = 0$  and  $\mu_A \circ f = (f \otimes f) \circ \mu_C$  allows us then to conclude that

$$(r \otimes (\eta_A \circ f \circ r)) \circ \mu_A \circ f = r \circ f = \text{Id}.$$

This shows that  $C$  is also a counital coalgebra and that  $f$  is counit preserving.

Let us finally prove that the induced isomorphism  $[r \ g]: A \xrightarrow{\sim} C \oplus B$  is a counit preserving coalgebra map. The commutative squares

$$\begin{array}{ccc} A \otimes A & \xrightarrow{r \otimes r} & C \otimes C \\ \uparrow \mu_A & & \uparrow \mu_C \\ A & \xrightarrow{r} & C \end{array} \quad \begin{array}{ccc} A \otimes A & \xrightarrow{g \otimes g} & B \otimes B \\ \uparrow \mu_A & & \uparrow \mu_B \\ A & \xrightarrow{g} & B \end{array}$$

imply automatically that  $[r \ g]$  is a coalgebra map. In what concerns the counit, one needs to prove that the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{[r \ g]} & C \oplus B \\
 \eta_A \downarrow & & \downarrow \eta_C \oplus \eta_B \\
 & & \mathbf{1} \oplus \mathbf{1} \\
 & & \downarrow \nabla \\
 \mathbf{1} & \xlongequal{\quad} & \mathbf{1} .
 \end{array}$$

Note first that since by hypothesis  $s$  is counit preserving we have  $\eta_B \circ g = \eta_A \circ s \circ g$ . On the other hand, by the above definition of  $\eta_C$  we have  $\eta_C \circ r = \eta_A \circ f \circ r$ . As a consequence, we obtain the equality

$$\eta_C \circ r + \eta_B \circ g = \eta_A(f \circ r + s \circ g) = \eta_A$$

and thus conclude that the above diagram commutes. This concludes the proof.  $\square$

#### ACKNOWLEDGMENTS

The authors are very grateful to Clark Barwick, Lars Hesselholt, Mike Mandell, and Markus Spitzweck for useful discussions. The paper was improved by a careful report from an anonymous referee.

#### REFERENCES

- [1] Gert Almkvist, *K-theory of endomorphisms*, J. Algebra **55** (1978), no. 2, 308–340, DOI 10.1016/0021-8693(78)90224-7. MR523461 (80i:18018)
- [2] Gert Almkvist, *The Grothendieck ring of the category of endomorphisms*, J. Algebra **28** (1974), 375–388. MR0432738 (55 #5721)
- [3] Benjamin Antieau and David Gepner, *Brauer groups and étale cohomology in derived algebraic geometry*, Geom. Topol. **18** (2014), no. 2, 1149–1244, DOI 10.2140/gt.2014.18.1149. MR3190610
- [4] C. Barwick, On the algebraic K-theory of higher categories. Preprint, arXiv:1204.3607.
- [5] C. Barwick, On exact  $\infty$ -categories and the Theorem of the Heart. Preprint, arXiv:1212.5232.
- [6] C. Barwick, On the Q-construction for exact  $\infty$ -categories. Preprint, arXiv:1301.4725.
- [7] Stanislaw Betley and Christian Schlichtkrull, *The cyclotomic trace and curves on K-theory*, Topology **44** (2005), no. 4, 845–874, DOI 10.1016/j.top.2005.02.004. MR2136538 (2006b:19005)
- [8] Andrew J. Blumberg, David Gepner, and Gonalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838, DOI 10.2140/gt.2013.17.733. MR3070515
- [9] Andrew J. Blumberg, David Gepner, and Gonalo Tabuada, *Uniqueness of the multiplicative cyclotomic trace*, Adv. Math. **260** (2014), 191–232, DOI 10.1016/j.aim.2014.02.004. MR3209352
- [10] Spencer Bloch, *Algebraic K-theory and crystalline cohomology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 187–268 (1978). MR488288 (81j:14011)
- [11] Andrew J. Blumberg and Michael A. Mandell, *The localization sequence for the algebraic K-theory of topological K-theory*, Acta Math. **200** (2008), no. 2, 155–179, DOI 10.1007/s11511-008-0025-4. MR2413133 (2009f:19003)
- [12] Andrew J. Blumberg and Michael A. Mandell, *Algebraic K-theory and abstract homotopy theory*, Adv. Math. **226** (2011), no. 4, 3760–3812, DOI 10.1016/j.aim.2010.11.002. MR2764905 (2012a:19007)
- [13] Andrew J. Blumberg and Michael A. Mandell, *Localization theorems in topological Hochschild homology and topological cyclic homology*, Geom. Topol. **16** (2012), no. 2, 1053–1120, DOI 10.2140/gt.2012.16.1053. MR2928988

- [14] M. V. Bondarko, *Weight structures vs.  $t$ -structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory **6** (2010), no. 3, 387–504, DOI 10.1017/is010012005jkt083. MR2746283
- [15] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284, DOI 10.1016/0022-4049(80)90049-3. MR579087 (81h:55018)
- [16] W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35, DOI 10.1016/0022-4049(80)90113-9. MR578563 (81h:55019)
- [17] W. G. Dwyer and D. M. Kan, *Function complexes for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 2, 139–147. MR705421 (85e:55038)
- [18] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004. MR2102294 (2005k:18027)
- [19] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, with an appendix by M. Cole. Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. MR1417719 (97h:55006)
- [20] T. Fiore and W. Lueck, *Waldhausen additivity: Classical and quasicategorical*. Preprint, arXiv:1207.6613.
- [21] S. Glasman, *Day convolution for  $\infty$ -categories*, Preprint, arXiv:1308.4940.
- [22] Daniel R. Grayson,  *$K_2$  and the  $K$ -theory of automorphisms*, J. Algebra **58** (1979), no. 1, 12–30, DOI 10.1016/0021-8693(79)90185-6. MR535839 (80g:16029)
- [23] Daniel R. Grayson, *The  $K$ -theory of endomorphisms*, J. Algebra **48** (1977), no. 2, 439–446. MR0480699 (58 #853)
- [24] Daniel R. Grayson, *Grothendieck rings and Witt vectors*, Comm. Algebra **6** (1978), no. 3, 249–255, DOI 10.1080/00927877808822245. MR484183 (81e:13014)
- [25] A. Grothendieck et al., SGA 6.
- [26] Michiel Hazewinkel, *Operations in the  $K$ -theory of endomorphisms*, J. Algebra **84** (1983), no. 2, 285–304, DOI 10.1016/0021-8693(83)90080-7. MR723394 (85k:13025)
- [27] Michiel Hazewinkel, *Witt vectors. I*, Handbook of algebra. Vol. 6, Handb. Algebr., vol. 6, Elsevier/North-Holland, Amsterdam, 2009, pp. 319–472, DOI 10.1016/S1570-7954(08)00207-6. MR2553661 (2011a:13045)
- [28] Lars Hesselholt and Ib Madsen, *Cyclic polytopes and the  $K$ -theory of truncated polynomial algebras*, Invent. Math. **130** (1997), no. 1, 73–97, DOI 10.1007/s002220050178. MR1471886 (98k:19002)
- [29] Lars Hesselholt, *On the  $p$ -typical curves in Quillen’s  $K$ -theory*, Acta Math. **177** (1996), no. 1, 1–53, DOI 10.1007/BF02392597. MR1417085 (98a:19001)
- [30] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134 (99h:55031)
- [31] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659 (2010j:18001)
- [32] J. Lurie, *Higher algebra*. Preprint, available at [www.math.harvard.edu/lurie](http://www.math.harvard.edu/lurie).
- [33] Ayelet Lindenstrauss and Randy McCarthy, *On the algebraic  $K$ -theory of formal power series*, J. K-Theory **10** (2012), no. 1, 165–189, DOI 10.1017/is012003003jkt186. MR2990565
- [34] M. A. Mandell and B. Shipley, *A telescope comparison lemma for  $THH$* , Topology Appl. **117** (2002), no. 2, 161–174, DOI 10.1016/S0166-8641(00)00121-8. MR1875908 (2002k:55019)
- [35] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR1812507 (2001k:18010)
- [36] Daniel Quillen, *Higher algebraic  $K$ -theory. I*, Algebraic  $K$ -theory, I: Higher  $K$ -theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973, pp. 85–147. MR0338129 (49 #2895)
- [37] S. Schwede, *Symmetric spectra*. Preprint, <http://www.math.uni-bonn.de/~schwede/>.
- [38] Stefan Schwede and Brooke Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), no. 1, 103–153, DOI 10.1016/S0040-9383(02)00006-X. MR1928647 (2003g:55034)
- [39] Jan Stienstra, *Cartier-Dieudonné theory for Chow groups*, J. Reine Angew. Math. **355** (1985), 1–66, DOI 10.1515/crll.1985.355.1. MR772483 (88c:14026a)

- [40] Jan Stienstra, *Operations in the higher K-theory of endomorphisms*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 59–115. MR686113 (84d:14012)
- [41] Gonçalo Tabuada, *Homotopy theory of spectral categories*, Adv. Math. **221** (2009), no. 4, 1122–1143, DOI 10.1016/j.aim.2009.01.014. MR2518634 (2010g:18020)
- [42] R. W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435, DOI 10.1007/978-0-8176-4576-2\_10. MR1106918 (92f:19001)
- [43] Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419, DOI 10.1007/BFb0074449. MR802796 (86m:18011)
- [44] E. Witt. Theorie der quadratischen Formen in beliebigen Körpern. J. Reine Angew. Math. **176** (1937), 31–44.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78703

*E-mail address:* blumberg@math.utexas.edu

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

*E-mail address:* dgepner@purdue.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139 – AND – DEPARTAMENTO DE MATEMÁTICA, FCT, UNL, PORTUGAL, CENTRO DE MATEMÁTICA E APLICAÇÕES (CMA), FCT, UNL, PORTUGAL

*E-mail address:* tabuada@math.mit.edu