Deligne categories and reduced Kronecker coefficients

J Algebr Comb (2016) 44:345–362
DOI 10.1007/s10801-016-0672-z

Deligne categories and reduced Kronecker coefficients

Inna Entova Aizenbud[1]

Received: 19 October 2014 / Accepted: 12 February 2016 / Published online: 24 February 2016

Abstract The Kronecker coefficients are the structural constants for the tensor categories of representations of the symmetric groups, namely, given three partitions λ, μ, τ of n, the multiplicity of λ in $\mu \otimes \tau$ is called the Kronecker coefficient $g^{\lambda}_{\mu,\tau}$. When the first part of each of the partitions is taken to be very large (the remaining parts being fixed), the values of the appropriate Kronecker coefficients stabilize; the stable value is called the reduced (or stable) Kronecker coefficient. These coefficients also generalize the Littlewood–Richardson coefficients and have been studied quite extensively. In this paper, we show that reduced Kronecker coefficients appear naturally as structure constants of Deligne categories $\underline{Rep}(S_t)$. This allows us to interpret various properties of the reduced Kronecker coefficients as categorical properties of Deligne categories $\underline{Rep}(S_t)$ and derive new combinatorial identities.

Keywords Representations of symmetric groups · Kronecker coefficients · Deligne categories

1 Introduction

The Kronecker coefficients are the structural constants for the semisimple categories $Rep(S_n)$. Namely, considering two irreducible representations μ, τ of S_n, we can decompose the tensor product $\mu \otimes \tau$ into a direct sum of irreducible representations of S_n. The multiplicity of λ in $\mu \otimes \tau$ is called *the Kronecker coefficient* $g^{\lambda}_{\mu,\tau}$ (good references for Kronecker coefficients are [11, Par. I.7], [16, Chapter 7]).

Consider three arbitrary Young diagrams λ, μ, τ. For $n >> 0$, denote by $\widetilde{\lambda}(n)$ the Young diagram of size n obtained by adding a top row of size $n - |\lambda|$ to λ

✉ Inna Entova Aizenbud
inna.entova@gmail.com

[1] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

(similarly for μ, τ). Such a diagram is defined whenever $n \geq \lambda_1 + |\lambda|$. It was noticed by Murnaghan in [13] that the sequence $\{g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)}\}_{n>>0}$ stabilizes, and the stable value of the sequence was called the *reduced Kronecker coefficient* $\bar{g}^{\lambda}_{\mu,\tau}$ associated with the triple (λ, μ, τ).

The reduced Kronecker coefficients have been studied extensively in [1,3,13,14], and other papers.

In particular, these coefficients have been linked to Geometric Complexity Theory (see [2,12]), stable representation theory and FI-modules (see [5,15]) and to the study of the Fourier-Deligne transform (see [9]).

It turns out that reduced Kronecker coefficients occur naturally in Deligne categories $\underline{Rep}(S_t)$, $t \in \mathbb{C}$, which are interpolations of the categories of finite-dimensional representations of the symmetric groups over the field $\mathbb{C}$ of complex numbers. These Karoubian rigid symmetric monoidal categories were defined and studied by Deligne [8], and subsequently by Comes and Ostrik [6,7].

A detailed description of the Deligne categories $\underline{Rep}(S_t)$, as well as their abelian envelopes, is given is Sect. 3. Below we give the main properties of these categories.

As it was said before, the categories $\underline{Rep}(S_t)$ interpolate the categories of representations of symmetric groups; namely, for $n \in \mathbb{Z}_+$, the category $\underline{Rep}(S_{t=n})$ admits a full, essentially surjective symmetric monoidal functor to the category of finite-dimensional representations of the symmetric group S_n.

For $t \notin \mathbb{Z}_+$, the category $\underline{Rep}(S_t)$ is a semisimple abelian tensor category, the simple objects being parameterized by arbitrary Young diagrams (of any size).

Note that for $n \in \mathbb{Z}_+$, the category $\underline{Rep}(S_{t=n})$ is not abelian. It can be embedded as a full monoidal subcategory into a tensor (i.e., rigid symmetric monoidal abelian) category $\underline{Rep}^{ab}(S_{t=n})$; the latter is not semisimple, but its structure can be described quite explicitly (see [10, Section 5]).

Our first main result is that for a generic value of t, the structural constants of $\underline{Rep}(S_t)$ as a tensor category turn out to be exactly the reduced Kronecker coefficients:

Theorem 1.1 *Denote by X_λ the simple object of $\underline{Rep}(S_t)$ ($t \notin \mathbb{Z}_+$) which corresponds to the Young diagram λ. Then*

$$X_\mu \otimes X_\tau = \bigoplus_{\lambda \text{ is a partition of arbitrary size}} \mathbb{C}^{\bar{g}^{\lambda}_{\mu,\tau}} \otimes X_\lambda$$

This approach provides a natural environment for the reduced Kronecker coefficients. It allows us to interpret various facts about reduced Kronecker coefficients in terms of Deligne categories, and obtain some new, previously unknown formulas involving reduced Kronecker coefficients, which we describe below.

In order to state these results, we will use the following definition:

Definition 1.1 Let λ be a Young diagram, and let $n \in \mathbb{Z}$ be such that $\widetilde{\lambda}(n)$ is a Young diagram (i.e. $|\lambda| + \lambda_1 \leq n$). We define the sequence of Young diagrams $\{\lambda^{(i)}\}_i$ corresponding to the pair (λ, n) by

$$\lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \dots$$
$$\text{and } \lambda^{(i+1)} \setminus \lambda^{(i)} = \text{ strip in row } i+1 \text{ of length } \lambda_i - \lambda_{i+1} + 1 \text{ for } i > 0,$$
$$\text{and } \lambda^{(1)} \setminus \lambda^{(0)} = \text{ strip in row 1 of length } n - |\lambda| - \lambda_1 + 1.$$

We now state the second main result (see Proposition 5.4).

Theorem 1.2 *Let λ, μ, τ be three Young diagrams. Let $n \in \mathbb{Z}$, $|\lambda| + \lambda_1 \leq n$ (i.e. $\widetilde{\lambda}(n)$ is defined).*

– *If $n \in \{|\mu| + \mu_l - l : l = 1, \dots, |\mu|\}$, then*

$$\sum_{i \geq 0} (-1)^i \bar{g}_{\mu,\tau}^{\lambda^{(i)}} = 0$$

– *Assume μ, τ satisfy the same condition as λ: $|\mu| + \mu_1, |\tau| + \tau_1 \leq n$.*
Consider the sequences $\{\mu^{(i)}\}_{i \geq 0}$ and $\{\tau^{(i)}\}_{i \geq 0}$ respectively (these sequences are not necessarily distinct), and let $k, l \geq 0$.
Then

$$\sum_{i \geq 0} (-1)^i \bar{g}_{\mu^{(k)},\tau^{(l)}}^{\lambda^{(i)}} = (-1)^{k+l} g_{\widetilde{\mu}(n),\widetilde{\tau}(n)}^{\widetilde{\lambda}(n)}$$

A special case of the second result when $k = l = 0$ recovers a formula due to Briand, Orellana and Rosas (see [3]) which describes how to recover the (standard) Kronecker coefficients from the reduced Kronecker coefficients.

1.1 Structure of the paper

In Sect. 3, we recall the relevant facts about Deligne's categories $\underline{Rep}(S_t)$.

In Sect. 4, we define the Deligne–Kronecker coefficients in terms of Deligne categories. We will later show that these coincide with the reduced Kronecker coefficients.

In Sect. 5 we prove some known properties of Kronecker coefficients using the machinery of Deligne categories; we also prove some previously unknown formulas in Sect. 5.5.

2 Notation and definitions

The base field throughout the paper will be $\mathbb{C}$.

2.1 Karoubian categories

Definition 2.1 (*Karoubian category*) We will call a category $\mathcal{A}$ Karoubian[1] if it is an additive category, and every idempotent morphism is a projection onto a direct factor.

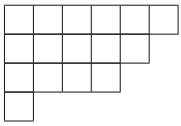
[1] Deligne calls such categories "pseudo-abelian" (see [8, 1.9]).

Definition 2.2 (*Block of a Karoubian category*) A block in an Karoubian category is a full subcategory generated by an equivalence class of indecomposable objects, defined by the minimal equivalence relation such that any two indecomposable objects with a non-zero morphism between them are equivalent.

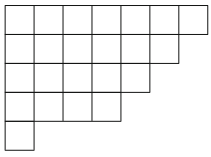
2.2 Symmetric group and young diagrams

Notation 2.3 – S_n will denote the symmetric group ($n \in \mathbb{Z}_+$).
- The notation λ will stand for a partition (weakly decreasing sequence of non-negative integers), a Young diagram λ, and the corresponding irreducible representation of $S_{|\lambda|}$. Here $|\lambda|$ is the sum of entries of the partition, or, equivalently, the number of cells in the Young diagram λ.
  The set of all Young diagrams will be denoted by $\mathcal{P}$.
- All the Young diagrams will be considered in the English notation, i.e. the lengths of the rows decrease from top to bottom.
- The length of the partition λ, i.e. the number of rows of Young diagram λ, will be denoted by $\ell(\lambda)$.
- The i-th entry of a partition λ, as well as the length of the i-th row of the corresponding Young diagram, will be denoted by λ_i (if $i > \ell(\lambda)$, then $\lambda_i := 0$).
- $\mathfrak{h}$ (in context of representations of S_n) will denote the permutation representation of S_n, i.e. the n-dimensional representation $\mathbb{C}^n$ with S_n acting by $g.e_j = e_{g(j)}$ on the standard basis $e_1, .., e_n$ of $\mathbb{C}^n$.
- For any Young diagram λ and an integer n such that $n \geq |\lambda| + \lambda_1$, we denote by $\widetilde{\lambda}(n)$ the Young diagram obtained by adding a row of length $n - |\lambda|$ on top of λ.

Example 2.1 Consider the Young diagram λ corresponding to the partition (6, 5, 4, 1):



The length of λ is 4, and $|\lambda| = 16$. For $n = 23$, we have:

$\widetilde{\lambda}(n) =$ 

3 Deligne categories

This section follows [6,8,10].

3.1 General description

For any $t \in \mathbb{C}$, the category $\underline{Rep}(S_t)$ is generated, as a $\mathbb{C}$-linear Karoubian tensor category, by one object, denoted $\mathfrak{h}$. This object is the analogue of the permutation representation of S_n, and any object in $\underline{Rep}(S_t)$ is a direct summand in a direct sum of tensor powers of $\mathfrak{h}$.

For $t \notin \mathbb{Z}_+$, $\underline{Rep}(S_t)$ is a semisimple abelian category.

Notation 3.1 We will denote Deligne's category for integer value $n \geq 0$ of t as $\underline{Rep}(S_{t=n})$, to distinguish it from the classical category $Rep(S_n)$ of representations of the symmetric group S_n. Similarly for other categories arising in this text.

If t is a non-negative integer, then the category $\underline{Rep}(S_t)$ has a tensor ideal $\mathfrak{I}_t$, called the ideal of negligible morphisms (this is the ideal of morphisms $f : X \longrightarrow Y$ such that $\mathrm{tr}(fu) = 0$ for any morphism $u : Y \longrightarrow X$). In that case, the classical category $Rep(S_n)$ of finite-dimensional representations of the symmetric group for $n := t$ is equivalent to $\underline{Rep}(S_{t=n})/\mathfrak{I}_t$ (equivalent as Karoubian rigid symmetric monoidal categories).

The full, essentially surjective functor $\underline{Rep}(S_{t=n}) \to Rep(S_n)$ defining this equivalence will be denoted by $\mathcal{S}_n$.

Note that $\mathcal{S}_n$ sends $\mathfrak{h}$ to the permutation representation of S_n.

Remark 3.1 Although $\underline{Rep}(S_t)$ is not semisimple and not even abelian when $t = n \in \mathbb{Z}_+$, a weaker statement holds (see [8, Proposition 5.1], Remark 3.1): consider the full subcategory $\underline{Rep}(S_{t=n})^{(n/2)}$ of $\underline{Rep}(S_t)$ whose objects are directs summands of sums of $\mathfrak{h}^{\otimes m}$, $0 \leq m \leq \frac{n}{2}$. This subcategory is abelian semisimple, and the restriction $\mathcal{S}_n|_{\underline{Rep}(S_{t=n})^{(n/2)}}$ is fully faithful.

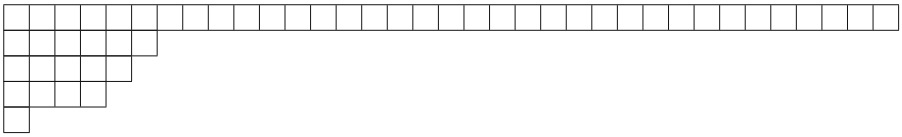
The indecomposable objects of $\underline{Rep}(S_t)$, regardless of the value of t, are parametrized (up to isomorphism) by all Young diagrams (of arbitrary size). We will denote the indecomposable object in $\underline{Rep}(S_t)$ corresponding to the Young diagram τ by X_τ.

For non-negative integer $t =: n$, we have: the partitions λ for which X_λ has a non-zero image in the quotient $\underline{Rep}(S_{t=n})/\mathfrak{I}_{t=n} \cong Rep(S_n)$ are exactly the λ for which $\lambda_1 + |\lambda| \leq n$.

If $\lambda_1 + |\lambda| \leq n$, then the image of λ in $Rep(S_n)$ is the irreducible representation of S_n corresponding to the Young diagram $\widetilde{\lambda}(n)$ (see Sect. 2).

This allows one to intuitively treat the indecomposable objects of $\underline{Rep}(S_t)$ as if they were parametrized by "Young diagrams with a very long top row". The indecomposable object X_λ would be treated as if it corresponded to $\widetilde{\lambda}(t)$, i.e. a Young diagram obtained by adding a very long top row ("of size $t - |\lambda|$"). This point of view is useful to understand how to extend constructions for S_n involving Young diagrams to $\underline{Rep}(S_t)$.

Example 3.1 The indecomposable object X_λ, where $\lambda =$  can be thought of as a Young diagram with a "very long top row of length $(t-16)$":



3.2 Lifting objects

We start with an equivalence relation on the set of all Young diagrams, defined in [6, Definition 5.1]:

Definition 3.2 Let λ be any Young diagram, and set

$$\mu_\lambda(t) = (t - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, \ldots)$$

Given two Young diagrams λ, λ', denote $\mu_\lambda(t) =: (\mu_0, \mu_1, \ldots), \mu_{\lambda'}(t) =: (\mu'_0, \mu'_1, \ldots)$.

We put $\lambda \overset{t}{\sim} \lambda'$ if there exists a bijection $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\mu_i = \mu'_{f(i)}$ for any $i \geq 0$.

We will call a $\overset{t}{\sim}$-class *trivial* if it contains exactly one Young diagram.

The following lemma is proved in [6, Corollary 5.6, Proposition 5.8]:

Lemma 3.1 *1. If $t \notin \mathbb{Z}_+$, then any Young diagram λ lies in a trivial $\overset{t}{\sim}$-class.*

2. The non-trivial $\overset{t}{\sim}$-classes are parametrized by all Young diagrams λ such that $\widetilde{\lambda}(t)$ is a Young diagram (in particular, $t \in \mathbb{Z}_+$), and are of the form $\{\lambda^{(i)}\}_i$, with

$$\lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \ldots$$
$$\text{and } \lambda^{(i+1)} \setminus \lambda^{(i)} = \text{ strip in row } i+1 \text{ of length } \lambda_i - \lambda_{i+1} + 1 \text{ for } i > 0,$$
$$\text{and } \lambda^{(1)} \setminus \lambda^{(0)} = \text{ strip in row } 1 \text{ of length } t - |\lambda| - \lambda_1 + 1.$$

We now consider Deligne's category $\underline{Rep}(S_T)$, where T is a formal variable (see [6, Section 3.2]). This category is $\mathbb{C}((T-t))$-linear, but otherwise it is very similar to Deligne's category $\underline{Rep}(S_t)$ for generic t. For instance, as a $\mathbb{C}((T-t))$-linear Karoubian tensor category, $\underline{Rep}(S_T)$ is generated by one object, again denoted by $\mathfrak{h}$.

One can show that $\underline{Rep}(S_T)$ is split semisimple and thus abelian, and its simple objects are parametrized by Young diagrams of arbitrary size.

In [6, Section 3.2], Comes and Ostrik defined a map

$$\text{lift}_t : \left\{\begin{matrix}\text{objects in } \underline{Rep}(S_t) \\ \text{up to isomorphism}\end{matrix}\right\} \to \left\{\begin{matrix}\text{objects in } \underline{Rep}(S_T) \\ \text{up to isomorphism}\end{matrix}\right\}$$

We will not give the precise definition of this map, but will list some of its useful properties. It is defined to be additive (i.e. $\mathrm{lift}_t(A \oplus B) \cong \mathrm{lift}_t(A) \oplus \mathrm{lift}_t(B)$ for any $A, B \in \underline{Rep}(S_t)$) and satisfies $\mathrm{lift}_t(\mathfrak{h}) \cong \mathfrak{h}$. Moreover, we have (see [6, Proposition 3.12]):

Proposition 3.1 *Let A, B be two objects in $\underline{Rep}(S_t)$.*

1. $\mathrm{lift}_t(A \otimes B) \cong \mathrm{lift}_t(A) \otimes \mathrm{lift}_t(B)$.
2. $\dim_{\mathbb{C}} \mathrm{Hom}_{\underline{Rep}(S_t)}(A, B) = \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(\mathrm{lift}_t(A), \mathrm{lift}_t(B))$.
3. *The map* $\widehat{\mathrm{lift}_t}$ *is injective.*

Remark 3.2 It was proved both in [8, Section 7.2] and in [6, Proposition 3.28] that the dimensions of the indecomposable objects X_λ in $\underline{Rep}(S_T)$ are polynomials in T whose coefficients depend on λ (given λ, this polynomial can be written down explicitly). Such polynomials are denoted by $P_\lambda(T)$.

Furthermore, it was proved in [6, Proposition 5.12] that given $d \in \mathbb{Z}_+$ and a Young diagram λ, λ belongs to a trivial $\overset{d}{\sim}$-class iff $P_\lambda(d) = 0$.

The following result is proved in [6, Lemma 5.20]:

Lemma 3.2 *(*Comes, Ostrik*) Consider the $\overset{t}{\sim}$-equivalence relation on Young diagrams.*

- *Whenever λ lies in a trivial $\overset{t}{\sim}$-class,* $\mathrm{lift}_t(X_\lambda) = X_\lambda$.
- *For a non-trivial $\overset{t}{\sim}$-class $\{\lambda^{(i)}\}_i$,*

$$\mathrm{lift}_t(X_{\lambda^{(0)}}) = X_{\lambda^{(0)}}, \qquad \mathrm{lift}_t(X_{\lambda^{(i)}}) = X_{\lambda^{(i)}} \oplus X_{\lambda^{(i-1)}} \ \ \forall i \geq 1$$

Based on Lemmas 3.1, 3.2, Comes and Ostrik give a full description of blocks in $\underline{Rep}(S_t)$ (see [6], [7, Proposition 2.7]):

3.3 Abelian envelope

As it was mentioned before, the category $\underline{Rep}(S_t)$ is defined as a Karoubian category. For $t \notin \mathbb{Z}_+$, it is semisimple and thus abelian, but for $t \in \mathbb{Z}_+$, it is not abelian. Fortunately, it has been shown that $\underline{Rep}(S_t)$ possesses an "abelian envelope", that is, that it can be embedded into an abelian tensor category, and this abelian tensor category has a universal mapping property (see [8, Conjecture 8.21.2], and [7, Theorem 1.2]).

An explicit construction of the category $\underline{Rep}^{ab}(S_{t=n})$ is given in [7]. We will only list the results which will be used in this paper.

Proposition 3.2 *The category $\underline{Rep}^{ab}(S_t)$ is a highest weight category corresponding to the (infinite) partially ordered set* ($\{$*Young diagrams*$\}, \geq$)*, where*

$$\lambda \geq \mu \text{ if } \lambda \overset{t}{\sim} \mu, \lambda \subset \mu$$

(namely, $\lambda^{(i)} \geq \lambda^{(j)}$ if $i \leq j$).

The category $\underline{Rep}(S_t)$ is the subcategory of tilting objects in $\underline{Rep}^{ab}(S_t)$. More specifically, we have:

Proposition 3.3 *The blocks of the abelian category $\underline{Rep}^{ab}(S_t)$ correspond to the $\overset{t}{\sim}$-classes:*

1. *Let λ lie in a trivial $\overset{t}{\sim}$-class. The corresponding block of $\underline{Rep}^{ab}(S_t)$ is semisimple and is generated by the simple projective object X_λ.*
2. *Let $\{\lambda^{(i)}\}_{i\geq 0}$ be a non-trivial $\overset{t}{\sim}$-class. The corresponding block of $\underline{Rep}^{ab}(S_t)$ is non-semisimple, and contains $X_{\lambda^{(i)}}$ for $i \geq 0$.*
   - *$X_{\lambda^{(0)}}$ is a simple, non-projective object.*
   - *For any $i \geq 1$, $X_{\lambda^{(i)}}$ is a projective object.*

4 Reduced Kronecker coefficients and Deligne's categories

Notation 4.1 Let $t \in \mathbb{C}$, and consider Deligne's category $\underline{Rep}(S_t)$. Let A be any object in $\underline{Rep}(S_t)$, and X_λ be an indecomposable object. Then A decomposes into a direct sum of indecomposable objects, and we denote by $[A : X_\lambda]_t$ the multiplicity of X_λ in this direct sum, i.e.

$$A \cong \bigoplus_{\lambda} [A : X_\lambda]_t X_\lambda$$

Similarly, given an object A in the $\mathrm{Frac}(\mathbb{C}[[T]])$-linear Deligne's category $\underline{Rep}(S_T)$, and an indecomposable object X_λ in the same category, we denote by $[A : X_\lambda]_T$ the multiplicity of X_λ in the decomposition of A into a direct sum of indecomposable objects. Since $\underline{Rep}(S_T)$ is semisimple, we have:

$$[A : X_\lambda]_T = \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(A, X_\lambda)$$

Definition 4.2 Consider the $\mathrm{Frac}(\mathbb{C}[[T]])$-linear Deligne's category $\underline{Rep}(S_T)$ (this category is semisimple). Let $\lambda, \mu, \tau \in \mathcal{P}$. We denote by $\bar{g}^\lambda_{\mu,\tau}$ the multiplicity of the simple object X_λ in $X_\mu \otimes X_\tau$:

$$\bar{g}^\lambda_{\mu,\tau} := [X_\mu \otimes X_\tau : X_\lambda]_T = \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(X_\mu \otimes X_\tau, X_\lambda)$$

The value $\bar{g}^\lambda_{\mu,\tau}$ will be called *the Deligne–Kronecker coefficient* corresponding to the triple of Young diagrams (λ, μ, τ).

Thus the Deligne–Kronecker coefficients are the structural constants of the Grothendieck rings $\mathfrak{R}(S_t)$ of Deligne's categories $\underline{Rep}(S_t)$ at generic values of t ($t \notin \mathbb{Z}_+$). These rings are all isomorphic to one another and do not depend on t:

Proposition 4.1 *Let $t \notin \mathbb{Z}_+$, and let $\lambda, \mu, \tau \in \mathcal{P}$. Consider the semisimple category $\underline{Rep}(S_t)$. Then the multiplicity of the simple object X_λ in $X_\mu \otimes X_\tau$ is $\bar{g}^\lambda_{\mu,\tau}$.*

Proof By Lemma 3.2, $\mathrm{lift}_t(X_\lambda) \cong X_\lambda$ for any Young diagram λ, and so $\mathrm{lift}_t(X_\mu \otimes X_\tau) \cong X_\mu \otimes X_\tau$. Thus we obtain:

$$\begin{aligned}\bar{g}^{\lambda}_{\mu,\tau} &:= [X_\mu \otimes X_\tau : X_\lambda]_T = \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(X_\mu \otimes X_\tau, X_\lambda) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{\underline{Rep}(S_t)}(X_\mu \otimes X_\tau, X_\lambda) = [X_\mu \otimes X_\tau : X_\lambda]_t.\end{aligned}$$

□

In fact, for a fixed triple (λ, μ, τ), the same is true for almost all values of t:

Proposition 4.2 *Fix λ, μ, τ, and let $N := \max\{|\lambda|, |\mu| + |\tau|\}$. Let $t \notin \{0, \ldots, 2N - 2\}$. Consider the category $\underline{Rep}(S_t)$. Then the multiplicity of the simple object X_λ in $X_\mu \otimes X_\tau$ is $\bar{g}^{\lambda}_{\mu,\tau}$.*

Proof Almost the same arguments as in Proposition 4.1 apply here:

By Lemma 3.2, $\mathrm{lift}_t(X_\lambda) \cong X_\lambda$ and

$$\mathrm{lift}_t(X_\mu \otimes X_\tau) \cong \mathrm{lift}_t(X_\mu) \otimes \mathrm{lift}_t(X_\tau) \cong X_\mu \otimes X_\tau$$

Again,

$$\begin{aligned}\bar{g}^{\lambda}_{\mu,\tau} &= \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(X_\mu \otimes X_\tau, X_\lambda) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{\underline{Rep}(S_t)}(X_\mu \otimes X_\tau, X_\lambda)\end{aligned}$$

The objects $X_\mu \otimes X_\tau$, X_λ are direct summands in $\mathfrak{h}^{\otimes|\mu|+|\tau|}$, $\mathfrak{h}^{\otimes|\lambda|}$ respectively. Therefore, they lie in $\underline{Rep}(S_t)^{(N)}$, which is semisimple (see Remark 3.1).

We conclude that

$$\bar{g}^{\lambda}_{\mu,\tau} = \dim_{\mathbb{C}} \mathrm{Hom}_{\underline{Rep}(S_t)}(X_\mu \otimes X_\tau, X_\lambda) = [X_\mu \otimes X_\tau : X_\lambda]_t.$$

□

To conclude this section, we prove a lemma which will be useful later on:

Lemma 4.1 *Let μ, τ, λ be three Young diagrams, and let $n \in \mathbb{Z}$, $n \geq |\lambda| + \lambda_1$. Denote by $\{\lambda^{(i)}\}_{i\geq 0}$ the $\overset{n}{\sim}$-class of λ ($\lambda = \lambda^{(0)}$ since $n \geq |\lambda| + \lambda_1$). Then*

$$\begin{aligned}[X_\mu \otimes X_\tau : X_\lambda]_{t=n} = \sum_{j\geq 0} (-1)^j \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(\mathrm{lift}_{t=n}(X_\mu) \\ \otimes \mathrm{lift}_{t=n}(X_\tau), X_{\lambda^{(j)}})\end{aligned}$$

Proof By definition, we have:

$$X_\mu \otimes X_\tau = \bigoplus_{\rho\in\mathcal{P}} [X_\mu \otimes X_\tau : X_\rho]_{t=n} X_\rho$$

so

$$\mathrm{lift}_{t=n}(X_\mu \otimes X_\tau) = \bigoplus_{\rho\in\mathcal{P}} [X_\mu \otimes X_\tau : X_\rho]_{t=n} \mathrm{lift}_{t=n}(X_\rho)$$

On the other hand,

$$[\mathrm{lift}_{t=n}(X_\mu \otimes X_\tau) \colon X_\rho] = \bigoplus_{\rho\in\mathcal{P}} \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(\mathrm{lift}_{t=n}(X_\mu) \otimes \mathrm{lift}_{t=n}(X_\tau), X_\rho)$$

Now, by Lemma 3.2,

$$\mathrm{lift}_{t=n}(X_\lambda) = X_\lambda, \quad \mathrm{lift}_{t=n}(X_{\lambda^{(i)}}) = X_{\lambda^{(i)}} \oplus X_{\lambda^{(i-1)}} \text{ for } i \geq 1$$

so for any $i \geq 0$,

$$\begin{aligned}&\dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(\mathrm{lift}_{t=n}(X_\mu) \otimes \mathrm{lift}_{t=n}(X_\tau), X_{\lambda^{(i)}}) \\ &\quad = [X_\mu \otimes X_\tau : X_{\lambda^{(i)}}]_{t=n} + [X_\mu \otimes X_\tau : X_{\lambda^{(i+1)}}]_{t=n}\end{aligned}$$

and thus

$$\begin{aligned}[X_\mu \otimes X_\tau : X_{\lambda^{(i)}}]_{t=n} = \sum_{j\geq 0} (-1)^j \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(\mathrm{lift}_{t=n}(X_\mu) \\ \otimes \mathrm{lift}_{t=n}(X_\tau), X_{\lambda^{(i+j)}}).\end{aligned}$$

The statement of the lemma is just the special case when $i = 0$. □

5 Properties of Deligne–Kronecker coefficients

5.1 Symmetry

The Deligne–Kronecker coefficient $\bar{g}^\lambda_{\mu,\tau}$ is symmetric in terms of the three partitions λ, μ, τ.

In the context of the Deligne category $\underline{Rep}(S_T)$, this corresponds to the fact that

$$\bar{g}^\lambda_{\mu,\tau} = \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(X_\lambda, X_\mu \otimes X_\tau)$$

and

$$\mathrm{Hom}_{\underline{Rep}(S_T)}(X_\lambda, X_\mu \otimes X_\tau) \cong \mathrm{Hom}_{\underline{Rep}(S_T)}(1, X_\lambda \otimes X_\mu \otimes X_\tau)$$

(since any object in $\underline{Rep}(S_T)$ is self-dual). The last expression is clearly symmetric in λ, μ, τ.

5.2 Murnaghan–Littlewood inequalities

We now give a sufficient condition on the Young diagrams for the Deligne–Kronecker coefficient to be zero. This is called the Murnaghan–Littlewood inequalities.

Lemma 5.1 *Let λ, μ, τ be three partitions of arbitrary sizes, and let $t \in \mathbb{C}$. Then $[X_\mu \otimes X_\tau : X_\lambda]_t = 0$ whenever $|\lambda| > |\mu| + |\tau|$.*

Proof Recall from the construction of Deligne's category ([6, Section 2]) that the indecomposable object X_ρ appears as a direct summand of $\mathfrak{h}^{\otimes|\rho|}$, but does not appear in smaller tensor powers of $\mathfrak{h}$. Thus $X_\mu \otimes X_\tau$ is a direct summand of $\mathfrak{h}^{\otimes(|\mu|+|\tau|)}$, while X_λ cannot appear as a direct summand in $\mathfrak{h}^{\otimes(|\mu|+|\tau|)}$ if $|\lambda| > |\mu| + |\tau|$. □

Applying this lemma to the case $t \notin \mathbb{Z}_+$, we immediately obtain the following well-known Murnaghan–Littlewood inequalities (c.f. [13]):

Corollary 5.1 *Let λ, μ, τ be three partitions of arbitrary sizes. Then $\bar{g}^\lambda_{\mu,\tau} \neq 0$ implies $|\lambda| \leq |\mu| + |\tau|$.*

Of course, since $\bar{g}^\lambda_{\mu,\tau}$ is symmetric with respect to the three Young diagrams, we conclude that $|\tau| \leq |\mu| + |\lambda|$, $|\mu| \leq |\lambda| + |\tau|$ as well.

5.3 Reduced Kronecker coefficients and Littlewood–Richardson coefficients

The following proposition is proved in [8, Proposition 5.11]:

Proposition 5.1 *When $|\lambda| = |\mu| + |\tau|$, the Deligne–Kronecker coefficient $\bar{g}^\lambda_{\mu,\tau}$ is equal to the Littlewood–Richardson coefficient*

$$c^\lambda_{\mu,\tau} := \dim_{\mathbb{C}} \mathrm{Hom}_{S_{|\mu|}\times S_{|\tau|}} \left(\mathrm{Res}^{S_{|\lambda|}}_{S_{|\mu|}\times S_{|\tau|}} \lambda, \mu \otimes \tau \right)$$

We give a sketch of the proof following [8, Sections 2, 5].

Sketch of proof The construction of any indecomposable object X_λ in $\underline{Rep}(S_t)$ can be done in two ways: one is to consider X_λ as a direct summand of $\mathfrak{h}^{\otimes|\lambda|}$, and the other is to consider X_λ as a direct summand of $\Delta_{|\lambda|}$ (here Δ_k is an object of $\underline{Rep}(S_t)$ which is an analogue of the S_n representation $\mathrm{Ind}^{S_n\times S_k}_{S_{n-k}\times S_k\times S_k}\mathbb{C}$).

Let $k = |\lambda|$. Consider the action of S_k on Δ_k^*, which is the largest direct summand of Δ_k having no common direct summands with Δ_{k-1}. The decomposition of Δ_k^* into irreducible S_k-representations with respect to this action gives us

$$\Delta_k^* = \bigoplus_{|\rho|=k} X_\rho \otimes \rho$$

Now, $X_\mu \otimes X_\tau$ is a direct summand of $\Delta_{|\mu|} \otimes \Delta_{|\tau|}$. The latter decomposes as a direct sum of Δ_k for $k \leq |\mu| + |\tau| = |\lambda|$, with $\Delta_{|\lambda|}$ appearing with multiplicity 1.

This allows us to conclude that

$$[X_\mu \otimes X_\tau : X_\lambda] = \dim_{\mathbb{C}} \mathrm{Hom}_{S_{|\mu|}\times S_{|\tau|}} \left(\mathrm{Res}^{S_{|\lambda|}}_{S_{|\mu|}\times S_{|\tau|}} \lambda, \mu \otimes \tau\right) =: c^\lambda_{\mu,\tau}.$$

□

Corollary 5.2 *For any $t \notin \mathbb{Z}_+$, fix a filtration on the Grothendieck ring $\mathfrak{R}(S_t)$ of $\underline{Rep}(S_t)$ by setting $X_\lambda \in \mathrm{Filtra}_{|\lambda|}(\mathfrak{R}(S_t))$. Then the associated graded ring $Gr(\mathfrak{R}(S_t))$ is isomorphic to the ring of symmetric functions, as defined in [11, Chapter I, Par. 5], with X_λ corresponding to the Schur function s_λ.*

Remark 5.1 The ring $Gr(\mathfrak{R}(S_t))$ is also isomorphic to the Grothendieck ring of the tensor category $\bigoplus_{n\geq 0} Rep(S_n)$, with the tensor product given by the Bernstein–Zelevinsky product

$$\mu \otimes \tau := \mathrm{Ind}^{S_{|\mu|+|\tau|}}_{S_{|\mu|}\times S_{|\tau|}} \mu \otimes \tau$$

One can show that the same is true for the all the abelian categories $\underline{Rep}^{ab}(S_t)$: taking an appropriate filtration on the Grothendieck ring of $\underline{Rep}^{ab}(S_t)$, the associated graded ring will be isomorphic to the ring of symmetric functions. The filtration should be taken so that the simple object corresponding to the Young diagram λ lies in the filtra $|\lambda|$.

5.4 Deligne–Kronecker coefficients and standard Kronecker coefficients

The following proposition shows that the Deligne–Kronecker coefficient $\bar{g}^\lambda_{\mu,\tau}$ is a stable value of a sequence of standard Kronecker coefficients, and thus coincides with *the reduced (*or *stable) Kronecker coefficient*. We also obtain a formula for recovering standard Kronecker coefficients from reduced Kronecker coefficients. This formula appears in [3, Theorem 1.1] in a slightly different form; we show that it can be obtained directly from object lifting for Deligne categories.

Proposition 5.2 *Let λ, μ, τ be three partitions of arbitrary sizes, and let $N := \max\{|\lambda| + \lambda_1, |\mu| + \mu_1, |\tau| + \tau_1\}$.*

1. *Consider the sequence*

$$\left\{ g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)} \right\}_{n\geq N}$$

   of standard Kronecker coefficients. This sequence stabilizes, and the stable value is the Deligne–Kronecker coefficient $\bar{g}^\lambda_{\mu,\tau}$. Thus $\bar{g}^\lambda_{\mu,\tau}$ is the reduced Kronecker coefficient corresponding to the triple (λ, μ, τ).
2. *Let $n \geq N$. Denote by $\{\lambda^{(i)}\}_{i\geq 0}$ the $\overset{n}{\sim}$-class of λ ($\lambda = \lambda^{(0)}$ since $n \geq |\lambda| + \lambda_1$). Then*

$$g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)} = [X_\mu \otimes X_\tau : X_{\lambda=\lambda^{(0)}}]_{t=n} = \sum_{i\geq 0} (-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau}$$

3. *The stable value* $\bar{g}^{\lambda}_{\mu,\tau}$ *is the maximum of the sequence* $\{g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)}\}_{n\geq N}$. *Namely,*

$$\bar{g}^{\lambda}_{\mu,\tau} \geq g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)}$$

Remark 5.2 Note that Corollary 5.1 implies that the sum $\sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau}$ is finite (due to the fact that $\{|\lambda^{(i)}|\}_i$ is a strongly increasing sequence).

Proof Let $n \geq N$. Recall the symmetric monoidal functor $\mathcal{S}_n : \underline{Rep}(S_{t=n}) \to Rep(S_n)$ described in Sect. 3. Due to the requirement on N, we have:

$$\mathcal{S}_n(X_\lambda) \cong \widetilde{\lambda}(n),\ \mathcal{S}_n(X_\mu) \cong \widetilde{\mu}(n),\ \mathcal{S}_n(X_\tau) \cong \widetilde{\tau}(n)$$

and since $\mathcal{S}_n$ preserves tensor products, we see that

$$[X_\mu \otimes X_\tau : X_\lambda]_{t=n} = g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)}$$

1. Let $n \geq 2(|\mu| + |\tau|)$. By Proposition 4.2,

$$\bar{g}^{\lambda}_{\mu,\tau} = [X_\mu \otimes X_\tau : X_\lambda]_{t=n}$$

and thus

$$\bar{g}^{\lambda}_{\mu,\tau} = g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)}$$

for any $n \geq 2(|\mu| + |\tau|)$ (in fact, one can use Lemma 3.2 to show that this stable value is reached when $n \geq |\mu| + |\tau| + \mu_1 + \tau_1$). This proves Part (1).

2. Let $n \geq N$. The objects X_λ, X_μ, X_τ are all minimal in their respective $\overset{n}{\sim}$-classes, so by Lemma 3.2,

$$\mathrm{lift}_{t=n}(X_\mu) \cong X_\mu,\ \mathrm{lift}_{t=n}(X_\tau) \cong X_\tau$$

By Lemma 4.1, we have:

$$[X_\mu \otimes X_\tau : X_{\lambda=\lambda^{(0)}}]_{t=n} = \sum_{i\geq 0}(-1)^i \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(\mathrm{lift}_{t=n}(X_\mu)$$
$$\otimes \mathrm{lift}_{t=n}(X_\tau), X_{\lambda^{(i)}}) = \sum_{i\geq 0}(-1)^i [X_\mu \otimes X_\tau : X_{\lambda^{(i)}}]_T = \sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau}$$

and so

$$g^{\widetilde{\lambda}(n)}_{\widetilde{\mu}(n),\widetilde{\tau}(n)} = [X_\mu \otimes X_\tau : X_\lambda]_{t=n} = \sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau}$$

which completes the proof of Part (2).

3. We proved in the proof of Lemma 4.1 that

$$[X_\mu \otimes X_\tau : X_{\lambda^{(1)}}]_{t=n} = \sum_{i\geq 1}(-1)^{i-1}[X_\mu \otimes X_\tau : X_{\lambda^{(i)}}]_T = \sum_{i\geq 1}(-1)^{i-1}\bar{g}_{\mu,\tau}^{\lambda^{(i)}}$$

This multiplicity is a non-negative number, and we just showed that

$$\sum_{i\geq 1}(-1)^{i-1}\bar{g}_{\mu,\tau}^{\lambda^{(i)}} = \bar{g}_{\mu,\tau}^{\lambda^{(0)}} - g_{\widetilde{\mu}(n),\widetilde{\tau}(n)}^{\widetilde{\lambda}(n)}$$

So

$$\bar{g}_{\mu,\tau}^{\lambda^{(0)}=\lambda} \geq g_{\widetilde{\mu}(n),\widetilde{\tau}(n)}^{\widetilde{\lambda}(n)}.$$

□

Remark 5.3 The fact that the sequence $\{g_{\widetilde{\mu}(n),\widetilde{\tau}(n)}^{\widetilde{\lambda}(n)}\}_n$ stabilizes was proved by Murnaghan, see [13,14]. The reduced Kronecker coefficients (sometimes also called "stable Kronecker coefficients") were originally defined as the stabilizing values of such sequences (see [3], for example). From now on we will use the more common term "reduced Kronecker coefficient" instead of "Deligne–Kronecker coefficient".

In addition, it was proved in [4] that this is a weakly increasing sequence (in particular, this implies that its stable value is its maximum).

Remark 5.4 Similarly to the proof of Proposition 5.2, Part (3), one can prove the following statement: consider the partial sums

$$P_k := \sum_{0\leq i\leq k}(-1)^i\,\bar{g}_{\mu,\tau}^{\lambda^{(i)}}$$

Then for any $k \geq 0$,

$$P_{2k} \leq g_{\widetilde{\mu}(n),\widetilde{\tau}(n)}^{\widetilde{\lambda}(n)} \leq P_{2k+1}$$

(the above proposition tells us that $g_{\widetilde{\mu}(n),\widetilde{\tau}(n)}^{\widetilde{\lambda}(n)}$ is the stable value of the sequence $\{P_k\}_{k\geq 0}$).

Let $u = (u_1, u_2, \ldots)$ be a sequence of integers. Denote

$$u^{\dagger i} := (u_1 + 1, \ldots, u_{i-1} + 1, u_{i+1}, \ldots)$$

(this is the sequence obtained from u by removing the i-th term and adding 1 to all the previous terms). Of course, if u was a weakly decreasing sequence (Young diagram), so is $u^{\dagger i}$.

Also, given a Young diagram λ, denote by $\bar{\lambda}$ the Young diagram obtained from λ by removing the top row (thus given $\lambda \vdash n$, $\mathcal{S}_n(X_{\bar{\lambda}}) = \lambda$). With these definitions, the

second statement of Proposition 5.2 can also be reformulated as follows (in this form it appears in [3]):

Proposition 5.3 *Let $n \in \mathbb{Z}_+$, and let λ, μ, τ be three partitions of n. Then*

$$g^{\lambda}_{\mu,\tau} = \sum_{i\geq 1}(-1)^{i+1}\bar{g}^{\lambda^{\dagger i}}_{\bar{\mu},\bar{\tau}}$$

Proof First of all, note that $\bar{\lambda}$ satisfies:

$$\left|\bar{\lambda}\right| + \bar{\lambda}_1 = n - \lambda_1 + \lambda_2 \leq n$$

Thus $\bar{\lambda}$ is the minimal element in a non-trivial $\overset{n}{\sim}$-class. All we need to do is check that $\{\lambda^{\dagger i}\}_{i\geq 1}$ is exactly the $\overset{n}{\sim}$-class of $\bar{\lambda}$.

Denote by $\{\bar{\lambda}^{(i)}\}_{i\geq 0}$ the $\overset{n}{\sim}$-class of $\bar{\lambda}$. Of course,

$$\bar{\lambda}^{(0)} = \bar{\lambda} = \lambda^{\dagger 1}$$

Now, by Lemma 3.1, we have:

$$\begin{aligned}\bar{\lambda}^{(1)} &= (\bar{\lambda}_1 + n - \left|\bar{\lambda}\right| - \bar{\lambda}_1 + 1, \bar{\lambda}_2, \bar{\lambda}_3, \ldots)\\ &= (n - (n - \lambda_1) + 1, \bar{\lambda}_2, \bar{\lambda}_3, \ldots) = (\lambda_1 + 1, \lambda_3, \lambda_4, \ldots) = \lambda^{\dagger 2}\end{aligned}$$

and proceeding by induction on i, we have (for $i > 1$):

$$\begin{aligned}\bar{\lambda}^{(i)} &= (\bar{\lambda}^{(i-1)}_1, \bar{\lambda}^{(i-1)}_2, \ldots, \bar{\lambda}^{(i-1)}_{i-1}, \bar{\lambda}^{(i-1)}_i - \bar{\lambda}_i + \bar{\lambda}_{i-1} + 1, \bar{\lambda}^{(i-1)}_{i+1}, \ldots)\\ &= (\lambda_1+1, \lambda_2+1, \ldots, \lambda_{i-2}+1, \lambda_{i-1}+1, \lambda_{i+1}-\bar{\lambda}_i+\bar{\lambda}_{i-1}+1, \lambda_{i+2}, \ldots) = \lambda^{\dagger(i+1)}\end{aligned}$$

Thus we proved that the sequences $\{\lambda^{\dagger i}\}_{i\geq 1}, \{\bar{\lambda}^{(i)}\}_{i\geq 0}$ of Young diagrams coincide, as wanted. □

5.5 New formulas involving reduced Kronecker coefficients

In this subsection we present close companions of the formula in Proposition 5.2, which are based on the following standard property of rigid symmetric monoidal abelian categories: given an object X and a projective object P, the tensor product $X \otimes P$ is again projective. Note that the second part of Proposition 5.4 is just a generalization of Proposition 5.2, Part (2).

Proposition 5.4 *Let λ, μ, τ be three Young diagrams. Let $n \in \mathbb{Z}$, $|\lambda| + \lambda_1 \leq n$.*

Denote by $\{\lambda^{(i)}\}_{i\geq 0}$ the $\overset{n}{\sim}$-class of λ ($\lambda = \lambda^{(0)}$ since $n \geq |\lambda| + \lambda_1$).

– *If X_μ lies in a trivial $\overset{n}{\sim}$-class (equivalently, if $n \in \{|\mu|+\mu_l-l : l=1,\dots,|\mu|\}$), then*

$$\sum_{i\geq 0}(-1)^i \bar{g}_{\mu,\tau}^{\lambda^{(i)}} = [X_\mu \otimes X_\tau : X_{\lambda=\lambda^{(0)}}]_{t=n} = 0$$

– *Assume both X_μ and X_τ lie in non-trivial $\overset{n}{\sim}$-classes, denoted by $\{\mu^{(i)}\}_{i\geq 0}$ and $\{\tau^{(i)}\}_{i\geq 0}$ respectively (these classes are not necessarily distinct). Let k,l be such that $\mu=\mu^{(k)}$, $\tau=\tau^{(l)}$. Then*

$$\sum_{i\geq 0}(-1)^i \bar{g}_{\mu,\tau}^{\lambda^{(i)}} = (-1)^{k+l} \widetilde{g^{\widetilde{\lambda}(n)}_{\mu^{(0)}(n),\tau^{(0)}(n)}}$$

Remark 5.5 Again, by Corollary 5.1 the above sums are finite (due to the fact that $\{|\lambda^{(i)}|\}_i$ is a strongly increasing sequence).

Proof First of all, recall that the case when $n \geq |\mu|+\mu_1, |\tau|+\tau_1$ has been studied in Proposition 5.2. So without loss of generality, we can assume that $n < |\mu|+\mu_1$.

Notice that the condition $n < |\mu|+\mu_1$ implies that μ is either in a trivial $\overset{n}{\sim}$-class, or a non-minimal element in a non-trivial $\overset{n}{\sim}$-class.

Now, consider the category $\underline{Rep}^{ab}(S_{t=n})$; this is a rigid symmetric monoidal abelian category. From Proposition 3.3, we know that X_μ is a projective object in $\underline{Rep}^{ab}(S_{t=n})$, while X_λ is a simple, but not projective object.

Thus $X_\mu \otimes X_\tau$ is again a projective object. Decomposing it as a sum of indecomposable objects in $\underline{Rep}(S_{t=n})$, we see that all the summands must be projective objects in $\underline{Rep}^{ab}(S_{t=n})$, and so

$$[X_\mu \otimes X_\tau : X_{\lambda=\lambda^{(0)}}]_{t=n} = 0$$

By Lemma 4.1, we have:

$$0 = [X_\mu \otimes X_\tau : X_\lambda]_{t=n} = \sum_{i\geq 0}(-1)^i \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(\mathrm{lift}_{t=n}(X_\mu) \otimes \mathrm{lift}_{t=n}(X_\tau), X_{\lambda^{(i)}})$$

– Assume X_μ lies in a trivial $\overset{n}{\sim}$-class. Then $\mathrm{lift}_{t=n}(X_\mu) = X_\mu$, and so

$$\sum_{i\geq 0}(-1)^i \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(X_\mu \otimes \mathrm{lift}_{t=n}(X_\tau), X_{\lambda^{(i)}}) = 0 \qquad (1)$$

First, assume $\mathrm{lift}_{t=n}(X_\tau) = X_\tau$. In this case, Eq (1) becomes

$$\sum_{i\geq 0}(-1)^i \dim_{\mathrm{Frac}(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(X_\mu \otimes X_\tau, X_{\lambda^{(i)}}) = \sum_{i\geq 0}(-1)^i \bar{g}_{\mu,\tau}^{\lambda^{(i)}} = 0$$

and we are done.

It remains to check the case when $\text{lift}_{t=n}(X_\tau) \neq X_\tau$. In this case, τ lies in a non-trivial $\overset{n}{\sim}$-class. Denote this class by $\{\tau^{(i)}\}_{i\geq 0}$. We will now prove that

$$\sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau^{(j)}} = 0$$

for any $j \geq 0$ by induction on j.

Base: The case when $j = 0$ has already been proved.

Step: Applying Eq. (1) to all $\tau^{(j)}$, $j \geq 1$, we obtain:

$$\begin{aligned} 0 &= \sum_{i\geq 0}(-1)^i \dim_{\text{Frac}(\mathbb{C}[[T]])} \text{Hom}_{\underline{Rep}(S_T)}(X_\mu \otimes X_{\tau^{(j)}}, X_{\lambda^{(i)}}) + \\ &\quad + \sum_{i\geq 0}(-1)^i \dim_{\text{Frac}(\mathbb{C}[[T]])} \text{Hom}_{\underline{Rep}(S_T)}(X_\mu \otimes X_{\tau^{(j)}}, X_{\lambda^{(i)}}) \\ &= \sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau^{(j)}} + \sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau^{(j-1)}} \end{aligned}$$

So assuming

$$\sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau^{(j-1)}} = 0$$

we obtain:

$$\sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau^{(j)}} = 0$$

and we are done.

– Assume both X_μ and X_τ lie in non-trivial $\overset{n}{\sim}$-classes, denoted by $\{\mu^{(i)}\}_{i\geq 0}$ and by $\{\tau^{(i)}\}_{i\geq 0}$ respectively. Let k, l be such that $\mu = \mu^{(k)}$, $\tau = \tau^{(l)}$.

Then

$$\begin{aligned} 0 = [X_\mu \otimes X_\tau : X_\lambda]_{t=n} = \sum_{i\geq 0}(-1)^i \dim_{\text{Frac}(\mathbb{C}[[T]])} \text{Hom}_{\underline{Rep}(S_T)}((X_{\mu^{(k)}} \oplus X_{\mu^{(k-1)}}) \\ \otimes (X_{\tau^{(l)}} \oplus X_{\tau^{(l-1)}}), X_{\lambda^{(i)}}) \end{aligned}$$

and so

$$\begin{aligned} &\sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu^{(k)},\tau^{(l)}} + \sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu^{(k)},\tau^{(l-1)}} + \sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu^{(k-1)},\tau^{(l)}} \\ &\quad + \sum_{i\geq 0}(-1)^i \bar{g}^{\lambda^{(i)}}_{\mu^{(k-1)},\tau^{(l-1)}} = 0 \end{aligned}$$

By induction on $k+l$, we can now prove that

$$\sum_{i\geq 0}(-1)^i \bar{g}_{\mu,\tau}^{\lambda(i)} = (-1)^{k+l} g_{\widetilde{\mu^{(0)}(n)},\widetilde{\tau^{(0)}(n)}}^{\widetilde{\lambda}(n)}$$

(the base case $k+l=0$ was proved in Proposition 5.2). □

References

1. Bowman, C., De Visscher, M., Orellana, R.: The partition algebra and the Kronecker coefficients. arXiv:1210.5579v5 [math.RT]
2. Briand, E., Orellana, R., Rosas, M.: Reduced Kronecker coefficients and counter-examples to mulmuley's strong saturation conjecture SH. Comput. Complex. **18**(4), 577–600, Birkhäuser-Verlag. arXiv:0810.3163 (2009)
3. Briand, E., Orellana, R., Rosas, M.: The stability of the Kronecker product of Schur functions. J. Algebra **331**(1), 11–27. arXiv:0907.4652v2 [math.RT] (2011)
4. Brion, M.: Stable properties of plethysm: on two conjectures of Foulkes. Manuscr. Math. **80**, 347–371 (1993)
5. Church, T., Ellenberg, J.S., Farb, B.: FI-modules and stability for representations of symmetric groups. Duke Math J. arXiv:1204.4533 [math.RT]
6. Comes, J., Ostrik, V.: On blocks of Deligne's category $\underline{Rep}(S_t)$. Adv. Math. **226**(2), 1331–1377. arXiv:0910.5695 [math.RT]. An appendix concerning reduced Kronecker coefficients appears on the website http://pages.uoregon.edu/jcomes/TensorAppendix (2011)
7. Comes, J., Ostrik, V.: On Deligne's category $Rep^{ab}(S_d)$. Algebra Number Theory **8**(2), 473-496. arXiv:1304.3491v1 [math.RT] (2014)
8. Deligne, P.: La categorie des representations du groupe symetrique S_t, lorsque t n'est pas un entier naturel. Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., pp 209–273. Mumbai. http://www.math.ias.edu/~phares/deligne/preprints.html (2007)
9. Dobrovolska, G.: Fourier-Deligne transform and representations of the symmetric group. Math. Res. Lett. **20**(6), 1059–1070. arXiv:1301.2157 [math.AG] (2013)
10. Entova-Aizenbud, I.: Schur–Weyl duality for Deligne categories. arXiv:1403.5509 [math.RT]
11. Macdonald, I.G.: Symmetric Functions and Hall polynomials. Oxford Mathematical Monographs, Oxford (1995)
12. Mulmuley, K.D.: Geometric complexity theory VI: the flip via saturated and positive integer programming in representation theory and algebraic geometry. Technical Report, Computer Science Department, The University of Chicago. arXiv:0704.0229 [cs.CC] (2011)
13. Murnaghan, F.D.: The analysis of the Kronecker product of irreducible representations of the symmetric group. Am. J. Math. **60**(3), 761–784 (1938)
14. Murnaghan, F.D.: On the analysis of the Kronecker product of irreducible representations of S_n. Proc. Natl. Acad. Sci. USA **41**, 515–518 (1955)
15. Sam, S.V., Snowden, A.: Stability patterns in representation theory. arXiv:1302.5859 [math.RT]
16. Stanley, R.P.: Enumerative Combinatorics, Volume 62 of Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, Cambridge (1999)