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Resonances with Applications to Black Holes

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Abstract: We describe asymptotic behavior of linear waves on Kerr(–de Sitter) black holes and more general Lorentzian manifolds, providing a quantitative analysis of the ringdown phenomenon. In particular we prove that if the initial data is localized at frequencies $\sim \lambda \gg 1$, then the energy norm of the solution is bounded by $\mathcal{O}(\lambda^{1/2} e^{-(\nu_{\min}-\varepsilon)t/2}) + \mathcal{O}(\lambda^{-\infty})$, for $t \leq C \log \lambda$, where $\nu_{\min}$ is a natural dynamical quantity. The key tool is a microlocal projector splitting the solution into a component with controlled rate of exponential decay and an $\mathcal{O}(\lambda e^{-(\nu_{\min}-\varepsilon)t}) + \mathcal{O}(\lambda^{-\infty})$ remainder. This splitting generalizes expansions into quasi-normal modes available in completely integrable settings. In the case of generalized Kerr(–de Sitter) black holes satisfying certain natural conditions, quasi-normal modes are localized in bands and satisfy a precise counting law.

1. Introduction

The subject of this paper is decay properties of solutions to the wave equation for the rotating Kerr (cosmological constant $\Lambda = 0$) and Kerr–de Sitter ($\Lambda > 0$) black holes, as well as for their stationary perturbations. These solutions are the linear scalar model for the *ringdown* phenomenon of black holes, and the results of this paper provide a quantitative understanding of the waves emitted at ringdown—see [DyZw] for a further discussion of applications to physics.

In the recent decade, there has been a lot of progress in understanding decay of linear waves, producing a polynomial decay rate $\mathcal{O}(t^{-3})$ for Kerr and an exponential decay rate $\mathcal{O}(e^{-\nu t})$ for Kerr–de Sitter (the latter is modulo constant functions). The weaker decay for $\Lambda = 0$ is explained by the presence of an asymptotically Euclidean infinite end; however, this polynomial decay comes from low frequency contributions.

In this paper we concentrate on the decay of solutions with initial data localized at *high frequencies* $\sim \lambda \gg 1$; it is related to the geometry of the *trapped set* $\widetilde{K}$, consisting of lightlike geodesics that never escape to the spatial infinity or through the event horizons.

The trapped set for both Kerr and Kerr–de Sitter metrics is *r-normally hyperbolic*, and this dynamical property is stable under stationary perturbations of the metric—see Sect. 3.6. The key quantities associated to such trapping are the minimal and maximal transversal expansion rates $0 < \nu_{\min} \leq \nu_{\max}$, see (2.9), (2.10). Using our recent work [Dy15], we show the exponential decay rate $\mathcal{O}(\lambda^{1/2} e^{-(\nu_{\min}-\varepsilon)t/2}) + \mathcal{O}(\lambda^{-\infty})$, valid for $t = \mathcal{O}(\log \lambda)$ (Theorem 1). This bound is new for the Kerr case, complementing Price's law.

Our methods give a more precise microlocal description of long time propagation of high frequency solutions. In Theorem 2, we split a solution $u(t)$ into two approximate solutions to the wave equation, $u_\Pi(t)$ and $u_R(t)$, with the rate of decay of $u_\Pi(t)$ between $e^{-(\nu_{\max}+\varepsilon)t/2}$ and $e^{-(\nu_{\min}-\varepsilon)t/2}$ and $u_R(t)$ bounded from above by $\lambda e^{-(\nu_{\min}-\varepsilon)t}$, all modulo $\mathcal{O}(\lambda^{-\infty})$ errors. The splitting is achieved using a Fourier integral operator quantizing a natural canonical relation and built in [Dy15] from the global dynamics of the flow. This result can be viewed as describing the *singularities of the least decaying component of the wave*.

For the $\Lambda > 0$ case, we furthermore study resonances, or *quasi-normal modes*, the complex frequencies z of solutions to the wave equation of the form $e^{-izt} v(x)$. Under a pinching condition $\nu_{\max} < 2\nu_{\min}$ which is numerically verified to be true for a large range of parameters (see Fig. 2a), we show existence of a band of quasi-normal modes satisfying a Weyl law—Theorem 3. In particular, this provides a large family of exact high frequency solutions to the wave equation that decay no faster than $e^{-(\nu_{\max}+\varepsilon)t/2}$. We finally compare our theoretical prediction on the imaginary parts of resonances in the band with the exact quasi-normal modes for Kerr computed by the authors of [BeCaSt], obtaining a remarkable agreement—see Fig. 2b.

Theorems 1–3 are related to the resonance expansion and quantization condition proved for the slowly rotating Kerr–de Sitter in [Dy12]. In this paper we only use dynamical assumptions stable under perturbations, rather than complete integrability of geodesic flow on Kerr(–de Sitter), and do not recover the precise results of [Dy12].

Statement of results. The Kerr(–de Sitter) metric, described in detail in Sect. 3.1, depends on three parameters, M (mass), a (speed of rotation), and Λ (cosmological
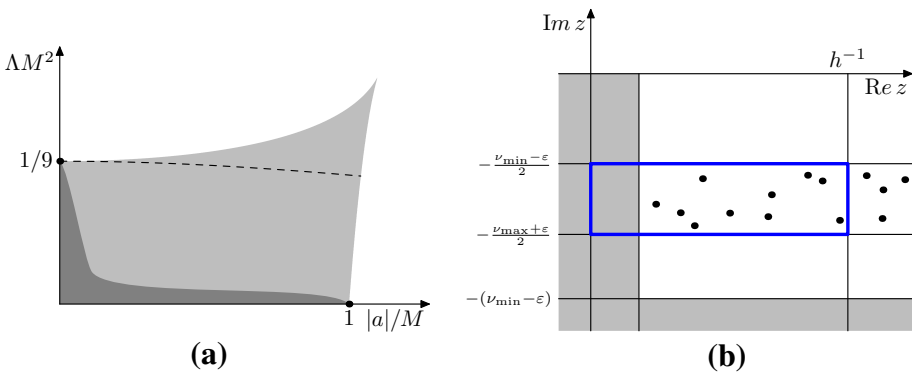
Fig. 1. a The numerically computed admissible range of parameters for the subextremal Kerr–de Sitter black hole (*light shaded*) and a schematic depiction of the range of parameters to which our results apply (*dark shaded*). The region *above the dashed line* is where the resolvent does not admit a meromorphic continuation, see (3.16). **b** An illustration of Theorem 3; (1.13) counts resonances in the *outlined box* and the *unshaded regions above and below the box* represent (1.11)

constant). We assume that the dimensionless quantities a/M and ΛM^2 lie in a small neighborhood (see Fig. 1a) of either the Schwarzschild(–de Sitter) case,

$$a = 0, \quad 9\Lambda M^2 < 1, \tag{1.1}$$

or the subextremal Kerr case

$$\Lambda = 0, \quad |a| < M. \tag{1.2}$$

(See the remark following Proposition 3.2 for a discussion of the range of parameters.) To facilitate the discussion of perturbations, we adopt the abstract framework of Sect. 2.2, with the spacetime $\widetilde{X}_0 = \mathbb{R}_t \times X_0$ and a Lorentzian metric $\widetilde{g}$ on $\widetilde{X}_0$ which is stationary in the sense that ∂_t is Killing. The space slice X_0 is noncompact because of the spatial infinity and/or event horizon(s); to measure the distance to those, we use a function $\mu \in C^\infty(X_0; (0, \infty))$, such that $X_\delta := \{\mu > \delta\}$ is compact for each $\delta > 0$. For the exact Kerr(–de Sitter metric), the function μ is defined in (3.6).

We study solutions to the wave equation in $\widetilde{X}_0$,

$$\Box_{\widetilde{g}} u(t) = 0, \ t \geq 0; \quad u|_{t=0} = f_0, \ \partial_t u|_{t=0} = f_1, \tag{1.3}$$

with $f_0, f_1 \in C_0^\infty(X_0)$ and the time variable shifted so that the metric continues smoothly past the event horizons—see (3.45). To simplify the statements, and because our work focuses on the phenomena caused by trapping, we only study the behavior of solutions in X_{δ_1} for some small $\delta_1 > 0$. Define the energy norm

$$\|u(t)\|_{\mathcal{E}} := \|u(t)\|_{H^1(X_{\delta_1})} + \|\partial_t u(t)\|_{L^2(X_{\delta_1})}. \tag{1.4}$$

Theorem 1. *Fix $T, N > 0$, $\varepsilon, \delta_1 > 0$, and let $(\widetilde{X}_0, \widetilde{g})$ be the Kerr(–de Sitter) metric with M, a, Λ near one of the cases* (1.1) *or* (1.2)*, or its small stationary perturbation as discussed in Sect.* 3.6 *(the maximal size of the perturbation depending on T, N).*

Assume that $f_0(\lambda), f_1(\lambda) \in C_0^\infty(X_{\delta_1})$ are localized at frequency $\sim \lambda \to \infty$ in the sense of (1.6)*. Then the solution u_λ to* (1.3) *with initial data f_0, f_1 satisfies the bound*

$$\|u_\lambda(t)\|_{\mathcal{E}} \leq C(\lambda^{1/2} e^{-(\nu_{\min} - \varepsilon)t/2} + \lambda^{-N})\|u_\lambda(0)\|_{\mathcal{E}}, \quad 0 \leq t \leq T \log \lambda. \tag{1.5}$$

Here we say that $f = f(\lambda)$ is localized at frequencies $\sim \lambda$, if for each coordinate neighborhood U in X_0 and each $\chi \in C_0^\infty(U)$, the Fourier transforms $\widehat{\chi f}(\xi)$ in the corresponding coordinate system satisfy for each N,

$$\int_{\mathbb{R}^3 \setminus \{C_{U,\chi}^{-1} \leq |\xi| \leq C_{U,\chi}\}} \langle \xi \rangle^N |\widehat{\chi f}(\xi)|^2 \, d\xi = \mathcal{O}(\lambda^{-N}), \tag{1.6}$$

where $C_{U,\chi} > 0$ is a constant independent of λ. For the proof, it is more convenient to use *semiclassical rescaling* of frequencies $\xi \mapsto h\xi$, where $h = \lambda^{-1} \to 0$ is the semiclassical parameter, and the notion of h-wavefront set $\mathrm{WF}_h(f) \subset T^*X_0$. The requirement that f_j is microlocalized at frequencies $\sim h^{-1}$ is then equivalent to stating that $\mathrm{WF}_h(f_j)$ is contained in a fixed compact subset of $T^*X_0 \setminus 0$, with 0 denoting the zero section; see Sect. 2.1 for details.

The main component of the proof of Theorem 1 is the following

Theorem 2. *Under the assumptions of Theorem* 1, *for each families* $f_0(h)$, $f_1(h) \in C_0^\infty(X_{\delta_1})$ *with* $\mathrm{WF}_h(f_j)$ *contained in a fixed compact subset of* $T^*X_0 \backslash 0$ *and* $u(h)$ *the corresponding solution to* (1.3), *for* t_0 *large enough there exists a decomposition*

$$u(t,x) = u_\Pi(t,x) + u_R(t,x), \quad t_0 \le t \le T\log(1/h), \; x \in X_{\delta/2},$$

such that $\Box_{\tilde{g}} u_\Pi(t)$, $\Box_{\tilde{g}} u_R(t)$ *are* $\mathcal{O}(h^N)_{H_h^N}$ *on* X_{δ_1} *uniformly in* $t \in [t_0, T\log(1/h)]$, *and we have uniformly in* $t_0 \le t \le T\log(1/h)$,

$$\|u_\Pi(t_0)\|_{\mathcal{E}} \le Ch^{-1/2}\|u(0)\|_{\mathcal{E}}, \tag{1.7}$$

$$\|u_\Pi(t)\|_{\mathcal{E}} \le Ce^{-(\nu_{\min}-\varepsilon)t/2}\|u_\Pi(t_0)\|_{\mathcal{E}} + Ch^N\|u(0)\|_{\mathcal{E}}, \tag{1.8}$$

$$\|u_\Pi(t)\|_{\mathcal{E}} \ge C^{-1}e^{-(\nu_{\max}+\varepsilon)t/2}\|u_\Pi(t_0)\|_{\mathcal{E}} - Ch^N\|u(0)\|_{\mathcal{E}}, \tag{1.9}$$

$$\|u_R(t)\|_{\mathcal{E}} \le C(h^{-1}e^{-(\nu_{\min}-\varepsilon)t} + h^N)\|u(0)\|_{\mathcal{E}}. \tag{1.10}$$

The decomposition $u = u_\Pi + u_R$ is achieved in Sect. 2.4 using the Fourier integral operator Π constructed for r-normally hyperbolic trapped sets in [Dy15]. The component u_Π enjoys additional microlocal properties, such as localization on the outgoing tail and approximately solving a pseudodifferential equation—see the proof of Theorem 4 in Sect. 2.4 and [Dy15, Sect. 8.5]. We note that (1.9) gives a *lower bound* on the rate of decay of the approximate solution u_Π, if $\|u_\Pi(t_0)\|_{\mathcal{E}}$ is not too small compared to $\|u(0)\|_{\mathcal{E}}$, and the existence of a large family of solutions with the latter property follows from the construction of u_Π.

We remark that Theorems 1 and 2 are completely independent from the behavior of linear waves at low frequency. In fact, we do not even use the boundedness in time of solutions for the wave equation, assuming merely that they grow at most exponentially (which is trivially true in our case); this suffices since $\mathcal{O}(h^\infty)$ remainders overcome such growth for $t = \mathcal{O}(\log(1/h))$. If a boundedness statement is available, then our results can be extended to all times, though the corresponding rate of decay stays fixed for $t \gg \log(1/h)$ because of the $\mathcal{O}(h^\infty)$ term.

To formulate the next result, we restrict to the case $\Lambda > 0$, or its small stationary perturbation. In this case, the metric has two event horizons and we consider the discrete set Res of *resonances*, as defined for example in [Va10]. As a direct application of [Dy15, Theorems 1 and 2], we obtain two gaps and a band of resonances in between with a Weyl law (see Fig. 1b).

Theorem 3. *Let* $(\widetilde{X}_0, \tilde{g})$ *be the Kerr–de Sitter metric with* M, a, Λ *near one of the cases* (1.1) *or* (1.2) *and* $\Lambda > 0$, *or its small stationary perturbation as discussed in Sect.* 3.6. *Fix* $\varepsilon > 0$. *Then:*

1. *For h small enough, there are no resonances in the region*

$$\{|\operatorname{Re} z| \ge h^{-1}, \; \operatorname{Im} z \in [-(\nu_{\min} - \varepsilon), 0] \backslash \tfrac{1}{2}(-(\nu_{\max} + \varepsilon), -(\nu_{\min} - \varepsilon))\} \tag{1.11}$$

and the corresponding semiclassical scattering resolvent, namely the inverse of the operator (3.56), *is bounded by* Ch^{-2} *for z in this region.*

2. *Under the pinching condition*

$$\nu_{\max} < 2\nu_{\min} \tag{1.12}$$

and for ε small enough so that $\nu_{\max} + \varepsilon < 2(\nu_{\min} - \varepsilon)$, *we have the Weyl law*

$$\#(\mathrm{Res} \cap \{0 \le \operatorname{Re} z \le h^{-1}, \; \operatorname{Im} z \in [-(\nu_{\min} - \varepsilon), 0]\}) = (2\pi h)^{1-n}(c_{\widetilde{K}} + o(1)) \tag{1.13}$$
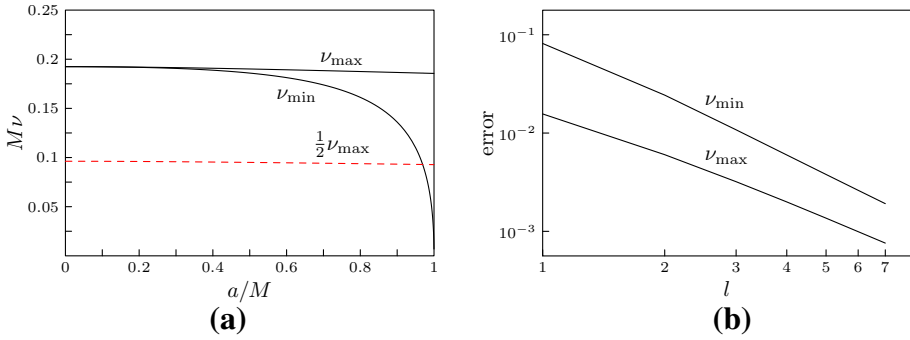
Fig. 2. a Numerically computed minimal and maximal transversal expansion rates for $\Lambda = 0$ and the range of a for which (1.12) holds. **b** A log-log plot of the relative errors $\frac{|\nu_{\min}^R(l) - \nu_{\min}/2|}{\nu_{\min}^R(l)}$, $\frac{|\nu_{\max}^R(l) - \nu_{\max}/2|}{\nu_{\max}^R(l)}$, where $\nu_{\min}^R(l)$, $\nu_{\max}^R(l)$ are the minimal/maximal imaginary parts of resonances in the first band defined by (1.14)

as $h \to 0$, where $c_{\widetilde{K}}$ is the symplectic volume of a certain part of the trapped set $\widetilde{K}$, see (2.16) and (3.42).

The pinching condition (1.12) is true for the non-rotating case $a = 0$, since $\nu_{\min} = \nu_{\max}$ there (see Proposition 3.8). However, it is violated for the nearly extremal case $M - |a| \ll M$, at least for Λ small enough; in fact, as $|a|/M \to 1$, $\nu_{\max}$ stays bounded away from zero, while $\nu_{\min}$ converges to zero—see Proposition 3.9 and Fig. 2a. Note that $(\nu_{\min} - \varepsilon)/2$ is the size of the resonance free strip and thus gives the minimal rate of exponential decay of linear waves on Kerr–de Sitter, modulo terms coming from finitely many resonances, by means of a resonance expansion—see for example [Va10, Lemma 3.1].

To demonstrate the sharpness of the size of the band of resonances $\{\operatorname{Im}\omega \in \frac{1}{2}[-\nu_{\max} - \varepsilon, -\nu_{\min} + \varepsilon]\}$, we use the exact quasi-normal modes for the Kerr metric computed (formally, since one cannot meromorphically continue the resolvent in the $\Lambda = 0$ case; however, one could consider the case of a very small positive Λ) by Berti–Cardoso–Starinets [BeCaSt]. Similarly to the quantization condition of [Dy12], these resonances ω_{mlk} are indexed by three integer parameters $m \geq 0$ (depth), $l \geq 0$ (angular energy), and $k \in [-l, l]$ (angular momentum). The parameter l roughly corresponds to the real part of the resonance and the parameter m, to its imaginary part. We define

$$\nu_{\min}^R(l) := \min_{k \in [-l,l]} (-\operatorname{Im}\omega_{0lk}), \quad \nu_{\max}^R(l) := \max_{k \in [-l,l]} (-\operatorname{Im}\omega_{0lk}). \tag{1.14}$$

We compare $\nu_{\min}^R(l)$, $\nu_{\max}^R(l)$ with $\nu_{\min}/2$, $\nu_{\max}/2$ and plot the supremum of the relative error over $a/M \in [0, 0.95]$ for different values of l; the error decays like $\mathcal{O}(l^{-1})$—see Fig. 2b.

Previous work. We give an overview of results on decay and non-decay on black hole backgrounds; for a more detailed discussion of previous results on normally hyperbolic trapped sets and resonance asymptotics, see the introduction to [Dy15].

The study of boundedness of solutions to the wave equation for the Schwarzschild ($\Lambda = a = 0$) black hole was initiated in [Wa, KaWa] and decay results for this case have been proved in [BlSt, DaRo09, MMTT, Lu10a]. The slowly rotating Kerr case ($\Lambda = 0$, $|a| \ll M$) was considered in [AnBl, DaRo11, DaRo08, Ta, TaTo, To, MeTaTo, Lu10b],

and the full subextremal Kerr case ($\Lambda = 0, |a| < M$) in [FKSY,FKSYErr,DaRo10, DaRo12,Sh1,Sh2]—see [DaRo12] for a more detailed overview. In either case the decay is polynomial in time, with the optimal decay rate $\mathcal{O}(t^{-3})$. A decay rate of $\mathcal{O}(t^{-2l-3})$, known as *Price's Law*, was proved in [DoScSo11,DoScSo12] for linear waves on the Schwarzschild black hole for solutions living on the lth spherical harmonic; the constant in the $\mathcal{O}(\cdot)$ depends on l. Our Theorem 1 improves on these decay rates in the high frequency regime $l = \lambda \gg 1$, for times $\mathcal{O}(\log \lambda)$.

The extremal Kerr case ($\Lambda = 0, |a| = M$) was recently studied for axisymmetric solutions in [Ar12], with a weaker upper bound due to the degeneracy of the event horizon. The earlier work [Ar11a,Ar11b] suggests that one cannot expect the $\mathcal{O}(t^{-3})$ decay to hold in the extremal case. In the high frequency regime studied here, we do not expect to get exponential decay due to the presence of slowly damped geodesics near the event horizon, see Fig. 2a above.

The Schwarzschild–de Sitter case ($\Lambda > 0, a = 0$) was considered in [SBZw,BoHä, DaRo07,MeSBVa], proving an exponential decay rate at all frequencies, a quantization condition for resonances, and a resonance expansion, all relying on separation of variables techniques. In [Dy11a,Dy11b,Dy12], a same flavor of results was proved for the slowly rotating Kerr–de Sitter ($\Lambda > 0, |a| \ll M$). The problem was then studied from a more geometric perspective, aiming for results that do not depend on symmetries and apply to perturbations of the metric—the resonance free strip of [WuZw] for normally hyperbolic trapping, the gluing method of [DaVa], and the analysis of the event horizons and low frequencies of [Va10] together give an exponential decay rate which is stable under perturbations, for $\Lambda > 0, |a| < \frac{\sqrt{3}}{2}M$, provided that there are no resonances in the upper half-plane except for the resonance at zero. Our Theorem 3 provides detailed information on the behavior of resonances below the resonance free strip of [WuZw], without relying on the symmetries of the problem.

Finally, we mention the Kerr–AdS case ($\Lambda < 0$). The metric in this case exhibits strong (elliptic) trapping, which suggests that the decay of linear waves is very slow because of the high frequency contributions. A logarithmic upper bound $\mathcal{O}(1/\log t)$ was proved in [HoSm11], and the existence of resonances exponentially close to the real axis and a logarithmic lower bound were established in [Ga,HoSm13].

Quasi-normal modes (QNMs) of black holes have a rich history of study in the physics literature, see [KoSc]. The exact QNMs of Kerr black holes were computed in [BeCaSt], which we use for Fig. 2b. The high-frequency approximation for QNMs, using separation of variables and WKB techniques, has been obtained in [YNZZZC,YZZNBC,YZZC, Ho]. In particular, for the nearly extremal Kerr case their size of the resonance free strip agrees with Proposition 3.9; moreover, they find a large number of QNMs with small imaginary parts, which correspond to a positive proportion of the Liouville tori on the trapped set lying close to the event horizon. See [YNZZZC] for an overview of the recent physics literature on the topic. We remark that the speed of rotation of an astrophysical black hole (NGC 1365) has recently been accurately measured in [RHMWBCCGHNSZ], yielding a high speed of rotation: $a/M \geq 0.84$ at 90 % confidence.

In the more recent physics work [YZZC], quasi-normal mode contributions for scalar waves on the Kerr metric are summed over separate bands to obtain a decomposition similar to the one in Theorem 2. We remark that [YZZC] relies on separation of variables techniques and WKB approximations for individual quasi-normal modes, not unlike the ones in [Dy12]; the contributions of QNMs are then summed using the Poisson formula. The present paper exhibits a similar phenomenon, relying instead on the geometry and the dynamics of the spacetime, and applies to perturbations of Kerr as well.

Structure of the paper. In Sect. 2, we study semiclassical properties of solutions to the wave equation on stationary Lorentzian metrics with noncompact space slices. We operate under the geometric and dynamical assumptions of Sect. 2.2. These assumptions are motivated by Kerr(–de Sitter) metrics and their stationary perturbations, but no explicit mention of these metrics is made at this point. The analysis of Sect. 2 works in a fixed compact subset of the space slice, and the results apply under microlocal assumptions in this compact subset. More precisely, we assume the outgoing property of solutions to the wave equation for Theorems 1–2 and meromorphic continuation of the scattering resolvent with an outgoing parametrix for Theorem 3 (which are verified for our specific applications in Sects. 3.4 and 3.5). In Sect. 2.3, we reduce the problem to the space slice via the stationary d'Alembert–Beltrami operator and show that some of the assumptions of [Dy15, Sects. 4.1, 5.1] are satisfied. In Sect. 2.4, we use the methods of [Dy15] to prove asymptotics of outgoing solutions to the wave equation.

Section 3 contains the applications of [Dy15] and Sect. 2 to the Kerr(–de Sitter) metrics and their perturbations. In Sect. 3.1, we define the metrics and establish their basic properties, verifying in particular the geometric assumptions of Sect. 2.2. In Sect. 3.2, we show that the trapping is r-normally hyperbolic, verifying the dynamical assumptions of Sect. 2.2. In Sect. 3.3, we study in greater detail trapping in the Schwarzschild(–de Sitter) case $a = 0$ and in the nearly extremal Kerr case $\Lambda = 0$, $a = M - \epsilon$, in particular showing that the pinching condition (1.12) is violated for the latter case. In the same subsection we study numerically some properties of the trapping for the general Kerr case, and give a formula for the constant in the Weyl law. In Sect. 3.4, we study solutions to the wave equation on Kerr(–de Sitter), using the results of Sect. 2.4 to prove Theorems 1 and 2. In Sect. 3.5, we use the results of [Dy15, Va10] to prove Theorem 3 for Kerr–de Sitter. Finally, in Sect. 3.6, we explain why our results apply to small smooth stationary perturbations of Kerr(–de Sitter) metrics.

2. General Framework for Linear Waves

2.1. Semiclassical preliminaries. We start by briefly reviewing some notions of semi-classical analysis, following [Dy15, Sect. 3]. For a detailed introduction to the subject, the reader is directed to [Zw].

Let X be an n-dimensional manifold without boundary. Following [Dy15, Sect. 3.1], we consider the class $\Psi^k(X)$ of all semiclassical pseudodifferential operators with classical symbols of order k. If X is noncompact, we impose no restrictions on how fast the corresponding symbols can grow at spatial infinity. The microsupport of a pseudodifferential operator $A \in \Psi^k(X)$, also known as its h-wavefront set $\mathrm{WF}_h(A)$, is a closed subset of the fiber-radially compactified cotangent bundle $\overline{T}^* X$. We denote by $\Psi^{\mathrm{comp}}(X)$ the class of all pseudodifferential operators whose wavefront set is a compact subset of $T^* X$ (and in particular lies away from the fiber infinity). Finally, we say that $A = \mathcal{O}(h^\infty)$ microlocally in some open set $U \subset \overline{T}^* X$, if $\mathrm{WF}_h(A) \cap U = \emptyset$; similar notions apply to tempered distributions and operators below.

Using pseudodifferential operators, we can study microlocalization of h-tempered distributions, namely families of distributions $u(h) \in \mathcal{D}'(X)$ having a polynomial in h bound in some Sobolev norms on compact sets, by means of the wavefront set $\mathrm{WF}_h(u) \subset \overline{T}^* X$. Using Schwartz kernels, we can furthermore study h-tempered operators $B(h) : C_0^\infty(X_1) \to \mathcal{D}'(X_2)$ and their wavefront sets $\mathrm{WF}_h(B) \subset \overline{T}^*(X_1 \times X_2)$. Besides pseudodifferential operators (whose wavefront set is this framework is the image under the diagonal embedding $\overline{T}^* X \to \overline{T}^*(X \times X)$ of the wavefront set used in the

previous paragraph) we will use the class $I^{\mathrm{comp}}(\Lambda)$ of compactly supported and compactly microlocalized Fourier integral operators associated to some canonical relation $\Lambda \subset T^*(X \times X)$, see [Dy15, Sect. 3.2]; for $B \in I^{\mathrm{comp}}(\Lambda)$, $\mathrm{WF}_h(B) \subset \Lambda$ is compact.

The h-wavefront set of an h-tempered family of distributions $u(h)$ can be characterized using the semiclassical Fourier transform

$$\mathcal{F}_h v(\xi) = (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{i}{h} x \cdot \xi} v(x)\, dx, \quad v \in \mathscr{S}'(\mathbb{R}^n).$$

We have $(x, \xi) \notin \mathrm{WF}_h(u)$ if and only if there exists a coordinate neighborhood U_x of x in X, a function $\chi \in C_0^\infty(U_x)$ with $\chi(x) \neq 0$, and a neighborhood U_ξ of ξ in $\overline{T}_x^* X$ such that if we consider χu as a function on $\mathbb{R}^n$ using the corresponding coordinate system, then for each N,

$$\int_{U_\xi} \langle \xi \rangle^N |\mathcal{F}_h(\chi u)(\xi)|^2\, d\xi = \mathcal{O}(h^N). \tag{2.1}$$

The proof is done analogously to [HöIII, Theorem 18.1.27].

One additional concept that we need is microlocalization of distributions depending on the time variable that varies in a set whose size can grow with h. Assume that $u(t; h)$ is a family of distributions on $(-\varepsilon, T(h) + \varepsilon) \times X$, where $\varepsilon > 0$ is fixed and $T(h) > 0$ depends on h. For $s \in [0, T(h)]$, define the shifted function

$$u_s(t; h) = u(s + t; h), \quad t \in (-\varepsilon, \varepsilon),$$

so that $u_s \in \mathcal{D}'((-\varepsilon, \varepsilon) \times X)$ is a distribution on a time interval independent of h. We then say that u is h-tempered uniformly in t, if u_s is h-tempered uniformly in s, that is, for each $\chi \in C_0^\infty((-\varepsilon, \varepsilon) \times X)$, there exist constants C and N such that $\|\chi u_s\|_{H_h^{-N}} \leq Ch^{-N}$ for all $s \in [0, T(h)]$. Next, we define the projected wavefront set $\widetilde{\mathrm{WF}_h}(u) \subset \overline{T^*X \times \mathbb{R}_\tau}$, where τ is the momentum corresponding to t and $\overline{T^*X \times \mathbb{R}_\tau}$ is the fiber-radial compactification of the vector bundle $T^*X \times \mathbb{R}_\tau$, with $\mathbb{R}_\tau$ part of the fiber, as follows: (x, ξ, τ) does not lie in $\widetilde{\mathrm{WF}_h}(u)$ if and only if there exists a neighborhood U of (x, ξ, τ) in $\overline{T^*X \times \mathbb{R}_\tau}$ such that

$$\sup_{s \in [0, T(h)]} \|A u_s\|_{L^2} = \mathcal{O}(h^\infty)$$

for each compactly supported $A \in \Psi^{\mathrm{comp}}((-\varepsilon, \varepsilon) \times X)$ such that $\mathrm{WF}_h(A) \cap ((-\varepsilon, \varepsilon)_t \times U) = \emptyset$. If $T(h)$ is independent of h, then $\widetilde{\mathrm{WF}_h}(u)$ is simply the closure of the projection of $\mathrm{WF}_h(u)$ onto the (x, ξ, τ) variables. The notion of $\widetilde{\mathrm{WF}_h}$ makes it possible to talk about u being microlocalized inside, or being $\mathcal{O}(h^\infty)$, on subsets of $\overline{T}^*((-\varepsilon, T(h) + \varepsilon) \times X)$ independent of t.

We now discuss restrictions to space slices. Assume that $u(h) \in \mathcal{D}'((-\varepsilon, T(h) + \varepsilon) \times X)$ is h-tempered uniformly in t and moreover, $\widetilde{\mathrm{WF}_h}(u)$ does not intersect the spatial fiber infinity $\{\xi = 0, \tau = \infty\}$. Then u (as well as all its derivatives in t) is a smooth function of t with values in $\mathcal{D}'(X)$, $u(t)$ is h-tempered uniformly in $t \in [0, T(h)]$, and

$$\mathrm{WF}_h(u(t)) \subset \{(x, \xi) \mid \exists \tau : (x, \xi, \tau) \in \widetilde{\mathrm{WF}_h}(u)\},$$

uniformly in $t \in [0, T(h)]$. One can see this using (2.1) and the formula for the Fourier transform of the restriction w of $v \in \mathscr{S}'(\mathbb{R}^{n+1})$ to the hypersurface $\{t = 0\}$:

$$\mathcal{F}_h w(\xi) = (2\pi h)^{-1/2} \int_{\mathbb{R}} \mathcal{F}_h v(\xi, \tau)\, d\tau.$$

2.2. General assumptions. In this section, we study Lorentzian metrics whose space slice is noncompact, and define r-normal hyperbolicity and the dynamical quantities $\nu_{\min}$, $\nu_{\max}$ in this case.

Geometric assumptions. We assume that:

(1) $(\widetilde{X}_0, \tilde{g})$ is an $n + 1$ dimensional Lorentzian manifold of signature $(1, n)$, and $\widetilde{X}_0 = \mathbb{R}_t \times X_0$, where X_0, the space slice, is a manifold without boundary;

(2) the metric $\tilde{g}$ is stationary in the sense that its coefficients do not depend on t, or equivalently, ∂_t is a Killing field;

(3) the space slices $\{t = \text{const}\}$ are spacelike, or equivalently, the covector dt is timelike with respect to the dual metric $\tilde{g}^{-1}$ on $T^*\widetilde{X}_0$;

The (nonsemiclassical) principal symbol of the d'Alembert–Beltrami operator $\Box_{\tilde{g}}$ (without the negative sign), denoted by $\tilde{p}(\tilde{x}, \tilde{\xi})$, is

$$\tilde{p}(\tilde{x}, \tilde{\xi}) = -\tilde{g}_{\tilde{x}}^{-1}(\tilde{\xi}, \tilde{\xi}), \tag{2.2}$$

here $\tilde{x} = (t, x)$ denotes a point in $\widetilde{X}_0$ and $\tilde{\xi} = (\tau, \xi)$ a covector in $T_{\tilde{x}}^*\widetilde{X}_0$. The Hamiltonian flow of $\tilde{p}$ is the (rescaled) geodesic flow on $T_{\tilde{x}}^*\widetilde{X}_0$; we are in particular interested in nontrivial lightlike geodesics, i.e. the flow lines of $H_{\tilde{p}}$ on the set $\{\tilde{p} = 0\}\backslash 0$, where 0 denotes the zero section.

Note that we do not assume that the vector field ∂_t is timelike, since this is false inside the ergoregion for rotating black holes. Because of this, the intersections of the sets $\{\tau = \text{const}\}$, invariant under the geodesic flow, with the energy surface $\{\tilde{p} = 0\}$ need not be compact in the ξ direction, and it is possible that ξ will blow up in finite time along a flow line of $H_{\tilde{p}}$, while x stays in a compact subset of X_0. [1] We consider instead the rescaled flow

$$\tilde{\varphi}^s := \exp(s H_{\tilde{p}}/\partial_\tau \tilde{p}) \quad \text{on } \{\tilde{p} = 0\}\backslash 0. \tag{2.3}$$

Here $\partial_\tau \tilde{p}(\tilde{x}, \tilde{\xi}) = -2\tilde{g}_{\tilde{x}}^{-1}(\tilde{\xi}, dt)$ never vanishes on $\{\tilde{p} = 0\}\backslash 0$ by assumption (3). Since $H_{\tilde{p}} t = \partial_\tau \tilde{p}$, the variable t grows linearly with unit rate along the flow $\tilde{\varphi}^s$. The flow lines of (2.3) exist for all s as long as x stays in a compact subset of X_0. The flow is homogeneous, which makes it possible to define it on the cosphere bundle $S^*\widetilde{X}_0$, which is the quotient of $T^*\widetilde{X}_0\backslash 0$ by the action of dilations. Finally, the flow preserves the restriction of the symplectic form to the tangent bundle of $\{\tilde{p} = 0\}$.

We next assume the existence of a 'defining function of infinity' μ on the space slice with a concavity property:

(4) there exists a function $\mu \in C^\infty(X_0)$ such that $\mu > 0$ on X_0, for $\delta > 0$ the set

$$X_\delta := \{\mu > \delta\} \subset X_0 \tag{2.4}$$

is compactly contained in X_0, and there exists $\delta_0 > 0$ such that for each flow line $\gamma(s)$ of (2.3), and with μ naturally defined on $T^*\widetilde{X}_0$,

$$\mu(\gamma(s)) < \delta_0, \; \partial_s \mu(\gamma(s)) = 0 \implies \partial_s^2 \mu(\gamma(s)) < 0. \tag{2.5}$$

We now define the trapped set.

---

[1] The simplest example of such behavior is $\tilde{p} = x\xi^2 + 2\xi\tau - \tau^2$, considering the geodesic starting at $x = t = \tau = 0, \xi = 1$.

Definition 2.1. Let $\gamma(s)$ be a maximally extended flow line of (2.3). We say that $\gamma(s)$ is trapped as $s \to +\infty$, if there exists $\delta > 0$ such that $\mu(\gamma(s)) > \delta$ for all $s \geq 0$ (and as a consequence, $\gamma(s)$ exists for all $s \geq 0$). Denote by $\widetilde{\Gamma}_-$ the union of all γ trapped as $s \to +\infty$; similarly, we define the union $\widetilde{\Gamma}_+$ of all γ trapped as $s \to -\infty$. Define the trapped set $\widetilde{K} := \widetilde{\Gamma}_+ \cap \widetilde{\Gamma}_- \subset \{\tilde{p} = 0\} \backslash 0$.

If $\mu(\gamma(s)) < \delta_0$ and $\partial_s \mu(\gamma(s)) \leq 0$ for some s, then it follows from assumption (4) that $\gamma(s)$ is not trapped as $s \to +\infty$. Also, if $\gamma(s)$ is not trapped as $s \to +\infty$, then $\mu(\gamma(s)) < \delta_0$ and $\partial_s \mu(\gamma(s)) < 0$ for $s > 0$ large enough. It follows that $\widetilde{\Gamma}_\pm$ are closed conic subsets of $\{\tilde{p} = 0\} \backslash 0$, and $\widetilde{K} \subset \{\mu \geq \delta_0\}$.

We next split the light cone $\{\tilde{p} = 0\} \backslash 0$ into the sets $\mathcal{C}_+$ and $\mathcal{C}_-$ of positively and negatively time oriented covectors:

$$\mathcal{C}_\pm = \{\tilde{p} = 0\} \cap \{\pm \partial_\tau \tilde{p} > 0\}. \tag{2.6}$$

Since $\partial_\tau \tilde{p}$ never vanishes on $\{\tilde{p} = 0\} \backslash 0$ by assumption (3), we have $\{\tilde{p} = 0\} \backslash 0 = \mathcal{C}_+ \sqcup \mathcal{C}_-$.

We fix the sign of τ on the trapped set, in particular requiring that $\widetilde{K} \subset \{\tau \neq 0\}$:

(5) $\widetilde{K} \cap \mathcal{C}_\pm \subset \{\pm \tau < 0\}$.

Dynamical assumptions. We now formulate the assumptions on the dynamical structure of the flow (2.3). They are analogous to the assumptions of [Dy15, Sect. 5.1] and related to them in Sect. 2.3 below. We start by requiring that $\widetilde{\Gamma}_\pm$ are regular:

(6) for a large constant r, $\widetilde{\Gamma}_\pm$ are codimension 1 orientable C^r submanifolds of $\{\tilde{p} = 0\} \backslash 0$;
(7) $\widetilde{\Gamma}_\pm$ intersect transversely inside $\{\tilde{p} = 0\} \backslash 0$, and the intersections $\widetilde{K} \cap \{t = \text{const}\}$ are symplectic submanifolds of $T^* \widetilde{X}_0$.

We next define a natural invariant decomposition of the tangent space to $\{\tilde{p} = 0\}$ at $\widetilde{K}$. Let $(T\widetilde{\Gamma}_\pm)^\perp$ be the symplectic complement of the tangent space to $\widetilde{\Gamma}_\pm$. Since $\widetilde{\Gamma}_\pm$ has codimension 2 and is contained in $\{\tilde{p} = 0\}$, $(T\widetilde{\Gamma}_\pm)^\perp$ is a two-dimensional vector subbundle of $T(T^* \widetilde{X}_0)$ containing $H_{\tilde{p}}$. Since $H_{\tilde{p}} t \neq 0$ on $\{\tilde{p} = 0\} \backslash 0$, we can define the one-dimensional vector subbundles of $T(T^* \widetilde{X}_0)$

$$\widetilde{\mathcal{V}}_\pm := (T\widetilde{\Gamma}_\pm)^\perp \cap \{dt = 0\}. \tag{2.7}$$

Since $\widetilde{\Gamma}_\pm$ is a codimension 1 submanifold of $\{\tilde{p} = 0\}$ and $H_{\tilde{p}}$ is tangent to $\widetilde{\Gamma}_\pm$, we see that $\widetilde{\Gamma}_\pm$ is coisotropic and then $\widetilde{\mathcal{V}}_\pm$ are one-dimensional subbundles of $T\widetilde{\Gamma}_\pm$; moreover, since $\partial_t \in T\widetilde{\Gamma}_\pm$, we find $\widetilde{\mathcal{V}}_\pm \subset \{d\tau = 0\}$. Since $\widetilde{K} \cap \{t = \text{const}\}$ is symplectic, we have

$$T_{\widetilde{K}}\widetilde{\Gamma}_\pm = T\widetilde{K} \oplus \widetilde{\mathcal{V}}_\pm|_{\widetilde{K}}, \quad T_{\widetilde{K}}\tilde{p}^{-1}(0) = T\widetilde{K} \oplus \widetilde{\mathcal{V}}_-|_{\widetilde{K}} \oplus \widetilde{\mathcal{V}}_+|_{\widetilde{K}}. \tag{2.8}$$

Since the flow $\tilde{\varphi}^s$ from (2.3) maps the space slice $\{t = t_0\}$ to $\{t = t_0 + s\}$ and $H_{\tilde{p}}$ is tangent to $T\widetilde{\Gamma}_\pm$, we see that the splittings (2.8) are invariant under $\tilde{\varphi}^s$.

We now formulate the dynamical assumptions on the linearization of the flow $\tilde{\varphi}^s$ with respect to the splitting (2.8). Define the minimal expansion rate in the transverse direction $\nu_{\min}$ as the supremum of all ν for which there exists a constant C such that

$$\sup_{\tilde{\rho} \in \widetilde{K}} \|d\tilde{\varphi}^{\mp s}(\tilde{\rho})|_{\mathcal{V}_\pm}\| \leq C e^{-\nu s}, \quad s \geq 0, \tag{2.9}$$

with $\| \cdot \|$ denoting any smooth t-independent norm on the fibers of $T(T^*\widetilde{X}_0)$, homogeneous of degree zero with respect to dilations on $T^*\widetilde{X}_0$. Similarly, define ν_{max} as the infimum of all ν for which these exists a constant $c > 0$ such that

$$\inf_{\tilde{\rho} \in \widetilde{K}} \|d\tilde{\varphi}^{\mp s}(\tilde{\rho})|_{\mathcal{V}_{\pm}}\| \geq ce^{-\nu s}, \quad s \geq 0. \tag{2.10}$$

We now formulate the dynamical assumption of r-normal hyperbolicity:

(8) $\nu_{min} > r\mu_{max}$, where μ_{max} is the maximal expansion rate of the flow along $\widetilde{K}$, defined as the infimum of all ν for which there exists a constant C such that

$$\sup_{\tilde{\rho} \in \widetilde{K}} \|d\tilde{\varphi}^{s}(\tilde{\rho})|_{T\widetilde{K}}\| \leq Ce^{\nu s}, \quad s \in \mathbb{R}. \tag{2.11}$$

The large constant r determines how many terms we need to obtain in semiclassical expansions, and how many derivatives of these terms need to exist—see [Dy15]. Theorem 3 simply needs r to be large (in principle, depending on the dimension), while Theorems 1 and 2 require r to be large enough depending on N, T. For exact Kerr(–de Sitter) metrics, our assumptions are satisfied for all r, but a small perturbation will satisfy them for some fixed large r depending on the size of the perturbation. See Sect. 3.6 for more details.

2.3. Reduction to the space slice.

We now put a Lorentzian manifold $(\widetilde{X}_0, \tilde{g})$ satisfying assumptions of Sect. 2.2 into the framework of [Dy15]. Consider the stationary d'Alembert–Beltrami operator $P_{\tilde{g}}(\omega), \omega \in \mathbb{C}$, the second order semiclassical differential operator on the space slice X_0 obtained by replacing hD_t by $-\omega$ in the semiclassical d'Alembert–Beltrami operator $h^2\square_{\tilde{g}}$. The principal symbol of $P_{\tilde{g}}(\omega)$ is given by

$$\mathbf{p}(x, \xi; \omega) = \tilde{p}(t, x, -\omega, \xi),$$

where $\tilde{p}$ is defined in (2.2) and the right-hand side does not depend on t. We will show that the operator $P_{\tilde{g}}(\omega)$ satisfies a subset of the assumptions of [Dy15, Sects. 4.1, 5.1].

First of all, we need to understand the solutions in ω to the equation $\mathbf{p} = 0$. Let

$$p(x, \xi) \in C^{\infty}(T^*X_0\backslash 0)$$

be the unique real solution ω to the equation $\mathbf{p}(x, \xi; \omega) = 0$ such that $(t, x, -\omega, \xi) \in \mathcal{C}_{+}$, with the positive time oriented light cone $\mathcal{C}_{+}$ defined in (2.6). The existence and uniqueness of such solution follows from assumption (3) in Sect. 2.2, and we also find from the definition of $\mathcal{C}_{+}$ that

$$\partial_{\omega}\mathbf{p}(x, \xi; p(x, \xi)) < 0, \quad (x, \xi) \in T^*X_0\backslash 0. \tag{2.12}$$

We can write $\mathcal{C}_{+}$ as the graph of p:

$$\mathcal{C}_{+} = \{(t, x, -p(x, \xi), \xi) \mid t \in \mathbb{R}, \ (x, \xi) \in T^*X_0\backslash 0\}.$$

The level sets of p are not compact if ∂_t is not timelike. To avoid dealing with the fiber infinity, we use assumption (5) in Sect. 2.2 to identify a bounded region in T^*X_0 invariant under the flow and containing the trapped set.

Lemma 2.2. *There exists an open conic subset $\mathcal{W} \subset \mathcal{C}_+$, independent of t, such that $\widetilde{K} \cap \mathcal{C}_+ \subset \mathcal{W}$, the closure of $\mathcal{W}$ in $\mathcal{C}_+$ is contained in $\{\tau < 0\}$, and $\mathcal{W}$ is invariant under the flow* (2.3).

Proof. Consider a conic neighborhood $\mathcal{W}_0$ of $\widetilde{K} \cap \mathcal{C}_+$ in $\mathcal{C}_+$ independent of t and such that the closure of $\mathcal{W}_0$ is contained in $\{\mu > \delta_0/2\} \cap \{\tau < 0\}$; this is possible by assumption (5) and since $\widetilde{K}$ is contained in $\{\mu \geq \delta_0\}$. Let $\mathcal{W} \subset \mathcal{C}_+$ be the union of all maximally extended flow lines of (2.3) passing through $\mathcal{W}_0$. Then $\mathcal{W}$ is an open conic subset of $\mathcal{C}_+$ containing $\widetilde{K} \cap \mathcal{C}_+$ and invariant under the flow (2.3). It remains to show that each point $(\tilde{x}, \tilde{\xi}) \in \mathcal{C}_+ \cap \{\tau \geq 0\}$ has a neighborhood that does not intersect $\mathcal{W}$. To see this, note that the corresponding trajectory $\gamma(s)$ of (2.3) does not lie in $\widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_-$ (as otherwise, the projection of $\gamma(s)$ onto the cosphere bundle would converge to $\widetilde{K}$ as $s \to +\infty$ or $s \to -\infty$, by [Dy15, Lemma 4.1]; it remains to use assumption (5) and the fact that τ is constant on $\gamma(s)$). We then see that $\gamma(s)$ escapes for both $s \to +\infty$ and $s \to -\infty$ and does not intersect the closure of $\mathcal{W}_0$ and same is true for nearby trajectories; therefore, a neighborhood of $(\tilde{x}, \tilde{\xi})$ does not intersect $\mathcal{W}$. $\square$

Arguing similarly (using an open conic subset $\mathcal{W}_0'$ of $\mathcal{C}_+$ such that $\overline{\mathcal{W}_0} \subset \mathcal{W}_0'$ and $\overline{\mathcal{W}_0'} \subset \{\mu > \delta_0/2\} \cap \{\tau < 0\}$), we construct an open conic subset $\mathcal{W}'$ of $\mathcal{C}_+$ independent of t and such that

$$\widetilde{K} \cap \mathcal{C}_+ \subset \mathcal{W}, \quad \overline{\mathcal{W}} \subset \mathcal{W}', \quad \overline{\mathcal{W}'} \subset \{\tau < 0\},$$

and $\mathcal{W}, \mathcal{W}'$ are invariant under the flow (2.3). Now, take small $\delta_1 > 0$ and define

$$\begin{aligned}\widetilde{\mathcal{U}} &:= \mathcal{C}_+ \cap \{|1 + \tau| < \delta_1\} \cap \mathcal{W} \cap \{\mu > \delta_1\}, \\ \widetilde{\mathcal{U}}' &:= \mathcal{C}_+ \cap \{|1 + \tau| < 2\delta_1\} \cap \mathcal{W}' \cap \{\mu > \delta_1/2\}.\end{aligned} \quad (2.13)$$

Then $\widetilde{\mathcal{U}}, \widetilde{\mathcal{U}}'$ are open subsets of $\mathcal{C}_+$ convex under the flow (2.3), $\widetilde{K} \cap \{|1 + \tau| < \delta_1\} \subset \widetilde{\mathcal{U}}$ (note that $\widetilde{K} \cap \{\tau < 0\} \subset \mathcal{C}_+$ by assumption (5)), and the closure of $\widetilde{\mathcal{U}}$ is contained in $\widetilde{\mathcal{U}}'$. Moreover, the projections of $\mathcal{U}, \mathcal{U}'$ onto the (x, τ, ξ) variables are bounded because $\mathcal{W}, \mathcal{W}'$ are conic and $\overline{\mathcal{W}}, \overline{\mathcal{W}'} \subset \{\tau \neq 0\}$.

Let $\mathcal{U} \Subset \mathcal{U}' \Subset T^*X_0$ be the projections of $\widetilde{\mathcal{U}}, \widetilde{\mathcal{U}}'$ onto the (x, ξ) variables, so that

$$\widetilde{\mathcal{U}} = \{(t, x, -p(x, \xi), \xi) \mid t \in \mathbb{R}, \ (x, \xi) \in \mathcal{U}\},$$

and similarly for $\mathcal{U}'$. Note that $\mathcal{U} \subset \{|p - 1| < \delta_1\}$ and $\mathcal{U}' \subset \{|p - 1| < 2\delta_1\}$. Since $\mathcal{U}'$ is bounded, and by (2.12), for $\delta_1 > 0$ small enough and $(x, \xi) \in \mathcal{U}'$, $p(x, \xi)$ is the only solution to the equation $\mathbf{p}(x, \xi; \omega) = 0$ in $\{\omega \in \mathbb{C} \mid |\omega - 1| < 2\delta_1\}$.

We now study the Hamiltonian flow of p. Since

$$\partial_{x,\xi} p(x, \xi) = -\frac{\partial_{x,\xi} \mathbf{p}(x, \xi, p(x, \xi))}{\partial_\omega \mathbf{p}(x, \xi, p(x, \xi))},$$

and for each t,

$$-\partial_\omega \mathbf{p}(x, \xi, p(x, \xi)) = \partial_\tau \tilde{p}(t, x, -p(x, \xi), \xi),$$

we see that the flow of H_p is the projection of the rescaled geodesic flow (2.3) on $\mathcal{C}_+$: for $(x, \xi) \in T^*X_0 \backslash 0$,

$$\tilde{\varphi}^s(t, x, -p(x, \xi), \xi) = (t + s, x(s), -p(x, \xi), \xi(s)), \quad (x(s), \xi(s)) = e^{sH_p}(x, \xi). \tag{2.14}$$

We now verify some of the assumptions of [Dy15, Sect. 4.1]. We let X be an n-dimensional manifold containing X_0 (for the Kerr–de Sitter metric it is constructed in Sect. 3.5) and consider the volume form $d\operatorname{Vol}$ on X_0 related to the volume form $d\widehat{\operatorname{Vol}}$ on $\widetilde{X}_0$ generated by $\tilde{g}$ by the formula $d\widehat{\operatorname{Vol}} = dt \wedge d\operatorname{Vol}$. The operator $P_{\tilde{g}}(\omega)$ is a semiclassical pseudodifferential operator depending holomorphically on $\omega \in \Omega := \{|\omega - 1| < 2\delta_1\}$ and $\mathbf{p}$ is its semiclassical principal symbol. We do not specify the spaces $\mathcal{H}_1, \mathcal{H}_2$ here and do not establish any mapping or Fredholm properties of $P_{\tilde{g}}(\omega)$; for our specific applications it is done in Sect. 3.5. Except for these mapping properties, the assumptions (1), (2), and (5)–(9) of [Dy15, Sect. 4.1] are satisfied, with $\mathcal{U}, \mathcal{U}'$ defined above, $[\alpha_0, \alpha_1] := [1 - \delta_1/2, 1 + \delta_1/2]$, and the incoming/outgoing tails $\Gamma_\pm$ on the space slice given by (for each t)

$$\Gamma_\pm = \{(x, \xi) \mid (t, x, -p(x, \xi), \xi) \in \widetilde{\Gamma}_\pm \cap \{|1 + \tau| \le \delta_1\} \cap \overline{\mathcal{W}} \cap \{\mu \ge \delta_1\}\}, \tag{2.15}$$

and similarly for the trapped set $K = \Gamma_+ \cap \Gamma_-$.

Finally, the dynamical assumptions of [Dy15, Sect. 5.1] are also satisfied, as follows directly from (2.14) and the dynamical assumptions of Sect. 2.2. Note that the subbundles $\mathcal{V}_\pm$ of $T\Gamma_\pm$ defined in [Dy15, Sect. 5.1] coincide with the subbundles $\widetilde{\mathcal{V}}_\pm$ of $T\widetilde{\Gamma}_\pm$ defined in Sect. 2.7 under the identification $T_{(x,\xi)}(T^*X_0) \simeq T_{(t,x,-p(x,\xi),\xi)}(T^*\widetilde{X}_0) \cap \{dt = d\tau = 0\}$, and the expansion rates $\nu_{\min}, \nu_{\max}, \mu_{\max}$ defined in (2.9)–(2.11) coincide with those defined in [Dy15, (5.1)–(5.3)].

To relate the constants for the Weyl laws in Theorem 3 and [Dy15, Theorem 2] , we note that for $[a, b] \subset (1 - \delta_1/2, 1 + \delta_1/2)$,

$$\operatorname{Vol}_\sigma(K \cap p^{-1}[a, b]) = \operatorname{Vol}_{\tilde{\sigma}}(\widetilde{K} \cap \{a \le -\tau \le b\} \cap \{t = \text{const}\}).$$

Here $\operatorname{Vol}_\sigma$ and $\operatorname{Vol}_{\tilde{\sigma}}$ stand for symplectic volume forms of order $2n - 2$ on T^*X_0 and $T^*\widetilde{X}_0$, respectively. The constant $c_{\widetilde{K}}$ from Theorem 3 is then given by

$$c_{\widetilde{K}} = \operatorname{Vol}_{\tilde{\sigma}}(\widetilde{K} \cap \{0 \le \tau \le 1\} \cap \{t = \text{const}\}). \tag{2.16}$$

2.4. Applications to linear waves. In this section, we apply the results of [Dy15] to understand the decay properties of linear waves; Theorem 4 below forms the base for the proofs of Theorems 1 and 2 in Sect. 3.4.

Consider a family of approximate solutions $u(h) \in \mathcal{D}'((-1, T(h) + 1)_t \times X_0)$ to the wave equation

$$h^2 \Box_{\tilde{g}} u(h) = \mathcal{O}(h^\infty)_{C^\infty}. \tag{2.17}$$

Here $h \ll 1$ is the semiclassical parameter and $T(h) > 0$ depends on h (for our particular application, $T(h) = T \log(1/h)$ for some constant T). We assume that u is h-tempered uniformly in t, as defined in Sect. 2.1. Then by the elliptic estimate (see for instance [Dy15, Proposition 3.2]), u is microlocalized on the light cone:

$$\widetilde{\operatorname{WF}_h}(u) \subset \{\tilde{p} = 0\}, \tag{2.18}$$
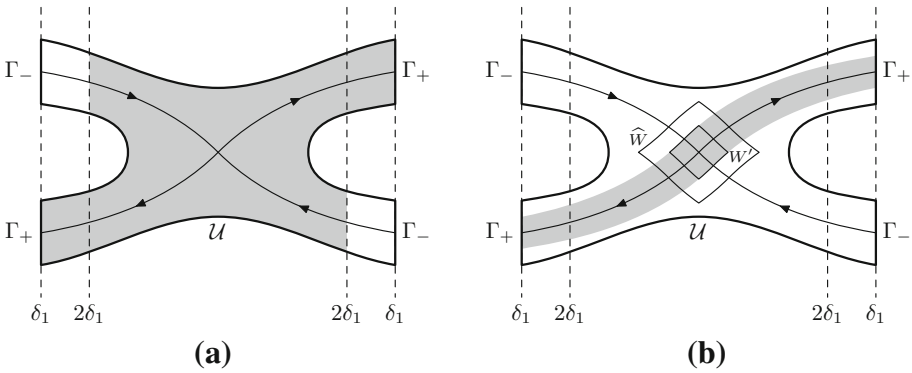
Fig. 3. The phase space picture of the flow, showing *shaded* $\mathrm{WF}_h(\mathbf{u}(t))$ for **a** all t and **b** all $t \geq t_1$, for u satisfying Definition 2.3. The *horizontal axis* corresponds to μ

where $\widetilde{\mathrm{WF}}_h(u)$ is defined in Sect. 2.1. By the restriction statement in Sect. 2.1, u is a smooth function of t with values in h-tempered distributions on X_0. Moreover, we obtain for $0 < \delta_1 < \delta_2$ small enough and each $t_0 \in [0, T(h)]$,

$$\|u(t_0)\|_{H^1_x(X_{\delta_2})} \leq C\|u\|_{H^1_{t,x}([t_0-1,t_0+1] \times X_{\delta_1})} + \mathcal{O}(h^\infty),$$
$$\|u\|_{L^2_t[t_0-1,t_0+1]L^2_x(X_{\delta_1})} \leq C\|u\|_{L^\infty_t[t_0-1,t_0+1]L^2_x(X_{\delta_1})} + \mathcal{O}(h^\infty). \tag{2.19}$$

The second of these inequalities is trivial; the first one is done by applying the standard energy estimate for the wave equation to the function $\chi(t - t_0)u$, with $\chi \in C_0^\infty(-\epsilon, \epsilon)$ equal to 1 near 0 and $\epsilon > 0$ small depending on δ_1, δ_2.

We furthermore restrict ourselves to the following class of outgoing solutions, see Fig. 3a.

Definition 2.3. Fix small $\delta_1 > 0$. A solution u to (2.17), h-tempered uniformly in $t \in (-1, T(h) + 1)$, is called outgoing, if its projected wavefront set $\widetilde{\mathrm{WF}}_h(u)$, defined in Sect. 2.1, satisfies (for $\widetilde{\mathcal{U}}$ defined in (2.13))

$$\widetilde{\mathrm{WF}}_h(u) \cap \{\mu > \delta_1\} \subset \widetilde{\mathcal{U}} \cap \{|\tau + 1| < \delta_1/4\}, \tag{2.20}$$

$$\widetilde{\mathrm{WF}}_h(u) \cap \{\delta_1 \leq \mu \leq 2\delta_1\} \subset \{H_{\tilde{p}}\mu \leq 0\}. \tag{2.21}$$

The main result of this section is

Theorem 4. *Fix $T, N, \varepsilon > 0$ and let the assumptions of Sect. 2.2 hold, including r-normal hyperbolicity with r large depending on T, N. Assume that u is an outgoing solution to (2.17), for $t \in (-1, T\log(1/h) + 1)$, and $\|u(t)\|_{H_h^{-N}(X_{\delta_1/2})} = \mathcal{O}(h^{-N})$ uniformly in t. Then for t_0 large enough and independent of h, we can write*

$$u(t, x) = u_\Pi(t, x) + u_R(t, x), \quad t_0 \leq t \leq T\log(1/h),$$

such that $h^2\Box_{\tilde{g}}u_\Pi, h^2\Box_{\tilde{g}}u_R$ are $\mathcal{O}(h^N)_{H_h^N}$ on X_{δ_1} and, with $\|\cdot\|_{\mathcal{E}}$ defined in (1.4),

$$\|u_\Pi(t_0)\|_{\mathcal{E}} \le Ch^{-1/2}\|u(0)\|_{\mathcal{E}} + \mathcal{O}(h^N), \tag{2.22}$$

$$\|u_\Pi(t)\|_{\mathcal{E}} \le Ce^{-(\nu_{\min}-\varepsilon)t/2}\|u_\Pi(t_0)\|_{\mathcal{E}} + \mathcal{O}(h^N), \tag{2.23}$$

$$\|u_\Pi(t)\|_{\mathcal{E}} \ge C^{-1}e^{-(\nu_{\max}+\varepsilon)t/2}\|u_\Pi(t_0)\|_{\mathcal{E}} - \mathcal{O}(h^N), \tag{2.24}$$

$$\|u_R(t)\|_{\mathcal{E}} \le Ch^{-1}e^{-(\nu_{\min}-\varepsilon)t}\|u(0)\|_{\mathcal{E}} + \mathcal{O}(h^N), \tag{2.25}$$

$$\|u(t)\|_{\mathcal{E}} \le Ce^{\varepsilon t}\|u(0)\|_{\mathcal{E}} + \mathcal{O}(h^N), \tag{2.26}$$

all uniformly in $t \in [t_0, T\log(1/h)]$.

For the proof, we assume that the metric is r-normally hyperbolic for all r, and prove the bounds for all T, N (so that $\mathcal{O}(h^N)$ becomes $\mathcal{O}(h^\infty)$); since semiclassical arguments require finitely many derivatives to work, the statement will be true for r large depending on T and N.

We first recall the factorization of [Dy15, Lemma 4.3]:

$$P_{\tilde{g}}(\omega) = \mathcal{S}(\omega)(P - \omega)\mathcal{S}(\omega) + \mathcal{O}(h^\infty) \quad \text{microlocally near } \mathcal{U}, \tag{2.27}$$

where $\mathcal{S}(\omega)$ is a family of pseudodifferential operators elliptic near $\mathcal{U}$, and such that $\mathcal{S}(\omega)^* = \mathcal{S}(\omega)$ for $\omega \in \mathbb{R}$, and P is a self-adjoint pseudodifferential operator, moreover we assume that it is compactly supported and compactly microlocalized. If we define the self-adjoint pseudodifferential operator $\widetilde{\mathcal{S}}$ on $\widetilde{X}_0$ by replacing ω by $-hD_t$ in $\mathcal{S}(\omega)$, then we get

$$h^2\Box_{\tilde{g}} = \widetilde{\mathcal{S}}(hD_t + P)\widetilde{\mathcal{S}} + \mathcal{O}(h^\infty) \quad \text{microlocally near } \widetilde{\mathcal{U}}. \tag{2.28}$$

We define

$$\mathbf{u}(t) := (\widetilde{\mathcal{S}}u)(t), \quad 0 \le t \le T\log(1/h),$$

note that $\mathbf{u}(t)$ and its t-derivatives are bounded uniformly in t with values in h-tempered distributions on X_0 by the discussion of restrictions to space slices in Sect. 2.1 and by (2.18). We have by (2.17), (2.20), (2.21), and (2.28),

$$(hD_t + P)\mathbf{u}(t) = \mathcal{O}(h^\infty) \quad \text{microlocally near } X_{\delta_1}, \tag{2.29}$$

$$\mathrm{WF}_h(\mathbf{u}(t)) \cap X_{\delta_1} \subset \{|p - 1| < \delta_1/4\}, \tag{2.30}$$

$$\mathrm{WF}_h(\mathbf{u}(t)) \cap \{\delta_1 \le \mu \le 2\delta_1\} \subset \{H_p\mu \le 0\}, \tag{2.31}$$

uniformly in $t \in [0, T\log(1/h)]$.

We next use the construction of [Dy15, Lemma 5.1], which (combined with the homogeneity of the flow) gives functions $\varphi_\pm$ defined in a conic neighborhood of K in T^*X_0, such that $\Gamma_\pm = \{\varphi_\pm = 0\}$ in this neighborhood, $\varphi_\pm$ are homogeneous of degree zero, and

$$H_p\varphi_\pm = \mp c_\pm\varphi_\pm, \quad \nu_{\min} - \varepsilon < c_\pm < \nu_{\max} + \varepsilon, \tag{2.32}$$

where $c_\pm$ are some smooth functions on the domain of $\varphi_\pm$. Then for small $\delta > 0$,

$$U_\delta := \{|\varphi_+| \le \delta, \ |\varphi_-| \le \delta\}$$

is a small closed conic neighborhood of K in $T^*X_0\backslash 0$.

We now fix δ small enough so that [Dy15, Theorem 3 in Sect. 7.1 and Proposition 7.1] apply, giving a Fourier integral operator $\Pi \in I_{\mathrm{comp}}(\Lambda^\circ)$ which satisfies the equations

$$\Pi^2 = \Pi + \mathcal{O}(h^\infty), \quad [P, \Pi] = \mathcal{O}(h^\infty) \tag{2.33}$$

microlocally near the set $\widehat{W} \times \widehat{W}$, with

$$\widehat{W} := U_\delta \cap \{|p - 1| \leq \delta_1/2\}. \tag{2.34}$$

Here $\Lambda^\circ \subset \Gamma_- \cap \Gamma_+$ is the canonical relation defined in [Dy15, (5.12)]. Also, we define

$$W' := U_{\delta/2} \cap \{|p - 1| \leq \delta_1/4\}. \tag{2.35}$$

We now derive certain conditions on the microlocalization of u for large enough times, see Fig. 3b (compare with [Dy15, Figure 5]).

Proposition 2.4. *For t_1 large enough independent of h, the function $\mathbf{u}(t)$ satisfies*

$$\mathrm{WF}_h(\mathbf{u}(t)) \cap \widehat{W} \subset \{|\varphi_+| < \delta/2\}, \tag{2.36}$$

$$\mathrm{WF}_h(\mathbf{u}(t)) \cap \Gamma_- \subset W', \tag{2.37}$$

uniformly in $t \in [t_1, T \log(1/h)]$.

Proof. Consider $(x, \xi) \in \mathrm{WF}_h(\mathbf{u}(t)) \cap X_{2\delta_1}$ for some $t \in [t_1, T \log(1/h)]$. Put $\gamma(s) = e^{sH_p}(x, \xi)$. Then by propagation of singularities (see for example [Dy15, Proposition 3.4]) for the equation (2.29), we see that either there exists $s_0 \in [-t_1, 0]$ such that $\gamma(s_0) \in \{\delta_1 \leq \mu \leq 2\delta_1\} \cap \mathrm{WF}_h(\mathbf{u}(t+s_0))$, or $\gamma(s) \in X_{2\delta_1}$ for all $s \in [-t_1, 0]$. However, in the first of these two cases, by (2.31) we have $\gamma(s_0) \in \{\mu \leq 2\delta_1\} \cap \{H_p\mu \leq 0\}$, which implies that $\gamma(0) \in \{\mu \leq 2\delta_1\}$ by assumption (4) in Sect. 2.2, a contradiction. Therefore,

$$e^{tH_p}(x, \xi) \in X_{2\delta_1}, \quad t \in [-t_1, 0].$$

It remains to note that for t_1 large enough,

$$e^{-t_1 H_p}(\widehat{W} \cap \{|\varphi_+| \geq \delta/2\}) \cap X_{2\delta_1} = \emptyset;$$
$$e^{t_1 H_p}(\Gamma_- \cap \{|p - 1| < \delta_1/4\} \cap X_{2\delta_1}) \subset W';$$

the first of these statements follows from the fact that $\widehat{W} \cap \{|\varphi_+| \geq \delta/2\}$ is a compact set not intersecting Γ_+, and the second one, from [Dy15, Lemma 4.1]. $\square$

By (2.29), (2.33), and (2.37), and since $\mathrm{WF}_h(\Pi) \subset \Gamma_- \times \Gamma_+$ we have uniformly in $t \in [t_1, T \log(1/h)]$,

$$(hD_t + P)\Pi\mathbf{u}(t) = \mathcal{O}(h^\infty) \quad \text{microlocally near } \widehat{W}. \tag{2.38}$$

By [Dy15, Proposition 6.1 and Sect. 6.2], we have

$$\|\Pi\mathbf{u}(t)\|_{L^2} \leq Ch^{-1/2}\|\mathbf{u}(t)\|_{L^2}. \tag{2.39}$$

We now use the methods of [Dy15, Sect. 8] to prove a microlocal version of Theorem 4 near the trapped set.

Proposition 2.5. *There exist compactly supported A_0, $A_1 \in \Psi^{\mathrm{comp}}(X_0)$ microlocalized inside $\widehat{W}$, elliptic on W', and such that for $t \in [t_1, T \log(1/h)]$,*

$$\|A_0\Pi\mathbf{u}(t)\|_{L^2} \leq Ce^{-(\nu_{\min}-\varepsilon)t/2}\|A_0\Pi\mathbf{u}(t_1)\|_{L^2} + \mathcal{O}(h^\infty), \tag{2.40}$$

$$\|A_0\Pi\mathbf{u}(t)\|_{L^2} \geq C^{-1}e^{-(\nu_{\max}+\varepsilon)t/2}\|A_0\Pi\mathbf{u}(t_1)\|_{L^2} - \mathcal{O}(h^\infty), \tag{2.41}$$

$$\|A_1(1 - \Pi)\mathbf{u}(t)\|_{L^2} \leq Ch^{-1}e^{-(\nu_{\min}-\varepsilon)t}\|A_0\mathbf{u}(t_1)\|_{L^2} + \mathcal{O}(h^\infty), \tag{2.42}$$

$$\|A_1\mathbf{u}(t)\|_{L^2} \leq Ce^{\varepsilon t}\|A_0\mathbf{u}(t_1)\|_{L^2} + \mathcal{O}(h^\infty). \tag{2.43}$$

Proof. We will use the operators $\Theta_\pm$, Ξ constructed in [Dy15, Proposition 7.1]. The microlocalization statements we make will be uniform in $t \in [t_1, T\log(1/h)]$.

We first prove (2.42), following the proof of [Dy15, Proposition 8.1]. Put

$$\mathbf{v}(t) := \Xi\mathbf{u}(t).$$

Then similarly to [Dy15, (8.14)], we find

$$(1 - \Pi)\mathbf{u}(t) = \Theta_-\mathbf{v}(t) + \mathcal{O}(h^\infty) \quad \text{microlocally near } \widehat{W}.$$

By (2.29) and (2.38),

$$(hD_t + P)(1 - \Pi)\mathbf{u}(t) = \mathcal{O}(h^\infty) \quad \text{microlocally near } \widehat{W}.$$

Similarly to [Dy15, Proposition 8.3], we use the commutation relation $[P, \Theta_-] = -ih\Theta_- Z_- + \mathcal{O}(h^\infty)$ together with propagation of singularities for the operator Θ_- to find

$$(hD_t + P - ihZ_-)\mathbf{v}(t) = \mathcal{O}(h^\infty) \quad \text{microlocally near } \widehat{W}. \tag{2.44}$$

Here $Z_- \in \Psi^{\mathrm{comp}}(X_0)$ satisfies $\sigma(Z_-) = c_-$ near $\widehat{W}$.

Let $\mathcal{X}_- \in \Psi^{\mathrm{comp}}(X_0)$ be the operator used in [Dy15, Sect. 8.2], satisfying $\mathrm{WF}_h(\mathcal{X}_-) \Subset \widehat{W}$, $\sigma(\mathcal{X}_-) \geq 0$ everywhere, and $\sigma(\mathcal{X}_-) > 0$ on W'. Similarly to [Dy15, (8.18)], we get

$$\frac{1}{2}\partial_t\langle\mathcal{X}_-\mathbf{v}(t), \mathbf{v}(t)\rangle + \langle\mathcal{Y}_-\mathbf{v}(t), \mathbf{v}(t)\rangle = \mathcal{O}(h^\infty), \tag{2.45}$$

where

$$\mathcal{Y}_- = \frac{1}{2}(Z_-^*\mathcal{X}_- + \mathcal{X}_- Z_-) + \frac{1}{2ih}[P, \mathcal{X}_-]$$

satisfies $\mathrm{WF}_h(\mathcal{Y}_-) \Subset \widehat{W}$, and similarly to [Dy15, (8.19)] we have

$$\sigma(\mathcal{Y}_-) \geq (\nu_{\min} - \varepsilon)\sigma(\mathcal{X}_-) \quad \text{near } \mathrm{WF}_h(\mathbf{v}(t)),$$

and the inequality is strict on W'. Similarly to [Dy15, Lemma 8.4], by sharp Gårding inequality we get

$$\langle(\mathcal{Y}_- - (\nu_{\min} - \varepsilon)\mathcal{X}_-)\mathbf{v}(t), \mathbf{v}(t)\rangle \geq \|A_1\mathbf{v}(t)\|_{L^2}^2 - Ch\|A_0'\mathbf{v}(t)\|_{L^2}^2 - \mathcal{O}(h^\infty) \tag{2.46}$$

for an appropriate choice of A_1 and some $A_0' \in \Psi^{\mathrm{comp}}(X_0)$ microlocalized inside $\widehat{W}$. Also similarly to [Dy15, Lemma 8.4], by propagation of singularities for the equation (2.44) we get for t_1 large enough,

$$\|A_0'\mathbf{v}(t)\|_{L^2}^2 \leq C\|A_0\mathbf{v}(t_1)\|_{L^2}^2 + \mathcal{O}(h^\infty), \quad t \in [t_1, 2t_1], \tag{2.47}$$

$$\|A_0'\mathbf{v}(t)\|_{L^2}^2 \leq C\|A_1\mathbf{v}(t - t_1)\|_{L^2}^2 + \mathcal{O}(h^\infty), \quad t \geq 2t_1, \tag{2.48}$$

for an appropriate choice of A_0. By (2.45) and (2.46), we see that

$$\begin{aligned}
\langle\mathcal{X}_-\mathbf{v}(t), \mathbf{v}(t)\rangle \leq{} & Ce^{-2(\nu_{\min} - \varepsilon)t}\langle\mathcal{X}_-\mathbf{v}(t_1), \mathbf{v}(t_1)\rangle \\
& - C^{-1}\int_{t_1}^t e^{-2(\nu_{\min} - \varepsilon)(t-s)}\|A_1\mathbf{v}(s)\|_{L^2}^2\,ds \\
& + Ch\int_{t_1}^t e^{-2(\nu_{\min} - \varepsilon)(t-s)}\|A_0'\mathbf{v}(s)\|_{L^2}^2\,ds + \mathcal{O}(h^\infty).
\end{aligned}$$

Breaking the second integral on the right-hand side in two pieces and estimating each of them separately by (2.47) and (2.48), we get for an appropriate choice of A_0,

$$\langle \mathcal{X}_- \mathbf{v}(t), \mathbf{v}(t) \rangle \leq C e^{-2(\nu_{\min} - \varepsilon)t} \| A_0 \mathbf{v}(t_1) \|_{L^2}^2 + \mathcal{O}(h^\infty).$$

We can moreover assume that $\mathcal{X}_-$ has the form $A_1^* A_1 + \mathcal{X}_1^* \mathcal{X}_1 + \mathcal{O}(h^\infty)$ for some pseudodifferential operator $\mathcal{X}_1$; this can be arranged since $\sigma(\mathcal{X}_-) > 0$ on $\mathrm{WF}_h(A_1)$ and the argument of [Dy15, Sect. 8.2] only depends on the principal symbol of $\mathcal{X}_-$, which can be taken to be the square of a smooth function. Then $\| A_1 \mathbf{v}(t) \|_{L^2}^2 \leq \langle \mathcal{X}_- \mathbf{v}(t), \mathbf{v}(t) \rangle + \mathcal{O}(h^\infty)$ and we get

$$\| A_1 \mathbf{v}(t) \|_{L^2} \leq C e^{-(\nu_{\min} - \varepsilon)t} \| A_0 \mathbf{v}(t_1) \|_{L^2} + \mathcal{O}(h^\infty). \tag{2.49}$$

To prove (2.42), it remains to note that $(1 - \Pi)\mathbf{u}(t) = \mathbf{v}(t) + \mathcal{O}(h^\infty)$ microlocally near $\widehat{W}$ and $\| \mathbf{v}(t_1) \|_{L^2} \leq C h^{-1} \| \mathbf{u}(t_1) \|_{L^2}$ by part 1 of [Dy15, Proposition 6.13].

To prove (2.43), we argue similarly to (2.45), but use the equation (2.29) instead of (2.44). We get

$$\frac{1}{2} \partial_t \langle \mathcal{X}_- \mathbf{u}(t), \mathbf{u}(t) \rangle + \langle \mathcal{Y}_-' \mathbf{u}(t), \mathbf{u}(t) \rangle = \mathcal{O}(h^\infty),$$

where

$$\mathcal{Y}_-' = \frac{1}{2ih} [P, \mathcal{X}_-]$$

satisfies $\mathrm{WF}_h(\mathcal{Y}_-') \Subset \widehat{W}$ and

$$\sigma(\mathcal{Y}_-') \geq -\varepsilon \sigma(\mathcal{X}_-) \quad \text{near } \mathrm{WF}_h(\mathbf{u}(t)),$$

and the inequality is strict on W'. The remainder of the proof of (2.43) proceeds exactly as the proof of (2.49).

Finally, we prove (2.40) and (2.41), following the proof of [Dy15, Proposition 8.2] . Let $\mathcal{X}_+ \in \Psi^{\mathrm{comp}}(X_0)$ be the operator defined in [Dy15, Sect. 8.3], satisfying in particular $\mathrm{WF}_h(\mathcal{X}_+) \Subset \widehat{W}$, $\sigma(\mathcal{X}_+) \geq 0$ everywhere, and $\sigma(\mathcal{X}_+) > 0$ on W'. Similarly to [Dy15, (8.33)], we get from (2.38) that for an appropriate choice of A_0,

$$\frac{1}{2} \partial_t \langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t) \rangle + \langle \mathcal{Z}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t) \rangle = \mathcal{O}(h) \| A_0 \Pi \mathbf{u}(t) \|_{L^2}^2 + \mathcal{O}(h^\infty), \tag{2.50}$$

where $\mathcal{Z}_+ \in \Psi^{\mathrm{comp}}(X_0)$, $\mathrm{WF}_h(\mathcal{Z}_+) \Subset \widehat{W}$,

$$\frac{\nu_{\min} - \varepsilon}{2} \sigma(\mathcal{X}_+) \leq \sigma(\mathcal{Z}_+) \leq \frac{\nu_{\max} + \varepsilon}{2} \sigma(\mathcal{X}_+) \quad \text{near } \mathrm{WF}_h(\Pi \mathbf{u}(t)),$$

and both inequalities are strict on $W' \cap \mathrm{WF}_h(\Pi \mathbf{u}(t))$. By [Dy15, Lemma 8.7], we deduce that

$$\langle \mathcal{Z}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t) \rangle \geq \frac{\nu_{\min} - \varepsilon}{2} \langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t) \rangle + \| A_0 \Pi \mathbf{u}(t) \|_{L^2}^2 - \mathcal{O}(h^\infty),$$

$$\langle \mathcal{Z}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t) \rangle \leq \frac{\nu_{\max} + \varepsilon}{2} \langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t) \rangle - \| A_0 \Pi \mathbf{u}(t) \|_{L^2}^2 + \mathcal{O}(h^\infty)$$

By (2.50), we find

$$(\partial_t + (\nu_{\min} - \varepsilon))\langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t)\rangle \leq \mathcal{O}(h^\infty),$$
$$(\partial_t + (\nu_{\max} + \varepsilon))\langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t)\rangle \geq -\mathcal{O}(h^\infty).$$

Therefore,

$$\langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t)\rangle \leq C e^{-(\nu_{\min} - \varepsilon)t} \langle \mathcal{X}_+ \Pi \mathbf{u}(t_1), \Pi \mathbf{u}(t_1)\rangle + \mathcal{O}(h^\infty),$$
$$\langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t)\rangle \geq C^{-1} e^{-(\nu_{\max} + \varepsilon)t} \langle \mathcal{X}_+ \Pi \mathbf{u}(t_1), \Pi \mathbf{u}(t_1)\rangle - \mathcal{O}(h^\infty).$$

To prove (2.39) and (2.40), it remains to note that

$$\langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t)\rangle \geq C^{-1} \|A_0 \Pi \mathbf{u}(t)\|_{L^2}^2 - \mathcal{O}(h^\infty),$$
$$\langle \mathcal{X}_+ \Pi \mathbf{u}(t), \Pi \mathbf{u}(t)\rangle \leq C \|A_0 \Pi \mathbf{u}(t)\|_{L^2}^2 + \mathcal{O}(h^\infty);$$

the first of these statements follows by [Dy15, Lemma 8.7] and the second one is arranged by choosing A_0 to be elliptic on $\mathrm{WF}_h(\mathcal{X}_+)$. □

Proof of Theorem 4. To construct the component $u_\Pi(t)$, we use $\Pi \mathbf{u}(t)$ together with the Schrödinger propagator $e^{-itP/h}$. Since $P^* = P$ and P is compactly supported and compactly microlocalized, the operator $e^{-itP/h}$ quantizes the flow e^{tH_p} in the sense of [Dy15, Proposition 3.1]. Since $\mathrm{WF}_h(\Pi \mathbf{u}(t)) \subset \Gamma_+$, we have by (2.38),

$$(hD_t + P)e^{-it_1 P/h}\Pi \mathbf{u}(t) = \mathcal{O}(h^\infty) \quad \text{on } X_{\delta_1}, \quad t \geq t_1, \tag{2.51}$$

if t_1 is large enough so that

$$e^{-t_1 H_p}(\Gamma_+ \cap X_{\delta_1} \cap \{|p - 1| < \delta_1/4\}) \subset W'; \tag{2.52}$$

such t_1 exists by [Dy15, Lemma 4.1]. We then take an elliptic parametrix $\widetilde{\mathcal{S}}'$ of $\widetilde{\mathcal{S}}$ near $\widetilde{\mathcal{U}}$ (see [Dy15, Proposition 3.3]) and define

$$u_\Pi(t) := \widetilde{\mathcal{S}}'(e^{-it_1 P/h}\Pi \mathbf{u}(t - t_1)), \quad t \in [t_0 - 1, T \log(1/h)], \quad t_0 := 2t_1 + 1. \tag{2.53}$$

Then by (2.28) and (2.51) we get

$$h^2 \square_{\tilde{g}} u_\Pi = \mathcal{O}(h^\infty) \quad \text{on } X_{\delta_1},$$

uniformly in $t \in [t_0, T \log(1/h)]$. Put

$$u_R(t) := u(t) - u_\Pi(t), \quad t \in [t_0, T \log(1/h)],$$

then $h^2 \square_{\tilde{g}} u_R = \mathcal{O}(h^\infty)$ on X_{δ_1} as well.

It remains to prove (2.22)–(2.26). Since $\mathrm{WF}_h(\Pi \mathbf{u}(t)) \subset \Gamma_+$ and by (2.52), we find

$$\|\widetilde{\mathcal{S}} u_\Pi(t)\|_{L^2(X_{\delta_1})} \leq C \|A_0 \Pi \mathbf{u}(t - t_1)\|_{L^2} + \mathcal{O}(h^\infty);$$

here A_0 is the operator from Proposition 2.5. Since $[P, \Pi] = \mathcal{O}(h^\infty)$ microlocally near $\widehat{W} \times \widehat{W}$, and by (2.29) and (2.37) (replacing t_1 by $s \in [0, t_1]$ in the definition of u_Π and differentiating in s) we get

$$\widetilde{\mathcal{S}} u_\Pi(t) = \Pi \mathbf{u}(t) + \mathcal{O}(h^\infty) \quad \text{microlocally near } \widehat{W}. \tag{2.54}$$

Therefore,

$$\|A_0 \Pi \mathbf{u}(t)\|_{L^2} \leq C \|\widetilde{\mathcal{S}} u_\Pi(t)\|_{L^2(X_{\delta_1})} + \mathcal{O}(h^\infty).$$

Next, by (2.21) each backwards flow line of e^{tH_p} starting in $X_{2\delta_1}$ either stays forever in $X_{2\delta_1}$ or reaches the complement of $\mathrm{WF}_h(\mathbf{u}(t))$—see the proof of Proposition 2.4. By propagation of singularities for the equation (2.29), we find

$$\|A \mathbf{u}(t_1)\|_{L^2} \leq C \|\mathbf{u}(0)\|_{L^2(X_{\delta_1})} + \mathcal{O}(h^\infty), \quad A \in \Psi^{\mathrm{comp}}(X_{2\delta_1}).$$

Also, for t_1 large enough, each flow line $\gamma(t)$, $t \in [-t_1, 0]$, of H_p such that $\gamma(0) \in X_{\delta_1}$ either satisfies $\gamma(-t_1) \in W'$ and $\gamma([-t_1, 0]) \subset X_{\delta_1}$, or there exists $s \in [-t_1, 0]$ such that $\gamma(s) \notin \mathrm{WF}_h(\mathbf{u}(t))$ for $t \in [t_1, T \log(1/h)]$ and $\gamma([s, 0]) \subset X_{\delta_1}$. This is true since if $\gamma(s) \in \mathrm{WF}_h(\mathbf{u}(t))$, then $\gamma([s - t_1, s]) \subset X_{\delta_1}$, see the proof of Proposition 2.4. By propagation of singularities for (2.29), we get

$$\|\mathbf{u}(t)\|_{L^2(X_{\delta_1})} \leq C \|A_1 \mathbf{u}(t - t_1)\|_{L^2} + \mathcal{O}(h^\infty), \quad t \in [2t_1, T \log(1/h)]. \tag{2.55}$$

By (2.29) and (2.51), we have $(hD_t + P)(\widetilde{\mathcal{S}} u_R(t)) = \mathcal{O}(h^\infty)$ on X_{δ_1}. Using propagation of singularities for this equation in a manner similar to (2.55), we obtain by (2.54)

$$\|\widetilde{\mathcal{S}} u_R(t)\|_{L^2(X_{\delta_1})} \leq C \|A_1(1 - \Pi)\mathbf{u}(t - t_1)\|_{L^2} + \mathcal{O}(h^\infty), \quad t \in [2t_1, T \log(1/h)].$$

Combining these estimates with (2.39)–(2.43), we arrive to

$$\|\widetilde{\mathcal{S}} u_\Pi(t_0)\|_{L^2(X_{\delta_1})} \leq C h^{-1/2} \|\mathbf{u}(0)\|_{L^2(X_{\delta_1})} + \mathcal{O}(h^\infty),$$
$$\|\widetilde{\mathcal{S}} u_\Pi(t)\|_{L^2(X_{\delta_1})} \leq C e^{-(\nu_{\min} - \varepsilon)t/2} \|\widetilde{\mathcal{S}} u_\Pi(t_0)\|_{L^2(X_{\delta_1})} + \mathcal{O}(h^\infty),$$
$$\|\widetilde{\mathcal{S}} u_\Pi(t)\|_{L^2(X_{\delta_1})} \geq C^{-1} e^{-(\nu_{\max} + \varepsilon)t/2} \|\widetilde{\mathcal{S}} u_\Pi(t_0)\|_{L^2(X_{\delta_1})} - \mathcal{O}(h^\infty),$$
$$\|\widetilde{\mathcal{S}} u_R(t)\|_{L^2(X_{\delta_1})} \leq C h^{-1} e^{-(\nu_{\min} - \varepsilon)t} \|\mathbf{u}(0)\|_{L^2(X_{\delta_1})} + \mathcal{O}(h^\infty),$$
$$\|\mathbf{u}(t)\|_{L^2(X_{\delta_1})} \leq C e^{\varepsilon t} \|\mathbf{u}(0)\|_{L^2(X_{\delta_1})} + \mathcal{O}(h^\infty),$$

holding uniformly in $t \in [t_0 - 1, T \log(1/h)]$. To obtain (2.22)–(2.26) from here, we need to remove the operator $\widetilde{\mathcal{S}}$ from the estimates; for that, we can use the fact that $\widetilde{\mathcal{S}}$ is bounded uniformly in h on $L^2_{t,x}$ together with the equivalency of the norms $h\|\cdot\|_{L^\infty_t \mathcal{E}_x}$ and $\|\cdot\|_{L^2_{t,x}}$ for solutions of the wave equation (2.17) given by (2.19) and the functions of interest being microlocalized at frequencies $\sim h^{-1}$. $\square$

3. Applications to Kerr(–de Sitter) Metrics

3.1. General properties. The Kerr(–de Sitter) metric in the Boyer–Lindquist coordinates is given by the formulas [Ca]

$$g = -\rho^2 \Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \Big) - \frac{\Delta_\theta \sin^2 \theta}{(1 + \alpha)^2 \rho^2} (a\, dt - (r^2 + a^2) d\varphi)^2$$
$$+ \frac{\Delta_r}{(1 + \alpha)^2 \rho^2} (dt - a \sin^2 \theta\, d\varphi)^2.$$

Here $M > 0$ denotes the mass of the black hole, a its angular speed of rotation, and $\Lambda \geq 0$ is the cosmological constant (with $\Lambda = 0$ in the Kerr case and $\Lambda > 0$ in the Kerr–de Sitter case);

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2Mr, \quad \Delta_\theta = 1 + \alpha \cos^2 \theta,$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \alpha = \frac{\Lambda a^2}{3}.$$

The metric is originally defined on

$$\widetilde{X}_0 := \mathbb{R}_t \times X_0, \quad X_0 := (r_-, r_+)_r \times \mathbb{S}^2,$$

here $\theta \in [0, \pi]$ and $\varphi \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ are the spherical coordinates on $\mathbb{S}^2$. The numbers $r_- < r_+$ are the roots of Δ_r defined below; in particular, $\Delta_r > 0$ on (r_-, r_+) and $\pm \partial_r \Delta_r(r_\pm) < 0$. The metric becomes singular on the surfaces $\{r = r_\pm\}$, known as the *event horizons*; however, this can be fixed by a change of coordinates, see Sect. 3.4.

The Kerr(–de Sitter) family admits the scaling $M \mapsto sM$, $\Lambda \mapsto s^{-2}\Lambda$, $a \mapsto sa$, $r \mapsto sr$, $t \mapsto st$ for $s > 0$; therefore, we often consider the parameters a/M and ΛM^2 invariant under this scaling. We assume that a/M, ΛM^2 lie in a neighborhood of the Schwarzschild(–de Sitter) case (1.1) or the Kerr case (1.2). Then for $\Lambda > 0$, Δ_r is a degree 4 polynomial with real roots $r_1 < r_2 < r_- < r_+$, with $r_- > M$. For $\Lambda = 0$, Δ_r is a degree 2 polynomial with real roots $r_1 < M < r_-$; we put $r_+ = \infty$. The general set of Λ and a for which Δ_r has all real roots as above was studied numerically in [AkMa, Sect. 3], and is pictured on Fig. 1a in the introduction. Note that in [AkMa], the roots are labeled $r_{--} < r_- < r_+ < r_C$; we do not adopt this (perhaps more standard) convention in favor of the notation of [Dy11a, Dy12, Va10], and since the roots r_1, r_2 are irrelevant in our analysis.

The symbol $\tilde{p}$ defined in (2.2) using the dual metric is (denoting by τ the momentum corresponding to t)

$$\tilde{p} = \rho^{-2}G, \quad G = G_r + G_\theta,$$

$$G_r = \Delta_r \xi_r^2 - \frac{(1+\alpha)^2}{\Delta_r}((r^2 + a^2)\tau + a\xi_\varphi)^2,$$

$$G_\theta = \Delta_\theta \xi_\theta^2 + \frac{(1+\alpha)^2}{\Delta_\theta \sin^2 \theta}(a \sin^2 \theta \, \tau + \xi_\varphi)^2.$$

Note that

$$\partial_{(t,\varphi,\theta,\xi_\theta)}G_r = 0, \quad \partial_{(t,\varphi,r,\xi_r)}G_\theta = 0, \tag{3.1}$$

therefore $\{G_r, G_\theta\} = 0$ and $G_\theta, \tau, \xi_\varphi$ are conserved quantities for the geodesic flow (2.3).

To handle the poles $\{\theta = 0\}$ and $\{\theta = \pi\}$, where the spherical coordinates (θ, φ) break down, introduce new coordinates (in a neighborhood of either of the poles)

$$x_1 = \sin \theta \cos \varphi, \quad x_2 = \sin \theta \sin \varphi; \tag{3.2}$$

note that $\sin^2 \theta = x_1^2 + x_2^2$ is a smooth function in this coordinate system. For the corresponding momenta ξ_1, ξ_2, we have

$$\xi_\theta = (x_1 \xi_1 + x_2 \xi_2) \cot \theta, \quad \xi_\varphi = x_1 \xi_2 - x_2 \xi_1,$$

note that ξ_φ is a smooth function vanishing at the poles. Then G_r, G_θ are smooth functions near the poles, with

$$G_\theta = (1+\alpha)(\xi_1^2 + \xi_2^2) \quad \text{when } x_1 = x_2 = 0. \tag{3.3}$$

The vector field ∂_t is not timelike inside the ergoregion, described by the inequality

$$\Delta_r \le a^2 \Delta_\theta \sin^2\theta. \tag{3.4}$$

For $a \ne 0$, this region is always nonempty. However, the covector dt is always timelike:

$$G|_{\xi=dt} = (1+\alpha)^2 \left(\frac{a^2 \sin^2\theta}{\Delta_\theta} - \frac{(r^2+a^2)^2}{\Delta_r} \right) < 0, \tag{3.5}$$

since $\Delta_r < r^2 + a^2$.

We now verify the geometric assumptions (1)–(4) of Sect. 2.2. Assumptions (1)–(3) have been established already; assumption (4) is proved by

Proposition 3.1. *Consider the function $\mu(r) \in C^\infty(r_-, r_+)$ defined by*

$$\mu(r) := \frac{\Delta_r(r)}{r^4}. \tag{3.6}$$

Then there exists $\delta_0 > 0$ such that for each $(\tilde{x}, \tilde{\xi}) \in T^*\widetilde{X}_0$,

$$\mu(\tilde{x}) < \delta_0, \ \tilde{\xi} \ne 0, \ \tilde{p}(\tilde{x}, \tilde{\xi}) = 0, \ H_{\tilde{p}}\mu(\tilde{x}, \tilde{\xi}) = 0 \implies H_{\tilde{p}}^2\mu(\tilde{x}, \tilde{\xi}) < 0. \tag{3.7}$$

Moreover, δ_0 can be chosen to depend continuously on M, Λ, a.

Proof. First of all, we calculate

$$\partial_r \mu(r) = -\frac{4\Delta_r - r\partial_r\Delta_r}{r^5}, \quad 4\Delta_r - r\partial_r\Delta_r = 2((1-\alpha)r^2 - 3Mr + 2a^2), \tag{3.8}$$

therefore $\partial_r \mu(r) < 0$ for $\alpha \le 1/2$ and $r > 6M$. Since $\partial_r \Delta_r(r_\pm) \ne 0$, we see that for δ_0 small enough and $\mu(r) < \delta_0$, we have $\partial_r \mu(r) \ne 0$. Therefore, we can replace the condition $H_{\tilde{p}}\mu = 0$ in (3.7) by $H_{\tilde{p}}r = 0$, which implies that $\xi_r = 0$; in this case, $H_{\tilde{p}}^2\mu$ has the same sign as $-\partial_r\mu\partial_r G_r$. We calculate for $\xi_r = 0$,

$$\partial_r G_r = -\frac{(1+\alpha)^2((r^2+a^2)\tau + a\xi_\varphi)}{\Delta_r^2}\Psi(r), \tag{3.9}$$

$$\Psi(r) := 4r\tau\Delta_r - ((r^2+a^2)\tau + a\xi_\varphi)\partial_r\Delta_r.$$

Next, denote

$$A := (r^2+a^2)\tau + a\xi_\varphi, \quad B := a\sin^2\theta\,\tau + \xi_\varphi, \tag{3.10}$$

then

$$\rho^2\tau = A - aB, \quad \Psi = \frac{(4r\Delta_r - \rho^2\partial_r\Delta_r)A - 4ar\Delta_r B}{\rho^2}. \tag{3.11}$$

Using the equation $\tilde{p} = 0$, we get

$$\frac{A^2}{\Delta_r} \ge \frac{B^2}{\Delta_\theta \sin^2\theta} \quad \text{on } \{\tilde{p} = \xi_r = 0\} \cap \{0 < \theta < \pi\}. \tag{3.12}$$

Since $\Delta_\theta \sin^2 \theta \le 1$ everywhere for $\alpha \le 1$, and $B = 0$ for $\sin \theta = 0$, we find

$$A^2 \ge \Delta_r B^2. \tag{3.13}$$

In particular, we see that $A \ne 0$, since otherwise $B = 0$, implying that $\tau = \xi_\varphi = 0$ and thus $\tilde{\xi} = 0$ since $\tilde{p} = \xi_r = 0$. Now, $H_{\tilde{p}}^2 \mu$ has the same sign as

$$\partial_r \mu((4r\Delta_r - \rho^2 \partial_r \Delta_r)A^2 - 4ar\Delta_r AB). \tag{3.14}$$

We now calculate by (3.8) and since $\partial_r \Delta_r \le 2r$, for $\alpha \le 1/2$

$$4r\Delta_r - \rho^2 \partial_r \Delta_r = 2r((1-\alpha)r^2 - 3Mr + 2a^2) - a^2 \cos^2 \theta \partial_r \Delta_r$$
$$\ge r(r^2 - 6Mr + 2a^2),$$

and thus, since $\Delta_r \le r^2 + a^2$ and $|a| < M$, and by (3.13),

$$(4r\Delta_r - \rho^2 \partial_r \Delta_r)A^2 - 4ar\Delta_r AB \ge A^2 r(r^2 - 6Mr + 2a^2 - 4|a|\sqrt{\Delta_r})$$
$$\ge A^2 r(r^2 - 10Mr - 4M^2).$$

We see that (3.7) holds for r large enough, namely $r > 14M$.

We now assume that $r \le 14M$ and $\mu < \delta_0$. Then (here the constants do not depend on δ_0 and are locally uniform in M, Λ, a)

$$\Delta_r = \mathcal{O}(\delta_0), \quad |\partial_r \Delta_r| \ge C^{-1}, \quad \partial_r \mu = r^{-4} \partial_r \Delta_r + \mathcal{O}(\delta_0).$$

Then for δ_0 small enough, by (3.13) the expression (3.14) has the same sign as

$$A^2 \partial_r \Delta_r (-\rho^2 \partial_r \Delta_r + \mathcal{O}(\sqrt{\delta_0})) < 0,$$

as required. $\square$

3.2. Structure of the trapped set.

We now study the structure of trapping for Kerr(–de Sitter) metrics, summarized in the following

Proposition 3.2. *For $(\Lambda M^2, a/M)$ in a neighborhood of the union of (1.1) and (1.2), assumptions (5)–(8) of Sect. 2.2 are satisfied, with $\mu_{\max} = 0$ (see (2.11)) and the trapped set (see Definition 2.1) given by*

$$\widetilde{K} = \{G = \xi_r = \partial_r G_r = 0, \; \tilde{\xi} \ne 0\} \subset T^* \widetilde{X}_0 \backslash 0. \tag{3.15}$$

Remark. The assumptions on M, Λ, a can quite possibly be relaxed. The only parts of the proof that need us to be in a neighborhood of (1.1) or (1.2) are (3.18) and (3.19). Several other statements require that α is small (in particular, (3.27) requires $\alpha < \frac{\sqrt{2}-1}{\sqrt{2}+1}$), but this is true for the full admissible range of parameters depicted on Fig. 1a in the introduction. However, if Λ, a are large enough so that

$$r \in (r_-, r_+), \; \partial_r \Delta_r(r) = 0 \implies \Delta_r(r) = a^2, \tag{3.16}$$

then the trapped set contains points with $\tau = 0$ (and also $\xi_r = \xi_\theta = 0, \theta = \pi/2, \xi_\varphi \ne 0$), which prevent us from having a meromorphic continuation of the resolvent and violate the required assumption (5) in Sect. 2.2—see the discussion preceding [Va10, (6.13)]. The set of values of Λ, a satisfying (3.16) is pictured as the dashed line on Fig. 1a.

Remark. Some parts of Proposition 3.2 have previously been verified in [Va10, Sect. 6.4] in the case $|a| < \frac{\sqrt{3}}{2}M$ and under the additional assumption [Va10, (6.13)].

We start by analysing the set $\widetilde{K}$ defined by (3.15); the fact that $\widetilde{K}$ is indeed the trapped set is established later, in Proposition 3.5. We first note that $\widetilde{K}$ is a closed conic subset of $\{\tilde{p} = 0\}\backslash 0$, invariant under the flow (2.3); indeed, $\xi_r = 0$ implies $H_{\tilde{p}}r = 0$, $\partial_r G_r = 0$ implies $H_{\tilde{p}}\xi_r = 0$, $H_{\tilde{p}}\tau = H_{\tilde{p}}\xi_\varphi = 0$ everywhere, and $\partial_r G_r$ depends only on $r, \xi_r, \tau, \xi_\varphi$.

By (3.9), and since $(r^2 + a^2)\tau + a\xi_\varphi = \tilde{p} = 0$ implies $\tilde{\xi} = 0$, we see that

$$\Psi = 0 \quad \text{on } \widetilde{K}. \tag{3.17}$$

Assumption (5) in Sect. 2.2 follows from the inequality

$$\tau((r^2 + a^2)\tau + a\xi_\varphi) > 0 \quad \text{on } \widetilde{K}. \tag{3.18}$$

For the Schwarzschild(–de Sitter) case (1.1), this is trivial (noting that $\tau = 0$ implies $\tilde{\xi} = 0$); for the Kerr case (1.2), it follows from (3.17) together with the fact that $\partial_r \Delta_r > 0$. The general case now follows by perturbation, using that, by Proposition 3.1, $\widetilde{K}$ is contained in a fixed compact subset of X_0.

We next claim that

$$\partial_r^2 G < 0 \quad \text{on } \widetilde{K}. \tag{3.19}$$

By (3.9), this is equivalent to requiring that $\tau \partial_r \Psi > 0$ on $\widetilde{K}$. Now, in either of the cases (1.1) or (1.2), we calculate

$$\Psi(r) = 2(\tau r^3 - 3M\tau r^2 + a(a\tau - \xi_\varphi)r + Ma(a\tau + \xi_\varphi)). \tag{3.20}$$

In particular,

$$\Psi(M) = 4M\tau(a^2 - M^2), \quad \partial_r^2 \Psi(r) = 12\tau(r - M).$$

Since $|a| < M$, we see that

$$\tau \Psi(M) < 0; \quad \tau \partial_r^2 \Psi(r) > 0 \text{ for } r > M.$$

Therefore, if $r > r_- > M$ and $\Psi(r) = 0$, then $\tau \partial_r \Psi(r) > 0$ and we get (3.19) in the cases (1.1) and (1.2); the general case follows by perturbation, similarly to (3.18).

To study the behavior of $\widetilde{K}$ in the angular variables, we introduce the equatorial set

$$\widetilde{K}_e := \widetilde{K} \cap \{\theta = \pi/2, \ \xi_\theta = 0\}. \tag{3.21}$$

This is a closed conic subset of $\widetilde{K}$ invariant under the flow (2.3) (which is proved similarly to the invariance of $\widetilde{K}$). We have

$$\partial_{\xi_\varphi} G \neq 0 \quad \text{on } \widetilde{K}_e. \tag{3.22}$$

Indeed,

$$\partial_{\xi_\varphi} G = 2(1 + \alpha)^2 \left(-\frac{a((r^2 + a^2)\tau + a\xi_\varphi)}{\Delta_r} + a\tau + \xi_\varphi \right) \quad \text{on } \{\theta = \pi/2\}. \tag{3.23}$$

Also, the equation $G = 0$ implies

$$\frac{((r^2 + a^2)\tau + a\xi_\varphi)^2}{\Delta_r} = \frac{(a \sin^2 \theta \, \tau + \xi_\varphi)^2}{\Delta_\theta \sin^2 \theta} \quad \text{on } \widetilde{K} \cap \{\xi_\theta = 0\}. \qquad (3.24)$$

Putting $\theta = \pi/2$ into (3.24), we solve for Δ_r and substitute it into (3.23), obtaining

$$\partial_{\xi_\varphi} G = 2(1 + \alpha)^2 \frac{r^2 \tau(a\tau + \xi_\varphi)}{(r^2 + a^2)\tau + a\xi_\varphi} \neq 0 \quad \text{on } \widetilde{K}_e, \qquad (3.25)$$

implying (3.22).

At the poles $\{\theta = 0, \pi\}$, we have

$$|\partial_{\xi_1} G| + |\partial_{\xi_2} G| > 0. \qquad (3.26)$$

This follows immediately from (3.3), as $\xi_1 = \xi_2 = 0$ would imply $G_\theta = 0$, which is impossible given that $\xi_r = 0$, $G = 0$, and $\tilde{\xi} \neq 0$.

Finally, we claim that

$$\widetilde{K} \cap \{\xi_\theta = \partial_\theta G = 0\} \cap \{0 < \theta < \pi\} = \widetilde{K}_e, \qquad (3.27)$$
$$\partial_\theta^2 G > 0 \quad \text{on } \widetilde{K}_e. \qquad (3.28)$$

To see this, note that $0 < \Delta_r < r^2 + a^2$, $\Delta_\theta \geq 1$, and $(r^2 + a^2)\tau + a\xi_\varphi \neq 0$ by (3.18); we get from (3.24)

$$((r^2 + a^2)\tau + a\xi_\varphi)^2 < (r^2 + a^2)\frac{(a \sin^2 \theta \, \tau + \xi_\varphi)^2}{\sin^2 \theta} \quad \text{on } \widetilde{K} \cap \{\xi_\theta = 0\},$$

or, using that $|a| < M < r$,

$$\frac{\xi_\varphi^2}{\sin^2 \theta} > (r^2 + a^2)\tau^2 > 2a^2\tau^2 \quad \text{on } \widetilde{K} \cap \{\xi_\theta = 0\}. \qquad (3.29)$$

Next, if $\xi_\theta = 0$, then

$$\partial_\theta G = \frac{2(1 + \alpha)^2 (a \sin^2 \theta \tau + \xi_\varphi) \cos \theta}{\Delta_\theta^2 \sin^3 \theta}((1 + \alpha)a \sin^2 \theta \, \tau - (1 + \alpha \cos(2\theta))\xi_\varphi).$$

In particular, using (3.29) we obtain (3.28) for $\alpha = 0$:

$$\partial_\theta^2 G = 2(\xi_\varphi^2 - a^2\tau^2) > 0 \quad \text{on } \widetilde{K}_e,$$

and the case of small α follows by perturbation. It remains to prove (3.27). Assume the contrary, that $\partial_\theta G = 0$, $\xi_\theta = 0$, but $\theta \neq \pi/2$. By (3.24), $a \sin^2 \theta \tau + \xi_\varphi \neq 0$; therefore, $(1 + \alpha)a \sin^2 \theta \, \tau = (1 + \alpha \cos(2\theta))\xi_\varphi$. Combining this with (3.29), we get $(1+\alpha) \sin \theta > \sqrt{2}(1+\alpha \cos(2\theta))$, which implies that $(1+\alpha) > \sqrt{2}(1-\alpha)$, a contradiction with the fact that α is small.

It follows from (3.19), (3.22), (3.26), and (3.27) that at each point of $\widetilde{K}$ the matrix of partial derivatives $G, \xi_r, \partial_r G$ in the variables $(r, \xi_r, *)$, where $*$ stands for one of $\theta, \xi_\theta, \xi_\varphi, \xi_1, \xi_2$, is invertible. This gives

Proposition 3.3. *The set $\widetilde{K}$ defined by* (3.15) *is a smooth codimension 2 submanifold of* $\{\tilde{p} = 0\}\backslash 0$, *and its projection $\widehat{K}$ onto the $\hat{x} = (t, \theta, \varphi), \hat{\xi} = (\tau, \xi_\theta, \xi_\varphi)$ variables is a smooth codimension 1 submanifold of $T^*(\mathbb{R} \times \mathbb{S}^2)$.*

We now study the global dynamics of the flow, relating it to the set $\widetilde{K}$. Take $(\tilde{x}^0, \tilde{\xi}^0) \in \{\tilde{p} = 0\}\backslash 0$ and let $(\tilde{x}(t), \tilde{\xi}(t))$ be the corresponding Hamiltonian trajectory of (2.3). Consider the function

$$\Phi^0(r) = G_r(\tilde{x}^0, \tilde{\xi}^0) + (1 + \alpha)^2 \frac{((r^2 + a^2)\tau^0 + a\xi_\varphi^0)^2}{\Delta_r(r)}.$$

Note that $G_r(\tilde{x}(t), \tilde{\xi}(t)), \tau(t), \xi_\varphi(t)$ are constant in t and $(r(t), \xi_r(t))$ is a rescaled Hamiltonian flow trajectory of

$$H^0(r, \xi_r) := \Delta_r(r)\xi_r^2 - \Phi^0(r);$$

in particular, $(r(t), \xi_r(t))$ solve the equation

$$\Delta_r(r)\xi_r^2 = \Phi^0(r). \tag{3.30}$$

The key property of Φ^0 is given by

Proposition 3.4. *For each $r \in (r_-, r_+)$,*

$$\Phi^0(r) \geq 0, \ \partial_r \Phi^0(r) = 0 \implies \partial_r^2 \Phi^0(r) > 0. \tag{3.31}$$

Proof. Assume that $\Phi^0(r) \geq 0$. Then we can find $(\tilde{x}^1, \tilde{\xi}^1) \in T^*\widetilde{X}^0$ such that $(t^1, \theta^1, \varphi^1) = (t^0, \theta^0, \varphi^0), \tilde{r}^1 = r, \tau^1 = \tau^0, \xi_\varphi^1 = \xi_\varphi^0, \xi_r^1 = 0$, and $\tilde{p}(\tilde{x}^1, \tilde{\xi}^1) = 0$; indeed, it suffices to start with $(\tilde{x}^0, \tilde{\xi}^0)$, put $r^1 = r, \xi_r^1 = 0$, and change the ξ_θ^1 component (or one of ξ_1^1, ξ_2^1 components if we are at the poles of the sphere) so that $G_\theta(\tilde{x}^1, \tilde{\xi}^1) = G_\theta(\tilde{x}^0, \tilde{\xi}^0) + \Phi^0(r)$. If additionally $\partial_r \Phi^0(r) = 0$, then $(\tilde{x}^1, \tilde{\xi}^1) \in \widetilde{K}$; it remains to apply (3.19). $\square$

We now consider the following two cases.

Case 1: $|\Phi^0(r)| + |\partial_r \Phi^0(r)| > 0$ for all $r \in (r_-, r_+)$. In this case, the set of solutions to (3.30) is a closed one-dimensional submanifold of $T^*(r_-, r_+)$ and the Hamiltonian field of H^0 is nonvanishing on this manifold. This manifold has no compact connected components, as the function $\Phi^0(r)$ cannot achieve a local maximum on it by (3.31). It follows that the geodesic $(\tilde{x}(t), \tilde{\xi}(t))$ escapes in both time directions.

Case 2: there exists $r' \in (r_-, r_+)$ such that $\Phi^0(r') = \partial_r \Phi^0(r') = 0$. Then

$$(t^0, r', \theta^0, \varphi^0, \tau^0, 0, \xi_\theta^0, \xi_\varphi^0) \in \widetilde{K},$$

therefore the projection $(\hat{x}^0, \hat{\xi}^0)$ lies in $\widehat{K}$ (see Proposition 3.3). By (3.31), we see that $\partial_r^2 \Phi^0(r') > 0$ and $(r - r')\partial_r \Phi^0(r) > 0$ for $r \neq r'$. Then the set of solutions to the equation (3.30) is equal to the union $\Gamma_+^0 \cup \Gamma_-^0$, where

$$\Gamma_\pm^0 = \{\xi_r = \mp \operatorname{sgn}(\tau^0) \operatorname{sgn}(r - r')\sqrt{\Phi^0(r)/\Delta_r(r)}\},$$

note that $\Gamma_\pm^0$ are smooth one-dimensional submanifolds of $T^*(r_-, r_+)$ intersecting transversely at $(r', 0)$. The trajectory $(\tilde{x}(t), \tilde{\xi}(t))$ is trapped as $t \to \mp\infty$ if and only if $(r^0, \xi_r^0) \in \Gamma_\pm^0$. Note that by (3.18), τ^0 is negative on $\mathcal{C}_+$ and positive on $\mathcal{C}_-$.

The analysis of the two cases above implies

Proposition 3.5. *The incoming/outgoing tails* $\widetilde{\Gamma}_\pm$ *(see Definition* 2.1*) are given by (here* $\widehat{K}$ *is defined in Proposition* 3.3*)*

$$\widetilde{\Gamma}_\pm := \{(r, \hat{x}, \xi_r, \hat{\xi}) \mid (\hat{x}, \hat{\xi}) \in \widehat{K}, \ \xi_r = \mp \operatorname{sgn}(\hat{\tau}) \operatorname{sgn}(r - r'_{\hat{x},\hat{\xi}}) \sqrt{\Phi_{\hat{x},\hat{\xi}}(r)/\Delta_r(r)}\},$$

where

$$\Phi_{\hat{x},\hat{\xi}}(r) = -G_\theta(\hat{x}, \hat{\xi}) + (1+\alpha)^2 \frac{((r^2 + a^2)\hat{\tau} + a\hat{\xi}_\varphi)^2}{\Delta_r(r)},$$

and $r'_{\hat{x},\hat{\xi}}$ *is the only solution to the equation* $\Phi_{\hat{x},\hat{\xi}}(r) = 0$*; moreover,* $\partial_r \Phi_{\hat{x},\hat{\xi}}(r'_{\hat{x},\hat{\xi}}) = 0$ *and* $\partial_r^2 \Phi_{\hat{x},\hat{\xi}}(r'_{\hat{x},\hat{\xi}}) > 0$. *Furthermore,* $\widetilde{\Gamma}_\pm$ *are conic smooth codimension 1 submanifolds of* $\{\tilde{p} = 0\}\backslash 0$ *intersecting transversely, and their intersection is equal to the set* $\widetilde{K}$ *defined in* (3.15).

We also see from (3.19) and the fact that $\partial_\tau \tilde{p} \neq 0$ on $\{\tilde{p} = 0\}\backslash 0$ (as follows from (3.5)) that the matrix of Poisson brackets of functions $G, \partial_r G, \xi_r, t$ on $\widetilde{K}$ is nondegenerate, which implies that the intersections $\widetilde{K} \cap \{t = \text{const}\}$ are symplectic submanifolds of $T^*\widetilde{X}_0$. Together with Proposition 3.5, this verifies assumptions (6) and (7) of Sect. 2.2.

It remains to verify r-normal hyperbolicity of the flow $\tilde{\varphi}^s$ defined in (2.3). We start by showing that the maximal expansion rate in the directions of the trapped set $\mu_{\max}$, defined in (2.11), is equal to zero.

Proposition 3.6. *For each* $\varepsilon > 0$*, there exists a constant* C *such that for each* $v \in T\widetilde{K}$*,*

$$|d\tilde{\varphi}^s v| \leq Ce^{\varepsilon|s|}|v|.$$

Here $|\cdot|$ *denotes any fixed smooth homogeneous norm on the fibers of* $T\widetilde{K}$*.*

Proof. Using the group property of the flow, it suffices to show that for each $\varepsilon > 0$ there exists $T > 0$ such that for each $v \in T\widetilde{K}$,

$$|d\tilde{\varphi}^T v| < e^{\varepsilon T}|v|. \tag{3.32}$$

Since $\widetilde{K}$ is a closed conic set, and $\widetilde{K} \cap \{\tau = 1\} \cap \{t = 0\}$ is compact, it suffices to show that for each flow line $\gamma(s)$ of (2.3) on $\widetilde{K}$, there exists T such that (3.32) holds for each $v = v(0)$ tangent to $\widetilde{K}$ at $\gamma(0)$. Denote $v(s) = d\tilde{\varphi}^s v(0)$.

If $\gamma(s)$ is a trajectory of (2.3) on $T\widetilde{K}$, then $r, \xi_r = 0, \tau$ are constant on $\gamma(s)$ and the generator of the flow does not depend on the variable t; therefore, it suffices to show (3.32) for the restriction of the matrix of $d\tilde{\varphi}^T$ to the $\partial_\theta, \partial_\varphi, \partial_{\xi_\theta}, \partial_{\xi_\varphi}$ variables. This is equivalent to considering the Hamiltonian flow of G in the $\theta, \varphi, \xi_\theta, \xi_\varphi$ variables only, on $T^*\mathbb{S}^2$. Recall that the equatorial set $\widetilde{K}_e = \widetilde{K} \cap \{\theta = \pi/2, \ \xi_\theta = 0\}$ defined in (3.21) is invariant under $\tilde{\varphi}^s$. We then consider two cases.

Case 1: $\gamma(s) \notin \widetilde{K}_e$ for all s. Then the differentials of G and ξ_φ are linearly independent by (3.26) and (3.27). Since $\{G, \xi_\varphi\} = 0$, by Arnold–Liouville theorem (see for example [Dy15, Proposition 2.8]), there is a local symplectomorphism from a neighborhood of $\gamma(s)$ in $T^*\mathbb{S}^2$ to $T^*\mathbb{T}^2$, where $\mathbb{T}^2$ is the two-dimensional torus, which conjugates G to some function $f(\eta_1, \eta_2)$; here $(y_1, y_2, \eta_1, \eta_2)$ are the canonical coordinates on $T^*\mathbb{T}^2$. The corresponding evolution of tangent vectors is given by $\partial_s v_y(s) = \nabla^2 f(\eta(s))v_\eta(s)$, $\partial_s v_\eta(s) = 0$, and (3.32) follows.

Case 2: $\gamma(s) \in \widetilde{K}_e$ for all s. Since $\partial_s v_{\xi_\varphi}(s) = 0$ and $\partial_s v$ does not depend on $v_{\varphi(s)}$, it suffices to estimate $v_\theta(s)$, $v_{\xi_\theta}(s)$. We then find

$$\partial_s v_\theta(s) = 2v_{\xi_\theta}(s), \quad \partial_s v_{\xi_\theta}(s) = -\partial_\theta^2 G(\gamma(s))v_\theta(s) - \partial_{\theta\xi_\varphi}^2 G(\gamma(s))v_{\varphi}(s).$$

Now, by (3.28), $\partial_\theta^2 G(\gamma(s))$ is a positive constant; (3.32) follows. $\square$

We finally show that the minimal expansion rate $v_{\min}$, defined in (2.9), is positive. By Proposition 3.5, $(\tilde{x}, \tilde{\xi}) \in \widetilde{\Gamma}_\pm$ if and only if

$$(\hat{x}, \hat{\xi}) \in \widehat{K}, \quad \tilde{\varphi}_\pm(\tilde{x}, \tilde{\xi}) = 0,$$

where

$$\tilde{\varphi}_\pm(\tilde{x}, \tilde{\xi}) = \xi_r \mp \operatorname{sgn}(\partial_\tau G) \operatorname{sgn}(r - r'_{\hat{x},\hat{\xi}}) \sqrt{\Phi_{\hat{x},\hat{\xi}}(r)/\Delta_r(r)}.$$

Since H_G is tangent to $\widetilde{\Gamma}_\pm$, we have $H_G\tilde{\varphi}_\pm = 0$ on $\widetilde{\Gamma}_\pm$; it follows that

$$\frac{H_G\tilde{\varphi}_\pm(\tilde{x}, \tilde{\xi})}{\partial_\tau G} = \mp\tilde{v}_\pm(\tilde{x}, \tilde{\xi})\tilde{\varphi}_\pm(\tilde{x}, \tilde{\xi}) \quad \text{when } (\hat{x}, \hat{\xi}) \in \widehat{K}, \tag{3.33}$$

for some functions $\tilde{v}_\pm$. By calculating $\partial_{\xi_r} H_G\tilde{\varphi}_\pm|_{\widetilde{K}}$, we find $\tilde{v}_+|_{\widetilde{K}} = \tilde{v}_-|_{\widetilde{K}} = \tilde{v}$, where

$$\tilde{v} = \frac{\sqrt{-2\Delta_r\partial_r^2 G_r}}{|\partial_\tau G|}; \tag{3.34}$$

note that $\partial_r^2 G_r < 0$ on $\widetilde{K}$ by (3.19) and $\partial_\tau G \neq 0$ on $\{\tilde{p} = 0\}\backslash 0$ by assumption (3) in Sect. 2.2.

Let $\widetilde{\mathcal{V}}_\pm$ be the one-dimensional subbundles of $T\widetilde{\Gamma}_\pm$ defined in (2.7), invariant under the flow $\tilde{\varphi}^s$. Since $d\tilde{\varphi}_\mp$ vanishes on $T\widetilde{K}$ and is not identically zero on $T_{\widetilde{K}}\widetilde{\Gamma}_\pm$, we can fix a basis $v_\pm$ of $\widetilde{\mathcal{V}}_\pm|_{\widetilde{K}}$ by requiring that

$$d\tilde{\varphi}_\mp \cdot v_\pm = 1.$$

Denote by $V = H_G/\partial_\tau G$ the generator of the flow $\tilde{\varphi}^s$. The Lie derivative $\mathcal{L}_V v_\pm$ is a multiple of $v_\pm$; to compute it, we use the identity

$$0 = V(d\tilde{\varphi}_\mp \cdot v_\pm) = \mathcal{L}_V(d\tilde{\varphi}_\mp) \cdot v_\pm + d\tilde{\varphi}_\mp \cdot \mathcal{L}_V v_\pm.$$

Since (3.33) holds on $\widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_-$, we get on vectors tangent to $\widetilde{\Gamma}_\pm$,

$$\mathcal{L}_V(d\tilde{\varphi}_\mp) = d(\pm\tilde{v}_\mp\tilde{\varphi}_\mp) = \pm\tilde{v}d\tilde{\varphi}_\mp \quad \text{on } \widetilde{K}.$$

It follows that

$$\partial_s(d\tilde{\varphi}^s v_\pm) = \pm(\tilde{v} \circ \tilde{\varphi}^s)v_\pm,$$

which implies immediately

Proposition 3.7. *The expansion rates defined in* (2.9) *and* (2.10) *are given by*

$$v_{\min} = \liminf_{T \to \infty} \inf_{(x,\xi)\in K} \langle\tilde{v}\rangle_T, \quad v_{\max} = \limsup_{T \to \infty} \sup_{(x,\xi)\in K} \langle\tilde{v}\rangle_T,$$

where $\tilde{v} > 0$ *is the function on* $\widetilde{K}$ *defined in* (3.34) *and*

$$\langle\tilde{v}\rangle_T := \frac{1}{T} \int_0^T \tilde{v} \circ \varphi^s \, ds.$$

Together, Propositions 3.6 and 3.7 verify assumption (8) of Sect. 2.2 and finish the proof of Proposition 3.2.

3.3. Trapping in special cases. We now establish some properties of the trapped set $\widetilde{K}$ and the local expansion rate $\tilde{\nu}$, defined in (3.34), in two special cases. We start with the Schwarzschild(–de Sitter) case (1.1), when everything can be described explicitly.

Proposition 3.8. *For $a = 0$, we have*

$$\widetilde{K} = \left\{ \xi_r = 0, \ r = 3M, \ \tau \neq 0, \ G_\theta = \frac{27M^2}{1 - 9\Lambda M^2} \tau^2 \right\}, \tag{3.35}$$

$$\tilde{\nu} = \frac{\sqrt{1 - 9\Lambda M^2}}{3\sqrt{3}M}. \tag{3.36}$$

Proof. We recall from (3.17) that $\widetilde{K}$ is given by the equations $G = 0, \xi_r = 0, \Psi = 0$, where Ψ is computed using (3.20):

$$\Psi(r) = 2\tau r^2(r - 3M).$$

Since $\tau \neq 0$ on $\widetilde{K}$ by (3.18), we see that $\Psi = 0$ is equivalent to $r = 3M$. Now, $\Delta_r(3M) = 3M^2(1 - 9\Lambda M^2)$; therefore, $G_r = -\frac{27M^2}{(1 - 9\Lambda M^2)}\tau^2$ for $\xi_r = 0$ and $r = 3M$ and we obtain (3.35). Next, by (3.9), we find

$$\partial_r^2 G_r = -\frac{r^2 \tau}{\Delta_r^2} \partial_r \Psi(r) = -\frac{18}{(1 - 9\Lambda M^2)^2} \tau^2 \quad \text{on } \widetilde{K}.$$

Finally, we compute

$$\partial_\tau G = -\frac{54M^2}{1 - 9\Lambda M^2} \tau \quad \text{on } \widetilde{K},$$

and (3.36) follows. $\square$

We next consider the case when $\Lambda = 0$ and a approaches the maximal rotation speed M from below, calculating the expansion rates on two equators to show that the pinching condition (1.12) is violated.

Proposition 3.9. *Fix M and assume that*

$$\Lambda = 0, \quad a = M - \epsilon, \quad 0 < \epsilon \ll 1.$$

Then $\widetilde{K}_e$, defined in (3.21), is the union of two conical sets

$$E_\pm = \{r = R_\pm(\epsilon), \ \xi_r = 0, \ \xi_\varphi = F_\pm(\epsilon)\tau, \ \theta = \pi/2, \ \xi_\theta = 0, \ \tau \neq 0\},$$

where $R_+(\epsilon), F_+(\epsilon)$ are smooth functions of ϵ, $R_-(\epsilon), F_-(\epsilon)$ are smooth functions of $\sqrt{\epsilon}$, and (see Fig. 4)

$$\begin{aligned} R_+(\epsilon) &= 4M + \mathcal{O}(\epsilon), \quad F_+(\epsilon) = 7M + \mathcal{O}(\epsilon); \\ R_-(\epsilon) &= M + \sqrt{8\epsilon M/3} + \mathcal{O}(\epsilon), \quad F_-(\epsilon) = -2M - \sqrt{6\epsilon M} + \mathcal{O}(\epsilon). \end{aligned} \tag{3.37}$$

Finally, the expansion rates $\tilde{\nu}$ defined in (3.34) are given by (see also Fig. 2a in the introduction)

$$\tilde{\nu} = \frac{3\sqrt{3}}{28M} + \mathcal{O}(\epsilon) \ \text{on } E_+; \quad \tilde{\nu} = \frac{\sqrt{\epsilon/2M}}{M} + \mathcal{O}(\epsilon) \ \text{on } E_-. \tag{3.38}$$
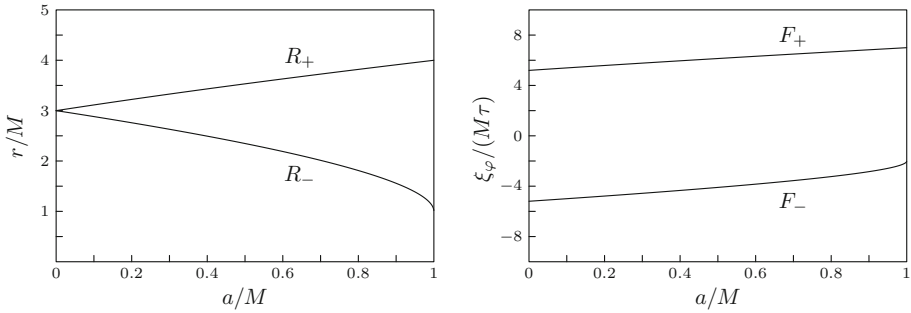
Fig. 4. The graphs of r and ξ_φ/τ on the trapped equators $E_\pm$, as functions of a for $\Lambda = 0$

Proof. The set $\widetilde{K}_e$ is defined by equations $\xi_r = \xi_\theta = 0$, $\theta = \pi/2$, and (see (3.17))

$$((r^2 + a^2)\tau + a\xi_\varphi)^2 = \Delta_r(r)(a\tau + \xi_\varphi)^2,$$
$$4r\tau\Delta_r(r) = ((r^2 + a^2)\tau + a\xi_\varphi)\partial_r\Delta_r(r). \tag{3.39}$$

Recall that $\Delta_r(r) = r^2 + a^2 - 2Mr$. Putting $A = (r^2 + a^2)\tau + a\xi_\varphi$ and $B = a\tau + \xi_\varphi$, we rewrite these as

$$A^2 = \Delta_r(r)B^2,$$
$$4(A - aB)\Delta_r(r) = Ar\partial_r\Delta_r(r).$$

The second equation can be written as $(r^2 + 2a^2 - 3Mr)A = 2a\Delta_r(r)B$. Solving for B and substituting into the first equation, we get

$$4a^2\Delta_r(r) - (r^2 + 2a^2 - 3Mr)^2 = 0. \tag{3.40}$$

This is a fourth order polynomial equation in r with coefficients depending on ϵ and with a root at $r = 0$; we will study the behavior of the other three roots as $\epsilon \to 0$. We write (3.40) as

$$(r - M)^2(r - 4M) = -8\epsilon M^2 + \mathcal{O}(\epsilon^2). \tag{3.41}$$

By the implicit function theorem, for ϵ small enough, the equation (3.40) has a solution $R = R_+(\epsilon) = 4M + \mathcal{O}(\epsilon)$. We next identify the two roots lying near $r = M$; they are solutions to the equations

$$r - M = \pm M\sqrt{\frac{8 + \mathcal{O}(\epsilon)}{4M - r}} \cdot \sqrt{\epsilon}.$$

The solution with the negative sign lies to the left of $r_- > M$, therefore we ignore it. The solution with the positive sign, which we denote by $R_-(\epsilon)$, exists for ϵ small enough by the implicit function theorem and we find $R_-(\epsilon) = M + \sqrt{8\epsilon M/3} + \mathcal{O}(\epsilon)$.

To find the values of ξ_φ/τ corresponding to $r = R_\pm(\epsilon)$, we use the second equation in (3.39); this completes the proof of (3.37). Finally, we calculate at $r = R_+(\epsilon)$, $\xi_\varphi = F_+(\epsilon)\tau$,

$$\Delta_r = 9M^2 + \mathcal{O}(\epsilon), \quad \partial_r^2 G = -\frac{32}{3}\tau^2 + \mathcal{O}(\epsilon), \quad \partial_\tau G = -\frac{224}{3}M^2\tau + \mathcal{O}(\epsilon),$$

and at $r = R_-(\epsilon)$, $\xi_\varphi = F_-(\epsilon)\tau$,

$$\Delta_r = \frac{2M}{3}\epsilon + \mathcal{O}(\epsilon^2), \quad \partial_r^2 G = -\frac{9M}{\epsilon}\tau^2 + \mathcal{O}(\epsilon^{-1/2}), \quad \partial_\tau G = -2\sqrt{\frac{6M}{\epsilon}}M^2\tau + \mathcal{O}(1);$$

(3.38) follows. $\quad\square$

We finally explain how to numerically compute the constant $c_{\widetilde{K}}$ from the Weyl law of Theorem 3, defined in (2.16). We can parametrize $\widetilde{K} \cap \{t = 0\} \cap \{\xi_\theta > 0\}$ by the variables $\tau, \xi_\varphi, \theta, \varphi$—indeed, we can find $r = r(\tau, \xi_\varphi)$ from the equation $\partial_r G = 0$, and find from the equation $G = 0$ that $\xi_\theta = \sqrt{\Theta(\tau, \xi_\varphi, \theta, \varphi)}$, where

$$\Theta = \frac{(1+\alpha)^2}{\Delta_\theta \Delta_r(r(\tau, \xi_\varphi))}((r(\tau, \xi_\varphi)^2 + a^2)\tau + a\xi_\varphi)^2 - \frac{(1+\alpha)^2}{\Delta_\theta^2 \sin^2\theta}(a\sin^2\theta\,\tau + \xi_\varphi)^2.$$

The domain of integration is $\{\Theta > 0\} \cap \{0 \le \tau \le 1\}$. Then (using the symmetry $\xi_\theta \mapsto -\xi_\theta$ and fact that Θ does not depend on φ)

$$c_{\widetilde{K}} = 4\pi \int_{\{\Theta > 0\} \cap \{0 \le \tau \le 1\}} d\theta \wedge d\xi_\theta \wedge d\xi_\varphi = 2\pi \int_{\{\Theta > 0\} \cap \{0 \le \tau \le 1\}} \frac{\partial_\tau \Theta}{\sqrt{\Theta}} d\theta d\xi_\varphi d\tau.$$

Now, Θ is homogeneous of degree 2 in the (τ, ξ_φ) variables; therefore, the integrand is homogeneous of degree 0 and we make the change of variables $\xi_\varphi = s\tau$ to get

$$c_{\widetilde{K}} = \pi \int_{\{\Theta > 0\} \cap \{\tau = 1\}} \frac{\partial_\tau \Theta}{\sqrt{\Theta}} d\theta d\xi_\varphi. \tag{3.42}$$

We also note that we can compute $\partial_\tau \Theta$ without involving $\partial_\tau r$, since $\partial_r G = 0$ on the trapped set.

For $a = 0$, we put $c_0 = \frac{3\sqrt{3}M}{\sqrt{1-9\Lambda M^2}}$ and compute (putting $\xi_\varphi = sc_0 \sin\theta$)

$$c_{\widetilde{K}} = 2\pi c_0^2 \int_0^\pi \int_{-1}^1 \frac{\sin\theta}{\sqrt{1-s^2}} ds d\theta = 4\pi^2 c_0^2.$$

3.4. Results for linear waves.

In this section, we apply Theorem 4 in Sect. 2.4 and the analysis of Sects. 3.1, 3.2 to obtain Theorems 1 and 2.

We start by formulating a well-posed problem for the wave equation on the Kerr–de Sitter background. For that, we in particular need to shift the time variable, see [Dy11a, Sect. 1] and [Dy12, Sect. 1.1]. Let μ be the defining function of the event horizons and/or spatial infinity defined in (3.6) and fix a small constant δ_1, used in Theorem 4 as well as in (2.13). To continue the metric smoothly past the event horizons, we make the change of variables

$$t = t^* + F_t(r), \quad \varphi = \varphi^* + F_\varphi(r), \tag{3.43}$$

where F_t, F_φ are smooth real-valued functions on (r_-, r_+) such that

- $F_t'(r) = \pm\frac{1+\alpha}{\Delta_r(r)}(r^2 + a^2) + f_\pm(r)$ and $F_\varphi'(r) = \pm\frac{1+\alpha}{\Delta_r(r)}a$ near $r = r_\pm$, where $f_\pm$ are smooth functions (for the Kerr case $\Lambda = 0$, we only require this at $r = r_-$)
- $F_t(r) = F_\varphi(r) = 0$ near $\{\mu \ge \delta_1/10\}$ (and also for r large enough in the Kerr case $\Lambda = 0$);

- the covector dt^* is timelike everywhere; equivalently, the level surfaces of t^* are spacelike.

See for example [Va10, Sect. 6.1 and (6.15)] for how to construct such F_t, F_φ. The metric in the coordinates $(t^*, r, \theta, \varphi^*)$ continues smoothly through $\{r = r_-\}$ and $\{r = r_+\}$ (the latter for $\Lambda > 0$), to an extension $\tilde{X}_{-\delta_1} := \{\mu > -\delta_1\}$ of X_0 past the event horizons. Since $F_t = F_\varphi = 0$ near $\{\mu \geq \delta_1/10\}$, our change of variables does not affect the arguments in Sect. 2.

The principal symbol of $h^2\Box_{\tilde{g}}$ in the new variables, denoted by $\tilde{p}^*$, is given by

$$\tilde{p}^*(r, \theta, \tau^*, \xi_r, \xi_\theta, \xi_{\varphi^*}) = \tilde{p}(r, \theta, \tau^*, \xi_r - \partial_r F_t(r)\tau^* - \partial_r F_\varphi(r)\xi_{\varphi^*}, \xi_\theta, \xi_{\varphi^*}).$$

In particular, if $\xi_\theta = \xi_{\varphi^*} = 0$, then for r close to r_- or to r_+ (the latter case for $\Lambda > 0$),

$$\tilde{p}^* = \Delta_r(\xi_r - f_\pm(r)\tau^*)^2 \mp 2(1+\alpha)(r^2+a^2)\tau^*(\xi_r - f_\pm(r)\tau^*) + \frac{(1+\alpha)^2 a^2 \sin^2\theta}{\Delta_\theta}(\tau^*)^2.$$

Then in the new coordinates,

$$\tilde{g}^{-1}(dr, dr) = -\Delta_r, \quad \tilde{g}^{-1}(dr, dt^*) = \pm(1+\alpha)(r^2+a^2) + \Delta_r f_\pm(r). \tag{3.44}$$

Therefore, the surfaces $\{r = r_0\}$ are timelike for $\mu(r_0) > 0$, lightlike for $\mu(r_0) = 0$, and spacelike for $\mu(r_0) < 0$, and $\tilde{g}^{-1}(d\mu, dt^*) < 0$ near the event horizon(s). Moreover, for $\Lambda = 0$ the d'Alembert–Beltrami operator

$$\Box_{\tilde{g}} = \frac{1}{\rho^2} D_r(\Delta_r D_r) + \frac{1}{\rho^2 \sin\theta} D_\theta(\sin\theta D_\theta)$$
$$+ \frac{(a\sin^2\theta\, D_t + D_\varphi)^2}{\rho^2 \sin^2\theta} - \frac{((r^2+a^2)D_t + aD_\varphi)^2}{\rho^2 \Delta_r}$$

belongs to Melrose's scattering calculus on the space slices near $r = \infty$ (see [VaZw, Sect. 2]) in the sense that it is a polynomial in the differential operators D_t, D_r, $r^{-1}D_\theta$, $r^{-1}D_\varphi$ with coefficients smooth up to $\{r^{-1} = 0\}$ in the r^{-1}, θ, φ variables (where of course θ, φ are replaced by a different coordinate system on $\mathbb{S}^2$ near the poles $\{\sin\theta = 0\}$).

Consider the initial-value problem for the wave equation (here $s \geq 0$ is integer)

$$\Box_{\tilde{g}}u = 0, \ t^* \geq 0; \quad u|_{t^*=0} = f_0, \ \partial_{t^*}u|_{t^*=0} = f_1;$$
$$f_0 \in H^{s+1}(X_{-\delta_1}), \ f_1 \in H^s(X_{-\delta_1}). \tag{3.45}$$

This problem is well-posed, based on standard methods for hyperbolic equations [Tay, Sect. 6.5] and the following crude energy estimate: if we consider functions on $\tilde{X}_{-\delta_1}$ as functions of t^* with values in functions on $X_{-\delta_1}$, then for $t' \geq 0$,

$$\|u(t')\|_{H^{s+1}(X_{-\delta_1})} + \|\partial_{t^*}u(t')\|_{H^s(X_{-\delta_1})}$$
$$\leq Ce^{Ct'}(\|u(0)\|_{H^{s+1}(X_{-\delta_1})} + \|\partial_{t^*}u(0)\|_{H^s(X_{-\delta_1})} + \|e^{-Ct^*}\Box_{\tilde{g}}u\|_{H^s((0,t')\times X_{-\delta_1})}). \tag{3.46}$$

To prove (3.46) for $s = 0$, we use the standard energy estimate on $\Omega = \tilde{X}_{-\delta_1} \cap \{0 \leq t^* \leq t'\}$ for hyperbolic equations (see [Tay, Sect. 2.8], [Dy11a, Proposition 1.1], or [Dy11b, Sect. 1.1]), with the timelike vector field $\mathcal{N}$ equal to ∂_t (a Killing field) for large r (in the case $\Lambda = 0$) and to $\tilde{g}^{-1}(dt^*)$ close to the event horizon(s); by (3.44), the boundary

$\partial\Omega$ is spacelike and $\mathcal{N}$ points inside of Ω on $\{t^* = 0\}$ and outside of it elsewhere on $\partial\Omega$. The higher order estimates are obtained from here as in [Tay, (6.5.14)], commuting with differential operators in the scattering calculus.

We now assume that $f_0 = f_0(h)$, $f_1 = f_1(h)$ are such that $\|f_0\|_{H^1(X_{-\delta_1})} + \|f_1\|_{L^2(X_{-\delta_1})}$ is bounded polynomially in h and f_0, f_1 are localized at frequencies $\sim h^{-1}$, namely (see the discussion in Sect. 2.1)

$$\mathrm{WF}_h(f_0) \cup \mathrm{WF}_h(f_1) \subset T^*X_{-\delta_1}\setminus 0.$$

Let u be the corresponding solution to (3.45) and assume that it is extended to small negative times (which can be done by taking a smaller δ_1 and using the local existence result backwards in time). By (3.46), we see that u is h-tempered uniformly for $t^* \in [0, T\log(1/h)]$. Similarly to (2.18), $\widetilde{\mathrm{WF}_h}(u) \subset \{\tilde{p}^* = 0\}$. Moreover, using standard microlocal analysis for hyperbolic equations, we get a pseudodifferential one-to-one correspondence between (f_0, f_1) and $(u_+(0), u_-(0))$, where $u_\pm$ are the components of u microlocalized on $\mathcal{C}_\pm$, the positive and negative parts of the light cone, each solving an equation of the form $(hD_t + P_\pm)u_\pm = \mathcal{O}(h^\infty)$ for some spatial pseudodifferential operators $P_\pm$ (similarly to (2.28)). This gives

$$\mathrm{WF}_h(u) \cap \{t^* = 0\} \subset \{(0, x, \tau, \xi) \mid \tilde{p}^*(x, \tau, \xi) = 0, \ (x, \xi) \in \mathrm{WF}_h(f_0) \cup \mathrm{WF}_h(f_1)\}.$$

In particular, we get

$$\mathrm{WF}_h(u) \cap \{t^* = 0\} \subset T^*\widetilde{X}_{-\delta_1}\setminus 0. \tag{3.47}$$

By the same correspondence, if $\mathrm{WF}_h(u) \cap \{t^* = 0\}$ is compact and covered by finitely many open subsets of $T^*\widetilde{X}_{-\delta_1}\setminus 0$, then we can apply the associated pseudodifferential partition of unity to f_0, f_1 to split u into several solutions to the wave equation such that the wavefront set of each solution at $t^* = 0$ is contained in one of the covering sets. The resulting solutions can then each be analysed separately.

We next assume that

$$\mathrm{supp}\, f_0 \cup \mathrm{supp}\, f_1 \subset X_{\delta_1}.$$

We obtain some restrictions on the microlocalization of u for large times. For that, we need to consider the dynamics of the geodesic flow on the extended spacetime $\widetilde{X}_{\delta_1}$. Define the flow $\hat{\varphi}^s$ similarly to (2.3), rescaling the geodesic flow so that the variable t^* is growing with speed 1. Since $t = t^*$, $\varphi = \varphi^*$ on $\widetilde{X}_{\delta_1/10}$, the flow lines of $\tilde{\varphi}^s$ and $\hat{\varphi}^s$ coincide on $\widetilde{X}_{\delta_1/10}$. If $\gamma(s)$ is a flow line of $\hat{\varphi}^s$ such that $\gamma(0) \in \widetilde{X}_{\delta_1/10}$ and γ is not trapped for positive times according to Definition 2.1, then either $\gamma(s)$ escapes to the Euclidean infinity (for $\Lambda = 0$) or $\gamma(s)$ crosses one of the event horizons at some fixed positive time s_0, and $\mu(\gamma(s)) < 0$ is strictly decreasing for $s > s_0$ (see the discussion following [Va10, (6.22)], verifying [Va10, (2.8)]); in the latter case, we say that γ escapes through the event horizons.

The next statement makes nontrivial use of the structure of the infinite ends (in particular, using [Me, VaZw, Da] for the asymptotically Euclidean end for $\Lambda = 0$) and is the key step for obtaining control on the escaping parts of the solution for long times:

Proposition 3.10. *Assume that all flow lines of $\hat{\varphi}^s$ starting on $\mathrm{WF}_h(u) \cap \{t^* = 0\}$ escape, either to the spatial infinity or through the event horizons. Then there exists $T_0 > 0$ such that uniformly in t^*,*

$$\|u(t^*)\|_{H^1(X_{\delta_1})} + \|\partial_{t^*}u(t^*)\|_{L^2(X_{\delta_1})} = \mathcal{O}(h^\infty), \quad t^* \in [T_0, T\log(1/h)]. \tag{3.48}$$

Proof. We first consider the case when $\mathrm{WF}_h(u) \cap \{t^* = 0\}$ is contained in a small neighborhood of some $(\tilde{x}, \tilde{\xi}) \in T^* X_{\delta_1} \backslash 0$, and, for $\gamma(s) = \hat{\varphi}^s(\tilde{x}, \tilde{\xi})$, there exists $T_0 > 0$ such that $\gamma([0, T_0]) \subset \widetilde{X}_{-3\delta_1/4}$ and $\gamma(T_0) \in \{\mu < -\delta_1/2\}$. By propagation of singularities for the wave equation (see for example [Dy15, Proposition 3.4]), we see that $\mathrm{WF}_h(u) \cap \{t^* = T_0\} \subset \{\mu < -\delta_1/2\}$; it follows that

$$\|u(T_0)\|_{H^1(X_{-\delta_1/2})} + \|\partial_{t^*} u(T_0)\|_{L^2(X_{-\delta_1/2})} = \mathcal{O}(h^\infty).$$

Then the same bound holds for $t^* \geq T_0$ in place of T_0 by (3.46) (replacing δ_1 by $\delta_1/2$).

For the remainder of this proof, we consider the opposite case, when $\Lambda = 0$ and each flow line of $\hat{\varphi}^s$ starting on $\mathrm{WF}_h(u) \cap \{t^* = 0\}$ escapes to the spatial infinity. Fix a large constant R_1; we require in particular that $X_{\delta_1} \subset \{r < R_1\}$. By propagation of singularities, similarly to the previous paragraph, we may shift the time parameter and assume that $\mathrm{WF}_h(u) \cap \{t^* = 0\}$ is contained in a small neighborhood of some $(\tilde{x}_0, \tilde{\xi}_0) \in T^* \widetilde{X}_0 \backslash 0$, where $r_0 > R_1$, $\partial_s \mu(\hat{\varphi}^s(\tilde{x}_0, \tilde{\xi}_0))|_{s=0} < 0$. In fact, by (3.46) and finite speed of propagation, we may assume that for t^* near 0, the support of u in x is contained in a compact subset of $\{r > R_1\}$. Without loss of generality, we assume that $\tau_0 < 0$. The trajectory $\hat{\varphi}^s(\tilde{x}_0, \tilde{\xi}_0)$ does not intersect $\{r \leq R_1\}$ for $s \geq 0$.

We replace the Kerr spacetime $(\widetilde{X}_0, \tilde{g})$ with a different spacetime $(\mathbb{R}_t \times \mathbb{R}^3_{r,\theta,\varphi}, \tilde{g}_1)$, where (r, θ, φ) are the spherical coordinates on $\mathbb{R}^3$ and $\tilde{g}_1$ is the stationary Lorentzian metric defined on $\mathbb{R}^4$ by

$$\tilde{g}_1^{-1} := \chi_1(r)\tilde{g}_0^{-1} + (1 - \chi_1(r))\tilde{g}^{-1},$$

where $\tilde{g}_0^{-1} = \tau^2 - \xi_r^2 - \xi_\theta^2/r^2 - \xi_\varphi^2/(r^2 \sin^2 \theta)$ is the Minkowski metric on $\mathbb{R}^4$, $\chi_1 \in C_0^\infty([0, R_1))$, $0 \leq \chi_1 \leq 1$ everywhere, and $\chi_1 = 1$ on $[0, R_1/2]$. The dual metrics $\tilde{g}^{-1}$ and $\tilde{g}_0^{-1}$ are close to each other for large r in the sense of scattering metrics, that is, as quadratic forms in $\tau, \xi_r, r^{-1}\xi_\theta, r^{-1}\xi_\varphi$, therefore for R_1 large enough, $\tilde{g}_1^{-1}$ is the dual to a Lorentzian metric, the surfaces $\{t = \text{const}\}$ are spacelike, and ∂_t is a timelike vector field. Note that the new spacetime no longer contains an event horizon.

We now show that $\tilde{g}_1^{-1}$ is nontrapping for large R_1 and a correct choice of χ_1, that is, each lightlike geodesic escapes to the spatial infinity in both time directions. It suffices to prove that if $\tilde{p}_1(\tilde{x}, \tilde{\xi}) = -\tilde{g}_{1,\tilde{x}}^{-1}(\tilde{\xi}, \tilde{\xi})$, then (compare with assumption (4) in Sect. 2.2)

$$r > 0, \ \tilde{p}_1 = 0, \ \tilde{\xi} \neq 0, \ H_{\tilde{p}_1} r(\tilde{x}, \tilde{\xi}) = 0 \implies H_{\tilde{p}_1}^2 r(\tilde{x}, \tilde{\xi}) > 0.$$

Indeed,

$$H_{\tilde{p}_1} r = 2\xi_r(\chi_1(r) + (1 - \chi_1(r))\Delta_r/\rho^2);$$

therefore, $H_{\tilde{p}_1} r = 0$ implies $\xi_r = 0$ and $H_{\tilde{p}_1}^2 r$ has the same sign as

$$-\partial_r \tilde{p}_1 = -\chi_1'(r)(\tilde{p}_0 - \tilde{p}) - \chi_1(r)\partial_r \tilde{p}_0 - (1 - \chi_1(r))\partial_r \tilde{p};$$

it remains to note that we can take $r\chi_1'(r)$ bounded by 3, $\tilde{p}_0 - \tilde{p}$ is small for large r in the sense of scattering metrics, and both $r\partial_r \tilde{p}_0$ and $r\partial_r \tilde{p}$ are homogeneous of degree 2 in $\tilde{\xi}$ and bounded from above by a negative constant on $\{\tau^2 + r^{-2}\xi_\theta^2 + r^{-2}\xi_\varphi^2 = 1\} \cap \{\tilde{p}_1 = \xi_r = 0\}$, uniformly in $r^{-1} \geq 0$ for $\partial_r \tilde{p}_0$ and uniformly in $r^{-1} \in [0, \delta_1)$ for $\partial_r \tilde{p}$.

Let u_1 be the solution to the wave equation on the new spacetime $(\mathbb{R}^4, \tilde{g}_1)$ such that $u_1|_{t=0} = u|_{t^*=0}$, $\partial_t u_1|_{t=0} = \partial_{t^*} u|_{t^*=0}$. It is enough to prove that, with $\widetilde{\mathrm{WF}}_h(u_1)$ defined in Sect. 2.1,

$$\widetilde{\mathrm{WF}}_h(u_1) \cap \{r < R_1\} = \emptyset, \quad 0 \le t \le T \log(1/h). \tag{3.49}$$

Indeed, in this case $\Box_{\tilde{g}}((1 - \chi_1(r))u_1) = \mathcal{O}(h^\infty)_{C^\infty}$; by (3.46), we have $\widetilde{\mathrm{WF}}_h(u_1) = \widetilde{\mathrm{WF}}_h(u)$ for $t \in [0, T \log(1/h)]$, and (3.48) follows since $X_{\delta_1} \subset \{r < R_1\}$.

To show (3.49), we use the Fourier transform in time,

$$\hat{u}_1(\lambda) = \int_{\mathbb{R}} e^{i\lambda t} \psi_1(t) u_1(t) \, dt, \quad \mathrm{Im}\, \lambda > 0.$$

Here $\psi_1(t)$ is supported in $[-\delta, \infty)$ and is equal to 1 on $[\delta, \infty)$, for some small fixed $\delta > 0$. The integral converges, as $\|u_1(t)\|_{H^2(\mathbb{R}^3)} \le C e^{\varepsilon t}$ for each $\varepsilon > 0$, as follows from the standard energy estimate for the wave equation (see the paragraph following (3.46)) applied for the timelike Killing vector field ∂_t.

Let $\widehat{P}(\lambda)$ be the stationary d'Alembert–Beltrami operator for the metric $\hat{g}$, constructed by replacing D_t by $-\lambda$ in the operator $\Box_{\hat{g}}$; the semiclassical version defined in Sect. 2.3 is given by the relation $\widehat{P}_h(\omega) = h^2 \widehat{P}(h^{-1}\omega)$. Then

$$\widehat{P}(\lambda)\hat{u}_1(\lambda) = \hat{f}_1(\lambda), \quad \mathrm{Im}\, \lambda > 0,$$

where $f_1 = [\Box_{\hat{g}}, \psi_1(t)]u_1(t)$. We note that $\mathrm{WF}_h(f_1)$ is contained in a small neighborhood of $(\tilde{x}_0, \tilde{\xi}_0)$ and f_1 is compactly supported; therefore, $\hat{f}_1(h^{-1}\omega + iE) = \mathcal{O}(h^\infty)_{\mathscr{S}_\omega C_0^\infty(\mathbb{R}^3)}$ for ω outside of a small neighborhood of $-\tau_0 > 0$, and $\mathrm{WF}_h(\hat{f}_1(h^{-1}\omega + iE))$ lies in a small neighborhood of (x_0, ξ_0) for all ω.

We now apply the results of [Me, VaZw, Da]. For this, note that for any fixed λ, the operator $\widehat{P}(\lambda)$ lies in Melrose's scattering calculus on the radially compactified $\mathbb{R}^3$, and for $\mathrm{Im}\, \lambda > 0$, the operator $\widehat{P}(\lambda)$ is elliptic in this calculus in the microlocal sense (that is, elliptic as ξ and/or r go to infinity)—in fact, near $r = \infty$ the operator $\widehat{P}(\lambda)$ is close to $\Delta_0 - \lambda^2$, where Δ_0 is the flat Laplacian on $\mathbb{R}^3$. Moreover, $\widehat{P}(\lambda)$ is a symmetric operator when $\lambda \in \mathbb{R}$. This implies that the proofs of [VaZw, Da] apply. Similarly to [Me, Theorem 2], for $\mathrm{Im}\, \lambda > 0$, the operator $\widehat{P}(\lambda)$ is Fredholm $H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ and invertible for λ outside of a discrete set; we can then fix $E > 0$ such that $\widehat{P}(\lambda + iE)$ is invertible for all $\lambda \in \mathbb{R}$.

Next, the Hamiltonian flow of the principal symbol $\hat{\mathbf{p}}(\omega)$ of $\widehat{P}_h(\omega)$ corresponds to lightlike geodesics of the metric $\hat{g}$, similarly to (2.14). Therefore, this flow is nontrapping at all energies $\omega \ne 0$. By [VaZw], we get for each $\chi_0 \in C_0^\infty(\mathbb{R}^3)$,

$$\|\chi_0 \widehat{P}(\lambda + iE)^{-1} \chi_0\|_{L^2 \to L^2} \le C\langle\lambda\rangle^{-1}, \quad \lambda \in \mathbb{R}; \tag{3.50}$$

in fact, the constant in the estimate is bounded as $E \to 0$, but we do not use this here. Finally, by [Da, Lemma 2], we see that $\widehat{P}_h(\omega + ihE)^{-1}$ is semiclassically outgoing for ω near $-\tau_0$, that is, the wavefront set of $\hat{u}_1(h^{-1}\omega + iE)$ is contained in the union of $\mathrm{WF}_h(\hat{f}_1(h^{-1}\omega + iE))$ and all Hamiltonian flow lines of $\hat{\mathbf{p}}(\omega)$ starting on $\mathrm{WF}_h(\hat{f}_1(h^{-1}\omega + iE)) \cap \{\hat{\mathbf{p}}(\omega) = 0\}$. Since no geodesic starting near $(\tilde{x}_0, \tilde{\xi}_0)$ intersects $\{r \le R_1\}$ for positive times, we get $\mathrm{WF}_h(\hat{u}_1(h^{-1}\omega + iE)) \cap \overline{T}^* X_{\delta_1} = \emptyset$ for ω in a neighborhood of $-\tau_0$. For ω outside of this neighborhood, we use the rapid decay of $\hat{f}_1(\omega)$ established before, together with (3.50), to get

$$\hat{u}_1(\lambda + iE) = \mathcal{O}(h^\infty\langle\lambda\rangle^{-\infty})_{C^\infty(\{r < R_1\})}.$$

It remains to use the Fourier inversion formula

$$u_1(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(\lambda + iE)t} \hat{u}_1(\lambda + iE) \, d\lambda$$

to get (3.49). □

Any solution satisfying (3.48) is trivial from the point of view of Theorems 1 and 2 (putting $u_\Pi = 0$). Therefore, we may assume that $\mathrm{WF}_h(u) \cap \{t^* = 0\}$ is contained in a small neighborhood of some $(\tilde{x}, \tilde{\xi})$ such that the corresponding geodesic does not escape. By assumption (5) in Sect. 2.2, see also Lemma 2.2, we may assume that

$$\mathrm{WF}_h(u) \cap \{t^* = 0\} \subset (\mathcal{C}_+ \cap \{\tau < 0\}) \cup (\mathcal{C}_- \cap \{\tau > 0\}),$$

here $\mathcal{C}_\pm$ are defined in (2.6). We can reduce the case $\mathrm{WF}(u) \cap \{t^* = 0\} \subset \mathcal{C}_-$ to the case $\mathrm{WF}_h(u) \cap \{t^* = 0\} \subset \mathcal{C}_+$ by taking the complex conjugate of u, and take a dyadic partition of unity together with the natural rescaling of the problem $\tilde{\xi} \mapsto s\tilde{\xi}, h \mapsto sh$, to reduce to the case

$$\mathrm{WF}_h(u) \cap \{t^* = 0\} \subset \mathcal{C}_+ \cap \{|1 + \tau| < \delta_1/8\}. \tag{3.51}$$

Proposition 3.11. *For* $\widetilde{\mathrm{WF}_h}(u)$ *defined in Sect.* 2.1, *we have*

$$\widetilde{\mathrm{WF}_h}(u) \subset \{|1 + \tau| < \delta_1/4\}. \tag{3.52}$$

Proof. Consider a function $\psi \in C_0^\infty(-1 - \delta_1/2, -1 + \delta_1/2)$ such that $\psi = 1$ near $[-1 - \delta_1/4, -1 + \delta_1/4]$. If u solves the wave equation on $(-\delta, \infty)$, then we extend it to a function on the whole $\widetilde{X}_{-\delta_1}$ smoothly and so that $\mathrm{supp}\, u \subset \{t^* > -2\delta\}$. Define

$$u' := (1 - \psi(hD_{t^*}))u,$$

then, since the metric is stationary, $\Box_{\tilde{g}} u' = (1 - \psi(hD_{t^*}))\Box_{\tilde{g}} u = \mathcal{O}(h^\infty)_{\mathscr{S}(\widetilde{X}_{-\delta_1} \cap \{t^* \geq -\delta/2\})}$. By (3.51), we get $\mathrm{WF}_h(u') \cap \{t^* = 0\} = \emptyset$. Then by the energy estimate (3.46), applied to u', we get $\widetilde{\mathrm{WF}_h}(u') = \emptyset$, uniformly in $t^* \in [0, T \log(1/h)]$. It remains to note that $\widetilde{\mathrm{WF}_h}(\psi(hD_{t^*})u) \subset \{|1 + \tau| < \delta_1/4\}$. □

We can now give

Proofs of Theorems 1 and 2. Without loss of generality (replacing δ_1 by $\delta_1/3$) we may assume that $\mathrm{supp}\, f_0 \cup \mathrm{supp}\, f_1 \subset X_{3\delta_1}$.

Choose small $t_\varepsilon > 0$ and a cutoff function $\chi = \chi(\mu)$, with $\mathrm{supp}\, \chi \subset \{\mu > 2\delta_1\}$ and $\chi = 1$ near $\{\mu \geq 3\delta_1\}$, such that, with the flow $\tilde{\varphi}^t$ defined in (2.3),

$$(\tilde{x}, \tilde{\xi}) \in \mathrm{supp}\, \chi, \ \tilde{\varphi}^{t_\varepsilon}(\tilde{x}, \tilde{\xi}) \in \mathrm{supp}(1 - \chi), \ \tilde{\xi} \neq 0 \implies \frac{H_{\tilde{p}}}{\partial_\tau \tilde{p}} \mu(\tilde{\varphi}^{t_\varepsilon}(\tilde{x}, \tilde{\xi})) < 0. \tag{3.53}$$

The existence of such χ and t_ε follows from Proposition 3.1, see the proof of [DyGu, Lemma 5.5(1)].

Take $N(h) = \lceil T \log(1/h)/t_\varepsilon \rceil$ and consider the functions $u^{(0)} := u$ and

$$u^{(j)} \in C^\infty(\widetilde{X}_{-\delta_1} \cap \{t^* \geq jt_\varepsilon\}), \quad 1 \leq j \leq N(h),$$
$$\Box_{\tilde{g}} u^{(j)} = 0, \ u^{(j)}(jt_\varepsilon) = \chi u^{(j-1)}(jt_\varepsilon), \ \partial_{t^*} u^{(j)}(jt_\varepsilon) = \chi \partial_{t^*} u^{(j-1)}(jt_\varepsilon).$$

By (3.46), $u^{(j)}$ are h-tempered uniformly in j and in $t^* \in [jt_\varepsilon, T\log(1/h) + 2]$. Moreover, similarly to Proposition 3.11, we get $\widetilde{\mathrm{WF}_h}(u^{(j)}) \subset \{|1 + \tau| < \delta_1/4\}$ uniformly in j. Then, $u^{(j)} - u^{(j-1)}$ are solutions to the wave equation with initial data $(1 - \chi)(u^{(j-1)}(jt_\varepsilon), \partial_{t^*} u^{(j-1)}(jt_\varepsilon))$, therefore by (3.53)

$$\mathrm{WF}_h(u^{(j)} - u^{(j-1)}) \cap \{t^* = jt_\varepsilon\} \subset \{|1 + \tau| < \delta_1/4\} \cap \{\mu > \delta_1\} \cap \left\{ \frac{H_{\tilde{p}}}{\partial_\tau \tilde{p}} \mu < 0 \right\}.$$

Then all the trajectories of $\hat{\varphi}^s$ starting on $\mathrm{WF}_h(u^{(j)} - u^{(j-1)}) \cap \{t^* = jt_\varepsilon\}$ escape as $s \to +\infty$; by Proposition 3.10, we see that

$$\widetilde{\mathrm{WF}_h}(u^{(j)} - u^{(j-1)}) \cap \{\mu > \delta_1\} = \emptyset, \quad t^* \in [jt_\varepsilon + T_0, T\log(1/h)],$$

uniformly in j, where T_0 is a fixed large constant. Adding these up, we get

$$\widetilde{\mathrm{WF}_h}(u^{(j)} - u) \cap \{\mu > \delta_1\} = \emptyset, \quad t^* \in [jt_\varepsilon + T_0, T\log(1/h)]. \tag{3.54}$$

By propagation of singularities for the wave equation and using that $\mathrm{WF}_h(u^{(j)}) \cap \{t^* = jt_\varepsilon\} \subset \{\mu > 2\delta_1\}$, we see, uniformly in j,

$$\mathrm{WF}_h(u) \cap \{\delta_1 \le \mu \le 2\delta_1\} \cap \{jt_\varepsilon \le t^* - T_0 \le (j+1)t_\varepsilon\} \subset \{|1 + \tau| < \delta_1/4\}$$
$$\cap \left\{ \frac{H_{\tilde{p}}}{\partial_\tau \tilde{p}} \mu < 0 \right\}.$$

Combining this with (3.54) (and another application of propagation of singularities for times up to T_0), we get uniformly in $t^* \in [0, T\log(1/h)]$,

$$\widetilde{\mathrm{WF}_h}(u) \cap \{\delta_1 \le \mu \le 2\delta_1\} \subset \{|1 + \tau| < \delta_1/4\} \cap \left\{ \frac{H_{\tilde{p}}}{\partial_\tau \tilde{p}} \mu < 0 \right\}. \tag{3.55}$$

This implies that for any bounded fixed T_1, the semiclassical singularities of $u(t + T_1)$ in X_{δ_1} come via propagation of singularities from the semiclassical singularities of $u(t)$ in X_{δ_1} —that is, no new singularities arrive from the outside. We can then apply propagation of singularities to see that $\widetilde{\mathrm{WF}_h}(u) \cap \{\mu > \delta_1\} \subset \mathcal{W}$ uniformly in $t^* \in [T_0, T\log(1/h)]$, where $\mathcal{W} \subset \mathcal{C}_+$ is constructed in Lemma 2.2; indeed, every trajectory of $\tilde{\varphi}^s$ starting on $\{|1 + \tau| < \delta_1/4\} \cap \{\mu > \delta_1\} \setminus \mathcal{W}$ escapes as $s \to +\infty$. Together with (3.52) and (3.55), this implies that for $t \ge T_0$, u satisfies the outgoing condition of Definition 2.3.

We can finally apply Theorem 4 in Sect. 2.4, giving Theorem 2 and additionally the bounds (the first one of which is a combination of (2.22), (2.23), and (2.25))

$$\|u(t)\|_{\mathcal{E}} \le C(h^{-1/2} e^{-(\nu_{\min} - \varepsilon)t/2} + h^{-1} e^{-(\nu_{\min} - \varepsilon)t} + h^N) \|u(0)\|_{\mathcal{E}},$$
$$\|u(t)\|_{\mathcal{E}} \le C e^{\varepsilon t} \|u(0)\|_{\mathcal{E}}.$$

The first of these bounds gives Theorem 1 for $(\nu_{\min} - \varepsilon)t \ge \log(1/h)$; the second one gives

$$\|u(t)\|_{\mathcal{E}} \le C h^{-1/2} e^{-(\nu_{\min} - 3\varepsilon)t/2} \|u(0)\|_{\mathcal{E}}, \quad (\nu_{\min} - \varepsilon)t \le \log(1/h),$$

which is the bound of Theorem 1 with ε replaced by 3ε. $\quad\square$

3.5. Results for resonances. In this section, we use the results of [Dy15] together with the analysis of Sects. 3.1, 3.2 to prove Theorem 3. As in the statement of this theorem, we consider the Kerr–de Sitter case $\Lambda > 0$.

We first use [Va10, Sect. 6] to define resonances for Kerr–de Sitter and put them into the framework of [Dy15, Sect. 4]. We use the change of variables (3.43); the metric in the coordinates $(t^*, r, \theta, \varphi^*)$ continues smoothly through the event horizons to $\widetilde{X}_{-\delta_1} = \{\mu > -\delta_1\}$, see [Va10, Sect. 6.1].

Following [Va10, Sect. 6.2] (but omitting the ρ^2 factor), we consider the stationary d'Alembert–Beltrami operator $P(z)$, obtained by replacing D_{t^*} by $-z \in \mathbb{C}$ in $\Box_{\tilde{g}}$. It is an operator on the space slice $X_{-\delta_1} = \{\mu > -\delta_1\}_r \times \mathbb{S}^2_{\theta,\varphi}$. We consider the semiclassical version

$$P_{\tilde{g}}(\omega) := h^2 P(h^{-1}\omega),$$

where $h \to 0$ is a small parameter; this definition agrees with the one used in Sect. 2.3.

Following [Va10, Sect. 6.5], we embed $X_{-\delta_1}$ as an open set into a compact manifold without boundary X, extend $P(z)$ to a second order differential operator on X depending holomorphically on z, and construct a complex absorbing operator $Q(z) \in \Psi^2(X)$, whose Schwartz kernel is supported inside the square of the nonphysical region $\{\mu < 0\}$. Then [Va10, Theorem 1.1] for $\mathrm{Im}\, z \geq -C_1$ and s large enough depending on C_1, $P(z) - iQ(z)$ is a holomorphic family of Fredholm operators $\mathcal{X}^s \to H^{s-1}(X)$, where

$$\mathcal{X}^s = \{u \in H^s(X) \mid (P(0) - iQ(0))u \in H^{s-1}(X)\},$$

and resonances are defined as the poles of its inverse. The semiclassical version is

$$\begin{aligned}
\mathcal{P}(\omega) &:= P_{\tilde{g}}(\omega) - h^2 Q(h^{-1}\omega) : \mathcal{X}^s_h \to H^{s-1}_h(X), \\
\|u\|_{\mathcal{X}^s_h} &= \|u\|_{H^s_h(X)} + \|(P(0) - iQ(0))u\|_{H^{s-1}_h(X)}.
\end{aligned} \tag{3.56}$$

We now claim that the operator $\mathcal{P}(\omega)$ satisfies all the assumptions of [Dy15, Sects. 4.1, 5.1]. Most of these assumptions have already been verified in Sect. 2.3, relying on the assumptions of Sect. 2.2 which in turn have been verified in Sects. 3.1, 3.2. Given the definition of the spaces $\mathcal{H}_1 := \mathcal{X}^s_h$, $\mathcal{H}_2 := H^{s-1}_h(X)$, and the Fredholm property discussed above, it remains to verify assumptions (10) and (11) of [Dy15, Sect. 4.1], namely the existence of an outgoing parametrix. This is done by modifying the proof of [Va10, Theorem 2.15] exactly as at the end of [Dy15, Sect. 4.4].

Theorem 3 now follows directly by [Dy15, Theorems 1 and 2]; the constant $c_{\widetilde{K}}$ is given by (2.16).

3.6. Stability. We finally discuss stability of Theorems 1–3, under perturbations of the metric. We assume that $(\tilde{X}_0, \tilde{g})$ is a Lorentzian manifold which is a small smooth metric perturbation of the exact Kerr(–de Sitter) (as described in Sect. 3.1 and with M, Λ, a in a small neighborhood of either (1.1) or (1.2)) and which is moreover stationary (that is, ∂_t is Killing). For perturbations of Kerr ($\Lambda = 0$) spacetime, we moreover assume that our perturbation coincides with the exact metric for large r (this assumption can be relaxed; in fact, all we need is for (3.46) and the analysis in Proposition 3.10 to apply, so we may take a small perturbation in the class of scattering metrics). We also assume that the perturbation continues smoothly across the event horizons in the coordinates (3.43). The initial value problem (3.45) is then well-posed, as $\{\mu = -\delta_1\}$ is still spacelike. The results of [Va10] still hold, as discussed in [Va10, Sect. 2.7].

It remains to verify that the assumptions of Sect. 2.2 still hold for the perturbed metric. Assumptions (1)–(3) are obviously true. Assumption (4) holds with the same function μ, at least for $\mu(\gamma(s)) \in (\delta, \delta_0)$, where δ_0 is fixed and $\delta > 0$ is small depending on the size of the perturbation; we take a small enough perturbation so that $\delta \ll \delta_1$, where $\delta_1 > 0$ is the constant used in Theorem 4 in Sect. 2.4 and in (2.13). Then the trapped set $\tilde{K}$ for the perturbed metric is close to the original trapped set, which implies assumption (5). Finally, the dynamical assumptions (6)–(8) still hold by the results of [HiPuSh] and the semicontinuity of $\nu_{\min}, \nu_{\max}, \mu_{\max}$, as discussed in [Dy15, Sect. 5.2].

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