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# Interval Bounds on the Solutions of Semi-Explicit Index-One DAEs. Part 1: Analysis

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**Abstract** This article presents two methods for computing interval bounds on the solutions of nonlinear, semi-explicit, index-one differential-algebraic equations (DAEs). Part 1 presents theoretical developments, while Part 2 discusses implementation and numerical examples. The primary theoretical contributions are (1) an interval inclusion test for existence and uniqueness of a solution, and (2) sufficient conditions, in terms of differential inequalities, for two functions to describe componentwise upper and lower bounds on this solution, point-wise in the independent variable. The first proposed method applies these results sequentially in a two-phase algorithm analogous to validated integration methods for ordinary differential equations (ODEs). The second method unifies these steps to characterize bounds as the solutions of an auxiliary system of DAEs. Efficient implementations of both are described using interval computations and demonstrated on numerical examples.

**Keywords** Differential-algebraic equations · Reachable set · Differential inequalities · Validated numerical integration · Interval Newton method

**Mathematics Subject Classification (2000)** 65L80, 65G20, 34A09, 34A40, 93B03

## 1 Introduction

This work explores the computation of interval bounds on the solutions of nonlinear, semi-explicit index-one differential-algebraic equations (DAEs) subject to a given set of initial conditions and model parameters. These parameters may represent uncertain constants in the model, as well as parametrized control inputs or disturbances.

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Computing enclosures of the reachable sets of dynamic systems is a classical problem with a wide variety of applications, including propagating uncertainty through dynamic models [7,33,26,27], solving state and parameter estimation problems [34, 13,25,11], safety verification and fault detection in dynamic systems [10,15], global optimization of dynamic systems [35,3,14,24], validated numerical integration [21], controller design and synthesis [23,16], and verification of continuous and hybrid systems [38,4,6]. However, nearly all available methods apply only to systems of explicit ordinary differential equations (ODEs). On the other hand, many dynamic systems encountered in applications are best modeled by DAEs [2,17].

For nonlinear ODEs, much work has been done on methods which compute a time-varying interval enclosure of the reachable set. These methods are primarily of two types. Taylor methods [21] use Taylor expansions and various interval techniques to approximate the ODE solutions and rigorously bound the approximation error. A key feature of these methods is that they produce *validated* enclosures, meaning that the enclosures are guaranteed even when computed on a finite precision machine. Some Taylor methods can be implemented very efficiently, but often produce extremely conservative enclosures. This conservatism can be greatly mitigated by using high-order Taylor expansions, or by using more sophisticated inclusion algebras, such as Taylor model arithmetic [1]. Unfortunately, these measures dramatically increase the computational cost, which in the latter case scales exponentially in the number of uncertain initial conditions and parameters. Methods of the second type use differential inequalities [40] and interval arithmetic to derive ODEs describing bounding trajectories, which are then integrated numerically [7,33,30,26,27,31]. These methods also suffer from potentially large overestimation [33], but are typically more efficient than Taylor methods, because state-of-the-art numerical integration software can be used. Moreover, it has recently been shown that overestimation in these methods can be dramatically reduced by exploiting simple solution invariants, without compromising efficiency [33,30,31]. While the enclosures produced by these methods are mathematically guaranteed, they are not validated. Therefore, they are inappropriate for investigating long-time behavior of unstable or oscillatory systems. Given the accuracy of modern numerical integration codes, however, these methods are effective for stable systems over modest integration times, especially when the reachable set is large compared to the expected numerical error owing to large parameter ranges.

In this article, we present two approaches for computing interval bounds on the solutions of semi-explicit index-one DAEs. The fact that such DAEs are equivalent to an explicit system of ODEs, the so-called underlying ODEs (see Remark 3.1), suggests that methods for ODEs could be applied directly. Unfortunately, this turns out to be unworkable, because ODE methods require explicit algebraic expressions for the right-hand side functions. For underlying ODEs, this necessitates an explicit expression for the inverse of the Jacobian of the algebraic equations, which would be very difficult to obtain in general (this requires the construction of the cofactor matrix, which has a factorial number of terms [36]). Moreover, the theoretical reduction to explicit ODEs is only valid locally around a given solution trajectory. This proves problematic for ODE methods, because the computed enclosures may come to contain regions of state space on which this reduction is invalid. For these reasons, it is necessary to develop a dedicated theory.

Part 1 of this article presents the major theoretical developments leading to the proposed bounding methods for DAEs, while Part 2 discusses the required computations. The first theoretical contribution is an interval inclusion test that verifies the existence and uniqueness of a DAE solution within a given interval. This test combines a well-known interval inclusion test for solutions of ODEs (used in standard Taylor methods) with an interval inclusion test for solutions of a system of nonlinear algebraic equations from the literature on interval Newton methods [22]. The second theoretical contribution is a pair of results using differential inequalities to derive bounding trajectories corresponding to the differential state variables; i.e., those state variables whose time derivatives are given explicitly by the DAE equations. Together, these contributions lead to the first bounding method proposed in Part 2. The final theoretical contribution is a result combining differential inequalities and interval Newton methods to compute bounds on both the differential and algebraic variables simultaneously. This result leads to the second method described in Part 2. Owing to the use of standard numerical integration codes in our implementation, the proposed methods produce enclosures that are mathematically guaranteed, but not validated. However, the existence and uniqueness test described above can be implemented in a validated manner, thus providing a key step towards validated bounding methods for DAEs.

A previous method for bounding the solutions of semi-explicit DAEs was proposed in [28]. This method is not based on differential inequalities, but it does involve an existence and uniqueness test based on an interval Newton method (the interval Krawczyk method). However, rather than combining the interval Krawczyk inclusion test with an interval inclusion tests for ODE solutions, as is done this work, the authors apply the interval Krawczyk inclusion test to the system of nonlinear integral equations obtained by replacing each instance of the differential variables in the original DAEs by the integrals of their time derivatives. The validity of this approach is unclear, since no justification is given for applying an inclusion test for real-valued solutions of algebraic equations to a system of functional equations defined on a function space.

The article [9] presents an algorithm for computing interval bounds on the solutions of implicit ODEs using Taylor models, which can be extended to treat DAEs as well. This method first computes a high-order polynomial approximation of the ODE solution, and then attempts to find a rigorous error bound by satisfying an inclusion test. Satisfying this inclusion test, which uses Taylor models rather than intervals, implies existence and uniqueness of an ODE solution near the polynomial approximation, i.e., within the validated error bound. This algorithm appears capable of computing very tight bounds, but requires the computation of a potentially very large number of Taylor coefficients. This method does not make use of differential inequalities. Furthermore, in addition to the use of Taylor models in place of intervals, the existence and uniqueness test proven in [9] is fundamentally different from the one presented here (and the one used in [28]) because it is derived through direct rearrangement of the implicit ODE equations into fixed-point form, rather than through application of the mean-value theorem, as is done in all interval Newton methods (see Remark 4.2).

Finally, in [5], a method for approximating the reachable sets of semi-explicit index-one DAEs is proposed, based on level set methods for ODEs [38]. Methods of this type are designed to provide an accurate approximation of the reachable set, rather than a rigorous enclosure of it. Accordingly, these methods are not appropriate for many applications of interest [34, 25, 10, 15, 35, 3].

The remainder of this article is organized as follows. Notation and relevant background material is presented in Section 2. Section 3 formally describes the DAEs considered in this work and presents basic results. In Section 4, an interval test for existence and uniqueness of solutions is described. Section 5 proves three results using differential inequalities to characterize bounding trajectories. Computational implementation of these results and case studies are presented in Part 2.

## 2 Preliminaries

### 2.1 Basic notation

Throughout this article, vector quantities are denoted in bold, while scalar quantities are written without emphasis. For any $\mathbf{v} \in \mathbb{R}^n$, the standard $p$-norms are denoted by $\|\mathbf{v}\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p}$, $1 \leq p < \infty$, and $\|\mathbf{v}\|_\infty = \max_i |v_i|$. Suppose that $\mathbf{w}, \mathbf{u} \in \mathbb{R}^n$ as well. The order relations $\mathbf{v} \leq \mathbf{w}$ and $\mathbf{v} < \mathbf{w}$ denote that these relations hold componentwise. Similarly, $\min(\mathbf{v}, \mathbf{w})$ and $\max(\mathbf{v}, \mathbf{w})$ denote the vectors with components $\min(v_i, w_i)$ and $\max(v_i, w_i)$, respectively, and $\text{mid}(\mathbf{v}, \mathbf{w}, \mathbf{u})$ denotes the vector where each component is the middle value of $v_i$, $w_i$ and $u_i$. For $V \subset \mathbb{R}^n$, the interior and boundary of $V$ are denoted by $\text{int}(V)$ and $\partial V$, respectively.

### 2.2 Intervals and natural interval extensions

If a set in $\mathbb{R}^n$ may be expressed as the Cartesian product of $n$ intervals in $\mathbb{R}$, it is referred to as an $n$-dimensional interval or simply an interval. For $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, the notation $[\mathbf{v}, \mathbf{w}]$ denotes the $n$-dimensional interval $[v_1, w_1] \times \ldots \times [v_n, w_n]$. The set of all nonempty compact interval subsets of $\mathbb{R}^n$ is denoted $\mathbb{IR}^n$. The set of all nonempty compact $n \times m$ interval matrices is denoted $\mathbb{IR}^{n\times m}$ and defined analogously; $A \in \mathbb{IR}^{n\times m}$ has elements $A_{ij} \in \mathbb{IR}$, for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, and, for any $\mathbf{A} \in \mathbb{R}^{n\times m}$ with elements $a_{ij}$, $\mathbf{A} \in A$ if $a_{ij} \in A_{ij}$ for all indices $i$ and $j$. For any $D \subset \mathbb{R}^n$, let $\mathbb{I}D$ denote the set $\{Z \in \mathbb{IR}^n : Z \subset D\}$. This notation is also used for $D \subset \mathbb{R}^{n\times m}$.

For $Z \equiv [\mathbf{z}^L, \mathbf{z}^U] \in \mathbb{IR}^n$, let $m(Z)$ denote the *midpoint* of $Z$, $m(Z) \equiv \mathbf{z}^L + (\mathbf{z}^U - \mathbf{z}^L)/2$. For $A \in \mathbb{IR}^{n\times m}$, $m(A)$ is a real-valued matrix defined analogously.

Let $D \subset \mathbb{R}^n$ and $\mathbf{f} : D \to \mathbb{R}^m$. An *interval extension* of $\mathbf{f}$ is a mapping $[\mathbf{f}] : \mathbb{I}D \to \mathbb{IR}^m$ such that, for any degenerate interval (i.e. singleton) $[\mathbf{x}, \mathbf{x}] \in \mathbb{I}D$, $[\mathbf{f}]([\mathbf{x}, \mathbf{x}]) = [\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x})]$. For any $X \in \mathbb{I}D$, the notation $[\mathbf{f}]^L(X)$ and $[\mathbf{f}]^U(X)$ is used to denote the lower and upper bounds of $[\mathbf{f}](X)$, respectively. An interval extension is *inclusion monotonic* if, for any $X, X' \in \mathbb{I}D$, $X' \subset X$ implies that $[\mathbf{f}](X') \subset [\mathbf{f}](X)$. It is easily verified that inclusion monotonic interval extensions satisfy $\mathbf{f}(X) \subset [\mathbf{f}](X)$, $\forall X \in \mathbb{I}D$, where

$\mathbf{f}(X)$ denotes the image of $X$ under $\mathbf{f}$. This image bounding property is fundamental to the use of interval extensions in this article.

Inclusion monotonic interval extensions are known for binary addition, subtraction, multiplication and division, and many common univariate functions including scalar multiplication, integer and fractional powers, logarithm, exponential, trigonometrics, etc. Throughout this work, the interval counterparts of the standard arithmetic operations $\{+,-,\times,/\}$ are implied; i.e., for $A,B \in \mathbb{IR}$, $AB$ denotes interval multiplication. Arithmetic operations between real numbers and intervals are carried out using interval arithmetic with real numbers identified with the corresponding degenerate interval in $\mathbb{IR}$. If $\mathbf{f}$ is defined by a computational graph, that is, by the recursive application of additions, subtractions, multiplications, divisions and compositions with common univariate functions, then it is referred to as a *factorable* function [18, 32], and each of these basic operations is called a *factor* of $\mathbf{f}$. For any factorable function $\mathbf{f}$, one can compute a particular interval extension called the *natural interval extension* by recursively applying the known interval extensions of the factors of $\mathbf{f}$. That is, each operation in the definition of $\mathbf{f}$ is replaced by its interval counterpart. Natural interval extensions are inclusion monotonic and thus satisfy the image bounding property of $[\mathbf{f}]$ discussed above. The reader is referred to [19] and [22] for further details on interval analysis.

## 2.3 Absolutely continuous and continuously differentiable functions

Let $I = [t_0, t_f]$. Recall that an absolutely continuous function $\phi : I \to \mathbb{R}$ is differentiable at almost every (a.e.) $t \in I$. The results in §5 involve some standard facts about absolutely continuous functions which are not reviewed here but can be found in [39]. Because it is central to many of the results in this work, we recall one standard monotonicity result below.

**Theorem 2.1** *If $\phi : I \to \mathbb{R}$ is absolutely continuous and $\phi'(t) \le 0$ for a.e. $t \in I$, then $\phi$ is non-increasing on $I$.*

*Proof* See Theorem 3.1 in [37]. □

For any open $D \subset \mathbb{R}^n$, $C^k(D,\mathbb{R}^m)$ denotes the set of $k$-times continuously differentiable mappings from $D$ into $\mathbb{R}^m$. For a general $D \subset \mathbb{R}^n$, $\boldsymbol{\phi} \in C^k(D,\mathbb{R}^m)$ if there exists an open set $\widetilde{D} \supset D$ and a function $\widetilde{\boldsymbol{\phi}} \in C^k(\widetilde{D},\mathbb{R}^m)$ such that $\widetilde{\boldsymbol{\phi}}|_D = \boldsymbol{\phi}$.

**Lemma 2.1** *Let $D \subset \mathbb{R}^n$ and $\boldsymbol{\phi} \in C^1(D,\mathbb{R}^m)$. Then, for any compact $K \subset D$, $\exists L_K \in \mathbb{R}_+$ such that $\|\boldsymbol{\phi}(\mathbf{z}) - \boldsymbol{\phi}(\hat{\mathbf{z}})\|_1 \le L_K \|\mathbf{z} - \hat{\mathbf{z}}\|_1$, $\forall (\mathbf{z},\hat{\mathbf{z}}) \in K \times K$.*

Let $D_s \subset \mathbb{R}^{n_s}$, $D_r \subset \mathbb{R}^{n_r}$, and $\boldsymbol{\ell} \in C^k(D_s \times D_r, \mathbb{R}^{n_r})$ with $k \ge 1$. For any $(\hat{\mathbf{s}},\hat{\mathbf{r}}) \in D_s \times D_r$, the Jacobian matrix of the mapping $\boldsymbol{\ell}(\hat{\mathbf{s}},\cdot)$ at $\hat{\mathbf{r}}$ is denoted by $\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}(\hat{\mathbf{s}},\hat{\mathbf{r}})$. The implicit function theorem is required below and stated here for reference.

**Theorem 2.2 (Implicit Function Theorem)** *Let $D_s \subset \mathbb{R}^{n_s}$ and $D_r \subset \mathbb{R}^{n_r}$ be open, and let $\boldsymbol{\ell} \in C^k(D_s \times D_r, \mathbb{R}^{n_r})$. Suppose that $(\mathbf{s}_0,\mathbf{r}_0) \in D_s \times D_r$ satisfies $\boldsymbol{\ell}(\mathbf{s}_0,\mathbf{r}_0) = \mathbf{0}$ and $\det \frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}(\mathbf{s}_0,\mathbf{r}_0) \neq 0$. Then there exists an open ball around $\mathbf{s}_0$, $V_0 \subset D_s$, an open ball around $\mathbf{r}_0$, $Q_0 \subset D_r$, and $\mathbf{h} \in C^k(V_0,Q_0)$ satisfying*

1. $\mathbf{h}(\mathbf{s}_0) = \mathbf{r}_0$,
2. *For any* $\mathbf{s} \in V_0$, *the vector* $\mathbf{r} = \mathbf{h}(\mathbf{s})$ *is the unique element of* $Q_0$ *satisfying* $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$,
3. $\det \frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}(\mathbf{s},\mathbf{r}) \neq 0$, $\forall (\mathbf{s},\mathbf{r}) \in V_0 \times Q_0$.

*Proof* See Theorem 9.2 in [20] and Theorem 9.28 in [29]. □

## 3 Problem Statement

In this section, the system of DAEs under consideration is defined and the problem of computing interval bounds is stated formally. Because we are interested in computing interval enclosures of the possible solutions of this system, it is necessary to have clear statements of the existence and uniqueness properties of these solutions. The basic local existence result is well-known [12] and is not proven here. On the other hand, certain arguments in this work require very particular properties related to uniqueness, so the relevant analysis is provided. In order to move quickly to the primary problem of computing interval bounds, detailed proofs are relegated to Appendix A.

### 3.1 Semi-explicit DAEs

Let $D_t \subset \mathbb{R}$, $D_p \subset \mathbb{R}^{n_p}$, $D_x \subset \mathbb{R}^{n_x}$ and $D_y \subset \mathbb{R}^{n_y}$ be open sets, and let $\mathbf{f} : D_t \times D_p \times D_x \times D_y \to \mathbb{R}^{n_x}$, $\mathbf{g} : D_t \times D_p \times D_x \times D_y \to \mathbb{R}^{n_y}$ and $\mathbf{x}_0 : D_p \to D_x$ be $C^1$ functions. Given some $t_0 \in D_t$, consider the initial value problem in semi-explicit differential-algebraic equations

$$\left.\begin{aligned} \dot{\mathbf{x}}(t,\mathbf{p}) &= \mathbf{f}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) \\ \mathbf{0} &= \mathbf{g}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) \end{aligned}\right\}, \tag{1a}$$

$$\mathbf{x}(t_0,\mathbf{p}) = \mathbf{x}_0(\mathbf{p}), \tag{1b}$$

where $t$ is the independent variable, $\mathbf{p}$ is a vector of problem parameters, $\dot{\mathbf{x}}(t,\mathbf{p})$ denotes the derivative of $\mathbf{x}(\cdot,\mathbf{p})$ at $t$, and $\mathbf{x}_0$ specifies the parametric initial conditions. A solution of (1) is defined below.

**Definition 3.1** Define the sets

$$\begin{aligned} \mathcal{G} &\equiv \{(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \in D_t \times D_p \times D_x \times D_y : \mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) = \mathbf{0}\}, \\ \mathcal{G}_0 &\equiv \{(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \in \mathcal{G} : \mathbf{x}_0(\mathbf{p}) = \mathbf{z}_x\}, \\ \mathcal{G}_R &\equiv \left\{(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \in D_t \times D_p \times D_x \times D_y : \det \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \neq 0\right\}. \end{aligned}$$

**Definition 3.2** Let $I \subset D_t$ be connected, and let $P \subset D_p$. A function $(\mathbf{x},\mathbf{y}) \in C^1(I \times P, D_x) \times C^1(I \times P, D_y)$ is called a *solution of* (1a) *on* $I \times P$ if (1a) holds for all $(t,\mathbf{p}) \in I \times P$. If in addition $(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) \in \mathcal{G}_R$, $\forall (t,\mathbf{p}) \in I \times P$, then $(\mathbf{x},\mathbf{y})$ is called *regular*. When $t_0 \in I$ is specified and $\mathbf{x}$ also satisfies (1b), $(\mathbf{x},\mathbf{y})$ it is called a (regular) solution of (1) on $I \times P$.

*Remark 3.1* In this work, the assumption that (1) has differential index 1 is not stated directly, but rather implied by restricting our results to *regular* solutions, as defined above. Indeed, these notions are identical in this case, since, for any regular solution of (1) on $I \times P$, a single differentiation of the algebraic equations $\mathbf{g}$ gives the underlying ODEs

$$\dot{\mathbf{x}}(t,\mathbf{p}) = \mathbf{f}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})), \tag{2}$$

$$\dot{\mathbf{y}}(t,\mathbf{p}) = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right)^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) + \frac{\partial \mathbf{g}}{\partial t}\right), \tag{3}$$

for all $(t,\mathbf{p}) \in I \times P$, where all partial derivatives of $\mathbf{g}$ are evaluated at $(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p}))$.

## 3.2 Existence and uniqueness

Existence of a solution of (1) can of course only be guaranteed locally. The main result is stated in terms of local solutions, defined as follows.

**Definition 3.3** For any $(t_0,\hat{\mathbf{p}},\hat{\mathbf{x}}_0,\hat{\mathbf{y}}_0) \in \mathcal{G}_0$, a mapping $(\mathbf{x},\mathbf{y}) \in C^1(I' \times P', D_x) \times C^1(I' \times P', D_y)$ is called a *solution* of (1) *local to* $(t_0,\hat{\mathbf{p}},\hat{\mathbf{x}}_0,\hat{\mathbf{y}}_0)$ if $I'$ and $P'$ are open balls containing $t_0$ and $\hat{\mathbf{p}}$, respectively, $\mathbf{x}$ and $\mathbf{y}$ satisfy (1) on $I' \times P'$, and $\mathbf{y}(t_0,\hat{\mathbf{p}}) = \hat{\mathbf{y}}_0$. If in addition $\mathbf{x}$ and $\mathbf{y}$ satisfy $(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) \in \mathcal{G}_R$, $\forall (t,\mathbf{p}) \in I' \times P'$, then $(\mathbf{x},\mathbf{y})$ is called *regular*.

**Theorem 3.1** *Let* $(t_0,\hat{\mathbf{p}},\hat{\mathbf{x}}_0,\hat{\mathbf{y}}_0) \in \mathcal{G}_0 \cap \mathcal{G}_R$. *There exists a regular solution of* (1) *local to* $(t_0,\hat{\mathbf{p}},\hat{\mathbf{x}}_0,\hat{\mathbf{y}}_0)$.

*Proof* See Theorems 4.13 and 4.18 in [12]. □

For any $(\mathbf{x},\mathbf{y}) \in C^1(I' \times P', D_x) \times C^1(I' \times P', D_y)$ satisfying (1), the initial value of $\mathbf{y}$ must obviously satisfy $\mathbf{g}(t_0,\mathbf{p},\mathbf{x}(t_0,\mathbf{p}),\mathbf{y}(t_0,\mathbf{p})) = \mathbf{0}$ for each $\mathbf{p} \in P'$. Therefore, these values cannot be specified arbitrarily. On the other hand, this equation may have multiple solutions in $D_y$, so that in general more information (in addition to (1)) is required to specify a solution uniquely. As will be shown below, uniqueness of regular local solutions follows from the additional condition $\mathbf{y}(t_0,\hat{\mathbf{p}}) = \hat{\mathbf{y}}_0$ in Definition 3.3. The following example demonstrates that uniqueness is not guaranteed in the absence of this condition.

*Example 3.1* Let $I \equiv [0,\delta] \subset D_t = \mathbb{R}$, $D_p = \emptyset$, $D_x = D_y = \mathbb{R}$, and define $g(t,z_x,z_y) = z_y^2 - z_x$. With fixed initial condition $x_0 = 1$ at $t_0 = 0$, there are two possible values for $y(t_0)$ satisfying $g(t_0,x(t_0),y(t_0)) = 0$; $y(t_0) = 1$ and $y(t_0) = -1$. Letting $f(t,z_x,z_y) = 1$, clearly $x(t) = 1+t$ satisfies $\dot{x}(t) = 1 = f(t,x(t),y(t))$ for any $y : I \to \mathbb{R}$. However, both $y(t) = \sqrt{1+t}$ and $y(t) = -\sqrt{1+t}$ result in $g(t,x(t),y(t)) = (y(t))^2 - x(t) = 0$. In particular, $y(t) = \sqrt{1+t}$ is a solution of (1) local to $(t_0,\hat{x}_0,\hat{y}_0) = (0,1,1)$, while $y(t) = -\sqrt{1+t}$ is a solution of (1) local to $(t_0,\hat{x}_0,\hat{y}_0) = (0,1,-1)$.

A detailed analysis of the uniqueness properties of solutions of (1) is given in Appendix A. The most relevant conclusion is the following.

**Corollary 3.1** *Let* $(\mathbf{x},\mathbf{y}) \in C^1(I\times P, D_x)\times C^1(I\times P, D_y)$ *and* $(\mathbf{x}^*,\mathbf{y}^*) \in C^1(\widetilde{I}\times\widetilde{P}, D_x)\times C^1(\widetilde{I}\times\widetilde{P}, D_y)$ *be solutions of* (1) *on* $I\times P$ *and* $\widetilde{I}\times\widetilde{P}$*, respectively, with some* $t_0 \in I\cap\widetilde{I}$*, and suppose that* $(\mathbf{x},\mathbf{y})$ *is regular. If* $\hat{P} \subset P\cap\widetilde{P}$ *is connected and* $\exists\hat{\mathbf{p}} \in \hat{P}$ *such that* $\mathbf{y}(t_0,\hat{\mathbf{p}}) = \mathbf{y}^*(t_0,\hat{\mathbf{p}})$*, then* $\mathbf{x}(t,\mathbf{p}) = \mathbf{x}^*(t,\mathbf{p})$ *and* $\mathbf{y}(t,\mathbf{p}) = \mathbf{y}^*(t,\mathbf{p})$*,* $\forall(t,\mathbf{p}) \in (I\cap\widetilde{I})\times\hat{P}$*.*

*Proof* See Appendix A. □

### 3.3 Interval bounds

The primary aim of this article is to compute interval bounds for the solutions of (1). Let $I = [t_0,t_f] \subset D_t$ and $P \subset D_p$ be intervals, and suppose that $(\mathbf{x},\mathbf{y}) \in C^1(I\times P, D_x)\times C^1(I\times P, D_y)$ is a regular solution of (1) on $I\times P$. Then, our objective is to compute functions $\mathbf{x}^L,\mathbf{x}^U : I \to \mathbb{R}^{n_x}$ and $\mathbf{y}^L,\mathbf{y}^U : I \to \mathbb{R}^{n_y}$ such that

$$\mathbf{x}^L(t) \le \mathbf{x}(t,\mathbf{p}) \le \mathbf{x}^U(t) \quad \text{and} \quad \mathbf{y}^L(t) \le \mathbf{y}(t,\mathbf{p}) \le \mathbf{y}^U(t), \quad \forall(t,\mathbf{p}) \in I\times P.$$

Recall that (1) may have multiple regular solutions on $I\times P$ corresponding to different solution branches of the algebraic equations (see Example 3.1). In the methods of this article, a single solution is specified for bounding through an interval, either provided as input or computed, which, for each $\mathbf{p} \in P$, contains exactly one initial condition for $\mathbf{y}$ which is consistent with $\mathbf{x}_0(\mathbf{p})$ (see Theorem 4.2). This interval specifies which solution branch defines $\mathbf{y}$ at $t_0$, and hence the solution is uniquely determined on $I\times P$ (Corollary 3.1). In principle, Theorem 5.1 provides bounds valid for all regular solutions of (1), but we do not pursue a method for computing such bounds.

## 4 An Interval Inclusion Test for DAE Solutions

This section presents an interval inclusion test which can computationally guarantee the existence and uniqueness of a solution of (1) over intervals $I'$ and $P'$ satisfying the test. When successful, the test provides intervals which are guaranteed to enclose the solutions $\mathbf{x}$ and $\mathbf{y}$ on $I'\times P'$. This test is very similar to the Phase 1 step of standard Taylor methods for ODEs [21]. The complicating factor here is of course the presence of the algebraic variables $\mathbf{y}$ and the fact that they are defined implicitly. To overcome this obstacle, a well-known interval inclusion test for existence and uniqueness of solutions of systems of nonlinear algebraic equations is used. This inclusion test is based on the interval Hansen-Sengupta method [22]. This method is described below, and its application to DAEs is discussed in §4.2.

### 4.1 The Interval Hansen-Sengupta Method

Let $D_s \subset \mathbb{R}^{n_s}$ and $D_r \subset \mathbb{R}^{n_r}$ be open, and let $\boldsymbol{\ell} \in C^k(D_s\times D_r, \mathbb{R}^{n_r})$. Given intervals $S \subset D_s$ and $R \subset D_r$, we are concerned with (i) determining if there exist points $\mathbf{r} \in R$ such that $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$ for some $\mathbf{s} \in S$, and (ii) computing a refined interval $R' \subset R$ which contains all such $\mathbf{r}$. Conceptually, this is done by using the mean value theorem to characterize the zeros of $\boldsymbol{\ell}$. For any $(\mathbf{s},\mathbf{r}) \in S \times R$ such that $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$, any $\widetilde{\mathbf{r}} \in R$, $\widetilde{\mathbf{r}} \neq \mathbf{r}$, and any index $i$, the mean value theorem states that $\exists \boldsymbol{\xi}^{[i]} \in R$ such that $\boldsymbol{\xi}^{[i]} = \widetilde{\mathbf{r}} + \lambda(\mathbf{r} - \widetilde{\mathbf{r}})$ for some $\lambda \in (0,1)$, and

$$\frac{\partial \ell_i}{\partial \mathbf{r}}(\mathbf{s},\boldsymbol{\xi}^{[i]})(\mathbf{r}-\widetilde{\mathbf{r}}) = -\ell_i(\mathbf{s},\widetilde{\mathbf{r}}). \tag{4}$$

Noting that $\boldsymbol{\xi}^{[i]} \in R$ because $\boldsymbol{\xi}^{[i]} = \widetilde{\mathbf{r}} + \lambda(\mathbf{r} - \widetilde{\mathbf{r}})$ and $\mathbf{r}, \widetilde{\mathbf{r}} \in R$, consider the interval linear equations

$$\left[\frac{\partial \ell_i}{\partial \mathbf{r}}\right](S,R)(\mathbf{r}-\widetilde{\mathbf{r}}) = -[\ell_i](S,\widetilde{\mathbf{r}}), \tag{5}$$

which can be written in matrix form, preconditioned by any $\mathbf{C} \in \mathbb{R}^{n_r \times n_r}$, as

$$\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R)(\mathbf{r}-\widetilde{\mathbf{r}}) = -\mathbf{C}[\boldsymbol{\ell}](S,\widetilde{\mathbf{r}}). \tag{6}$$

The solution set of (6) is the set of all $\boldsymbol{\rho} \in \mathbb{R}^{n_r}$ such that $\mathbf{A}\boldsymbol{\rho} = \mathbf{b}$ for some $\mathbf{A} \in \mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R)$ and $\mathbf{b} \in -\mathbf{C}[\boldsymbol{\ell}](S,\widetilde{\mathbf{r}})$. Clearly, any $\mathbf{r} \in R$ satisfying $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$ for some $\mathbf{s} \in S$ must correspond to an element $(\mathbf{r}-\widetilde{\mathbf{r}}) = \boldsymbol{\rho}$ of this solution set. Thus, we are interested in computing an interval enclosure of the solution set of (6).

For $Q \subset \mathbb{R}$, let hull$(Q)$ denote the *interval hull* of $Q$; i.e, the smallest interval containing $Q$. To state the Hansen-Sengupta method formally, the following definition is useful.

**Definition 4.1** For all $A,B,Z \in \mathbb{IR}$, let

$$\Gamma(A,B,Z) \equiv \mathrm{hull}\left(\{z \in Z : az = b \text{ for some } (a,b) \in A \times B\}\right).$$

The following lemma provides a way to evaluate $\Gamma$ computationally.

**Lemma 4.1** *For all $A,B,Z \in \mathbb{IR}$,*

$$\Gamma(A,B,Z) = \begin{cases} B/A \cap Z & \text{if } 0 \notin A \\ \mathrm{hull}\left(Z \backslash \mathrm{int}([b^L/a^L, b^L/a^U])\right) & \text{if } 0 \in A \text{ and } b^L > 0 \\ \mathrm{hull}\left(Z \backslash \mathrm{int}([b^U/a^U, b^U/a^L])\right) & \text{if } 0 \in A \text{ and } b^U < 0 \\ Z & \text{if } 0 \in A \text{ and } 0 \in B \end{cases}, \tag{7}$$

*where $B/A$ denotes interval division,*

$$B/A = [\min(b^L/a^L, b^U/a^L, b^L/a^U, b^U/a^U), \max(b^L/a^L, b^U/a^L, b^L/a^U, b^U/a^U)].$$

*Proof* See Proposition 4.3.1 in [22]. □

For any $A,B,Z \in \mathbb{IR}$, either $\Gamma(A,B,Z) \in \mathbb{IR}$ or $\Gamma(A,B,Z) = \emptyset$. For convenience, the definition of $\Gamma$ is extended so that $\Gamma(A,B,Z) = \emptyset$ when any of $A$, $B$, or $Z$ is empty. Furthermore, we adopt the convention that any arithmetic operation between an element of $\mathbb{IR}$ and $\emptyset$ returns $\emptyset$, and any Cartesian product involving $\emptyset$ is equivalent to $\emptyset$. The following definition generalizes $\Gamma$ for application to $n$ dimensional linear systems.

**Definition 4.2** For $A \in \mathbb{IR}^{n\times n}$, $B, Z \in \mathbb{IR}^n$, let

$$W_i \equiv \Gamma\left(A_{ii}, B_i - \sum_{j<i} A_{ij}W_j - \sum_{j>i} A_{ij}Z_j, Z_i\right),$$

for all $i = 1,\ldots,n$. Define $\Gamma(A,B,Z) \equiv W_1 \times \ldots \times W_n$.

Applying $\Gamma$ to (6) gives the following variant of the well-known result Theorem 5.1.8 in [22].

**Theorem 4.1** *Let $S \in \mathbb{I}D_s$, $R \in \mathbb{I}D_r$, $\widetilde{\mathbf{r}} \in R$, $\mathbf{C} \in \mathbb{R}^{n_r\times n_r}$, and let*

$$\mathcal{H}(S,R,\widetilde{\mathbf{r}},\mathbf{C}) \equiv \widetilde{\mathbf{r}} + \Gamma\left(\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R), -\mathbf{C}\left[\boldsymbol{\ell}\right](S,\widetilde{\mathbf{r}}), (R-\widetilde{\mathbf{r}})\right).$$

*With $R' \equiv \mathcal{H}(S,R,\widetilde{\mathbf{r}},\mathbf{C})$, the following conclusions hold:*

1. *If $(\mathbf{s},\mathbf{r}) \in S\times R$ satisfies $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$, then $\mathbf{r} \in R'$.*
2. *If $R' = \emptyset$, then $\nexists(\mathbf{s},\mathbf{r}) \in S\times R$ such that $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$.*
3. *If $\widetilde{\mathbf{r}} \in \mathrm{int}(R)$ and $\emptyset \neq R' \subset \mathrm{int}(R)$, then $\exists \mathbf{H} \in C^k(S,R')$ such that, for every $\mathbf{s} \in S$, $\mathbf{r} = \mathbf{H}(\mathbf{s})$ is the unique element of $R$ satisfying $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$. Moreover, the interval matrix $\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R)$ does not contain a singular matrix and does not contain zero in any of its diagonal elements.*

*Proof* Suppose first that $S$ is a singleton, $S \equiv [\mathbf{s},\mathbf{s}]$, for some $\mathbf{s} \in D_s$. Then, noting that $[\boldsymbol{\ell}]([\mathbf{s},\mathbf{s}],\widetilde{\mathbf{r}}) = \boldsymbol{\ell}(\mathbf{s},\widetilde{\mathbf{r}})$ by the definition of an interval extension, applying Corollary 5.1.5 and Theorem 5.1.8 in [22] to the function $\boldsymbol{\ell}(\mathbf{s},\cdot)$ proves the theorem (the properties of $\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R)$ in Conclusion 3 result from Theorem 4.4.5 (ii) in [22]). Next, suppose that $S$ is not a singleton. Fix any $\mathbf{s} \in S$ and suppose that $\mathbf{r} \in R$ satisfies $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$. Since the theorem holds for $[\mathbf{s},\mathbf{s}]$ as shown above, we must have $\mathbf{r} \in \mathcal{H}([\mathbf{s},\mathbf{s}],R,\widetilde{\mathbf{r}},\mathbf{C})$. But, by the inclusion monotonicity of natural interval extensions, $\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right]([\mathbf{s},\mathbf{s}],R) \subset \mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R)$ and $-\mathbf{C}[\boldsymbol{\ell}]([\mathbf{s},\mathbf{s}],\widetilde{\mathbf{r}}) \subset -\mathbf{C}[\boldsymbol{\ell}](S,\widetilde{\mathbf{r}})$. Then Proposition 4.3.4 in [22] gives

$$\mathcal{H}([\mathbf{s},\mathbf{s}],R,\widetilde{\mathbf{r}},\mathbf{C}) = \widetilde{\mathbf{r}} + \Gamma\left(\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right]([\mathbf{s},\mathbf{s}],R), -\mathbf{C}\left[\boldsymbol{\ell}\right]([\mathbf{s},\mathbf{s}],\widetilde{\mathbf{r}}), (R-\widetilde{\mathbf{r}})\right), \tag{8}$$

$$\subset \widetilde{\mathbf{r}} + \Gamma\left(\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R), -\mathbf{C}\left[\boldsymbol{\ell}\right](S,\widetilde{\mathbf{r}}), (R-\widetilde{\mathbf{r}})\right), \tag{9}$$

$$= \mathcal{H}(S,R,\widetilde{\mathbf{r}},\mathbf{C}). \tag{10}$$

Therefore, $\mathbf{r} \in R'$, which proves 1, and 2 is an immediate consequence.

To prove Conclusion 3, suppose that $\widetilde{\mathbf{r}} \in \mathrm{int}(R)$, and $\emptyset \neq R' \subset \mathrm{int}(R)$. Theorem 4.4.5 (ii) in [22] again establishes the properties of $\mathbf{C}\left[\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right](S,R)$. By Theorem 5.5.1 in [22] (see also Corollary 5.1.5), there exists a continuous function $\mathbf{H} : S \to R$ such that, for every $\mathbf{s} \in S$, $\mathbf{r} = \mathbf{H}(\mathbf{s})$ is the unique element of $R$ satisfying $\boldsymbol{\ell}(\mathbf{s},\mathbf{r}) = \mathbf{0}$. By Conclusion 1 of the present theorem, $\mathbf{H} : S \to R'$. It only remains to show that $\mathbf{H} \in C^k(S,R')$.

Choosing any $\hat{\mathbf{s}} \in S$, Theorem 2.2 can be applied at the point $(\hat{\mathbf{s}},\mathbf{H}(\hat{\mathbf{s}}))$ to conclude that there exists an open ball around $\hat{\mathbf{s}}$, $V_{\hat{\mathbf{s}}} \subset D_s$, an open ball around $\mathbf{H}(\hat{\mathbf{s}})$, $Q_{\hat{\mathbf{s}}}$, and $\mathbf{h}_{\hat{\mathbf{s}}} \in$

$C^k(V_{\hat{\mathbf{s}}}, Q_{\hat{\mathbf{s}}})$ such that $\mathbf{h}_{\hat{\mathbf{s}}}(\hat{\mathbf{s}}) = \mathbf{H}(\hat{\mathbf{s}})$ and, for every $\mathbf{s} \in V_{\hat{\mathbf{s}}}$, $\mathbf{r} = \mathbf{h}_{\hat{\mathbf{s}}}(\mathbf{s})$ is the unique element of $Q_{\hat{\mathbf{s}}}$ satisfying $\boldsymbol{\ell}(\mathbf{s}, \mathbf{r}) = \mathbf{0}$. By continuity of $\mathbf{H}$, it is possible to choose an open ball $U_{\hat{\mathbf{s}}}$ around $\hat{\mathbf{s}}$ small enough that $\mathbf{H}$ maps $U_{\hat{\mathbf{s}}} \cap S$ into $Q_{\hat{\mathbf{s}}}$. Then, by the uniqueness property of $\mathbf{h}_{\hat{\mathbf{s}}}$ in $Q_{\hat{\mathbf{s}}}$, $\mathbf{H} = \mathbf{h}_{\hat{\mathbf{s}}}$ on $U_{\hat{\mathbf{s}}} \cap S$. The fact that $\mathbf{H} \in C^k(S, R')$ now follows from Lemma 23.1 in [20].

*Remark 4.1* Theorem 4.1 does not require that the preconditioner $\mathbf{C}$ is nonsingular. However, singular preconditioners are not useful in the sense that the inclusion test $\emptyset \neq R' \subset \text{int}(R)$ in Conclusion 3 will never be satisfied if $\mathbf{C}$ is singular. Nonetheless, we will often choose $\mathbf{C}$ as the value of $\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}$ at a point in $S \times R$, and it is convenient that we do not need to check invertibility before applying Theorem 4.1. If the inclusion test in Conclusion 3 is satisfied, invertibility follows.

*Remark 4.2* The interval inclusion test given in part 3 of Theorem 4.1 is based on a characterization of the zeros of $\boldsymbol{\ell}$ derived from the mean-value theorem. Alternatively, an inclusion test can be derived from Brouwer's fixed point theorem without using the mean value theorem. This requires deriving a fixed point equation, $\mathbf{r} = \boldsymbol{\phi}(\mathbf{s}, \mathbf{r})$, with the same solutions as the original equations. For example, assuming that $\left(\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right)$ is nonsingular on $S \times R$, let

$$\boldsymbol{\phi}(\mathbf{s}, \mathbf{r}) \equiv \mathbf{r} - \left(\frac{\partial \boldsymbol{\ell}}{\partial \mathbf{r}}\right)^{-1} (\mathbf{s}, \mathbf{r}) \boldsymbol{\ell}(\mathbf{s}, \mathbf{r}). \tag{11}$$

Brouwer's fixed point theorem can be used to show that the inclusion $[\boldsymbol{\phi}](S, R) \subset R$ guarantees the existence of $\mathbf{H} : S \to R$ satisfying $\mathbf{H}(\mathbf{s}) = \boldsymbol{\phi}(\mathbf{s}, \mathbf{H}(\mathbf{s}))$, and hence $\boldsymbol{\ell}(\mathbf{s}, \mathbf{H}(\mathbf{s})) = \mathbf{0}$, for all $\mathbf{s} \in S$. However, it is easily demonstrated that this inclusion will almost never be satisfied when the natural interval extension of $\boldsymbol{\phi}$ is used. Denoting the natural interval extension of the second term on the right-hand side of (11) over $S \times R$ by $M$, the natural interval extension of $\boldsymbol{\phi}$ is computed as $[\boldsymbol{\phi}](S, R) := R - M$. If $\exists (\mathbf{s}, \mathbf{r}) \in S \times R$ satisfying $\boldsymbol{\ell}(\mathbf{s}, \mathbf{r}) = \mathbf{0}$, then we must have $\mathbf{0} \in M$, and hence $[\boldsymbol{\phi}](S, R) \supset R$. Therefore, the desired inclusion will only hold when $[\boldsymbol{\phi}](S, R) = R$. This requires $M = [\mathbf{0}, \mathbf{0}]$, which can only occur in trivial cases.

## 4.2 An interval existence and uniqueness test for DAEs

Applying Theorem 4.1 to the algebraic equations in (1) gives the following corollary.

**Corollary 4.1** *Let* $(I, P, Z_x, Z_y) \in \mathbb{I}D_t \times \mathbb{I}D_p \times \mathbb{I}D_x \times \mathbb{I}D_y$, $\tilde{\mathbf{z}}_y \in Z_y$, $\mathbf{C} \in \mathbb{R}^{n_y \times n_y}$ *and define*

$$\mathcal{H}(I, P, Z_x, Z_y, \tilde{\mathbf{z}}_y, \mathbf{C}) \equiv \tilde{\mathbf{z}}_y + \Gamma\left(\mathbf{C}\left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right](I, P, Z_x, Z_y), -\mathbf{C}\,[\mathbf{g}]\,(I, P, Z_x, \tilde{\mathbf{z}}_y), (Z_y - \tilde{\mathbf{z}}_y)\right).$$

*With* $Z'_y \equiv \mathcal{H}(I, P, Z_x, Z_y, \tilde{\mathbf{z}}_y, \mathbf{C})$, *the following conclusions hold:*

1. *If* $(t, \mathbf{p}, \mathbf{z}_x, \mathbf{z}_y) \in I \times P \times Z_x \times Z_y$ *satisfies* $\mathbf{g}(t, \mathbf{p}, \mathbf{z}_x, \mathbf{z}_y) = \mathbf{0}$, *then* $\mathbf{z}_y \in Z'_y$.
2. *If* $Z'_y = \emptyset$, *then* $\nexists (t, \mathbf{p}, \mathbf{z}_x, \mathbf{z}_y) \in I \times P \times Z_x \times Z_y$ *such that* $\mathbf{g}(t, \mathbf{p}, \mathbf{z}_x, \mathbf{z}_y) = \mathbf{0}$.

3. *If* $\widetilde{\mathbf{z}}_y \in \mathrm{int}(Z_y)$ *and* $\emptyset \neq Z'_y \subset \mathrm{int}(Z_y)$*, then* $\exists \mathbf{H} \in C^1(I \times P \times Z_x, Z'_y)$ *such that, for every* $(t,\mathbf{p},\mathbf{z}_x) \in I \times P \times Z_x$*,* $\mathbf{z}_y = \mathbf{H}(t,\mathbf{p},\mathbf{z}_x)$ *is the unique element of* $Z_y$ *satisfying* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) = \mathbf{0}$*. Moreover, the interval matrix* $\mathbf{C}\left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right](I,P,Z_x,Z_y)$ *does not contain a singular matrix and does not contain zero in any of its diagonal elements.*

*Proof* The result follows immediately from Theorem 4.1. □

The following theorem is the main result of this section.

**Theorem 4.2** *Let* $(I,P,Z_x,Z_y) \in \mathbb{I}D_t \times \mathbb{I}D_p \times \mathbb{I}D_x \times \mathbb{I}D_y$*,* $\widetilde{\mathbf{z}}_y \in Z_y$*,* $\mathbf{C} \in \mathbb{R}^{n_y \times n_y}$*, and define* $\mathcal{H}(I,P,Z_x,Z_y,\widetilde{\mathbf{z}}_y,\mathbf{C})$ *as in Corollary 4.1. Furthermore, let* $X_0 \in \mathbb{IR}^{n_x}$ *satisfy* $\mathbf{x}_0(P) \subset X_0$ *and denote* $I = [t_0, t_f]$*. If the inclusions*

$$\widetilde{\mathbf{z}}_y \in \mathrm{int}(Z_y), \tag{12}$$

$$\emptyset \neq Z'_y \equiv \mathcal{H}(I,P,Z_x,Z_y,\widetilde{\mathbf{z}}_y,\mathbf{C}) \subset \mathrm{int}(Z_y), \tag{13}$$

$$X_0 + [0, t_f - t_0]\,[\mathbf{f}]\,(I,P,Z_x,Z'_y) \subset Z_x, \tag{14}$$

*hold, then there exists a regular solution of* (1) *on* $I \times P$ *satisfying* $(\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) \in Z_x \times Z'_y$ *for all* $(t,\mathbf{p}) \in I \times P$*. Furthermore, for any connected* $\widetilde{I} \subset I$ *containing* $t_0$*, any connected* $\widetilde{P} \subset P$*, and any solution* $(\mathbf{x}^*,\mathbf{y}^*)$ *of* (1) *on* $\widetilde{I} \times \widetilde{P}$*, either* $(\mathbf{x}^*,\mathbf{y}^*) = (\mathbf{x},\mathbf{y})$ *on* $\widetilde{I} \times \widetilde{P}$*, or* $\mathbf{y}^*(t_0,\mathbf{p}) \notin Z_y$*,* $\forall \mathbf{p} \in \widetilde{P}$*.*

*Proof* By Conclusion 3 of Corollary 4.1, $\mathbf{C}\left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right](I,P,Z_x,Z_y)$ contains no singular matrix and $\exists \mathbf{H} \in C^1(I \times P \times Z_x, Z'_y)$ such that, for every $(t,\mathbf{p},\mathbf{z}_x) \in I \times P \times Z_x$, $\mathbf{z}_y = \mathbf{H}(t,\mathbf{p},\mathbf{z}_x)$ is the unique element of $Z_y$ satisfying $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) = \mathbf{0}$.

Choose any $\mathbf{x}^0 \in C^1(I \times P, Z_x)$ and define the sequence $\{\mathbf{x}^k\}$ by

$$\mathbf{x}^{k+1}(t,\mathbf{p}) = \mathbf{x}_0(\mathbf{p}) + \int_{t_0}^{t} \mathbf{f}(s,\mathbf{p},\mathbf{x}^k(s,\mathbf{p}),\mathbf{H}(s,\mathbf{p},\mathbf{x}^k(s,\mathbf{p})))ds, \quad \forall (t,\mathbf{p}) \in I \times P. \tag{15}$$

If $\mathbf{x}^k \in C^1(I \times P, Z_x)$, which is true for $k = 0$, then $\mathbf{x}^{k+1}$ is well-defined and

$$\mathbf{x}^{k+1}(t,\mathbf{p}) \in X_0 + [0, t_f - t_0]\,[\mathbf{f}]\,(I,P,Z_x,Z'_y) \subset Z_x, \quad \forall (t,\mathbf{p}) \in I \times P. \tag{16}$$

Then, by induction, $\mathbf{x}^k \in C^1(I \times P, Z_x)$, $\forall k \in \mathbb{N}$.

Noting that both $\mathbf{f}$ and $\mathbf{H}$ are continuously differentiable, the mapping $(t,\mathbf{p},\mathbf{z}_x) \mapsto \mathbf{f}(t,\mathbf{p},\mathbf{z}_x,\mathbf{H}(t,\mathbf{p},\mathbf{z}_x))$ is Lipschitz on $I \times P \times Z_x$ by Lemma 2.1. Then, a standard inductive argument (see [8], Ch. II, Thm. 1.1) shows that $\{\mathbf{x}^k\}$ converges uniformly on $I \times P$ to a continuous limit function, denoted $\mathbf{x}$, and $\mathbf{x}$ satisfies

$$\dot{\mathbf{x}}(t,\mathbf{p}) = \mathbf{f}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{H}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))), \quad \mathbf{x}(t_0,\mathbf{p}) = \mathbf{x}_0(\mathbf{p}), \quad \forall (t,\mathbf{p}) \in I \times P. \tag{17}$$

Since $\dot{\mathbf{x}}$ is continuous on $I \times P$, $\mathbf{x} \in C^1(I \times P, Z_x)$. Then, we may define $\mathbf{y} : I \times P \to D_y$ by $\mathbf{y}(t,\mathbf{p}) \equiv \mathbf{H}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))$. With this definition, $\mathbf{y} \in C^1(I \times P, Z'_y)$ and

$$\mathbf{g}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) = \mathbf{g}\Big(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{H}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))\Big) = \mathbf{0}, \quad \forall (t,\mathbf{p}) \in I \times P. \tag{18}$$

Therefore, $(\mathbf{x},\mathbf{y})$ is a solution of (1) on $I \times P$. Since $\mathbf{C}\left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right](I,P,Z_x,Z_y)$, and hence $\left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right](I,P,Z_x,Z_y)$, contains no singular matrix, $(\mathbf{x},\mathbf{y})$ must be regular.

Now consider any connected $\widetilde{I} \subset I$ containing $t_0$, any connected $\widetilde{P} \subset P$, and any solution $(\mathbf{x}^*,\mathbf{y}^*)$ of (1) on $\widetilde{I} \times \widetilde{P}$. If $\mathbf{y}^*(t_0,\mathbf{p}) \in Z_y$ for some $\mathbf{p} \in \widetilde{P}$, then the fact that $\mathbf{H}(t_0,\mathbf{p},\mathbf{x}_0(\mathbf{p}))$ satisfies $\mathbf{g}(t_0,\mathbf{p},\mathbf{x}_0(\mathbf{p}),\mathbf{H}(t_0,\mathbf{p},\mathbf{x}_0(\mathbf{p}))) = \mathbf{0}$ uniquely among elements of $Z_y$ implies that $\mathbf{y}^*(t_0,\mathbf{p}) = \mathbf{H}(t_0,\mathbf{p},\mathbf{x}_0(\mathbf{p})) = \mathbf{y}(t_0,\mathbf{p})$. Then the fact that $(\mathbf{x},\mathbf{y}) = (\mathbf{x}^*,\mathbf{y}^*)$ on $\widetilde{I} \times \widetilde{P}$ follows from Corollary 3.1. □

By checking some relatively simple inclusions, Theorem 4.2 provides a computational means to verify existence and uniqueness of a solution of (1) on given intervals $I \times P$, and provides a valid interval enclosure of this solution. In Part 2 of this article, an efficient numerical procedure for satisfying these inclusions is presented. In the following section, this result is used to develop computationally useful characterizations of bounding trajectories for the solutions of (1).

## 5 Bounding DAE Solutions using Differential Inequalities

This section presents three comparison theorems which provide sufficient conditions, in terms of differential inequalities, for mappings $\mathbf{v},\mathbf{w} : I \to \mathbb{R}^{n_x}$ to satisfy

$$\mathbf{v}(t) \leq \mathbf{x}(t,\mathbf{p}) \leq \mathbf{w}(t), \quad \forall (t,\mathbf{p}) \in I \times P, \tag{19}$$

for some solution of (1) on $I \times P$. The first such theorem (Theorem 5.1) is very general, but does not suggest a complete computational bounding procedure for reasons discussed below. The remaining two results are modifications of Theorem 5.1 that address these issues. Since these results are proven by similar methods, three lemmas are first proven to minimize repeated arguments.

**Lemma 5.1** *Let $I = [t_0,t_f] \subset D_t$ and $P \subset D_p$ be intervals and let $(\mathbf{x},\mathbf{y})$ be a regular solution of (1) on $I \times P$. Choose any continuous $\mathbf{v},\mathbf{w} : I \to \mathbb{R}^{n_x}$ and any $\hat{\mathbf{p}} \in P$ and define*

$$\bar{\mathbf{x}}(t,\hat{\mathbf{p}}) \equiv \mathrm{mid}(\mathbf{v}(t),\mathbf{w}(t),\mathbf{x}(t,\hat{\mathbf{p}})). \tag{20}$$

*For any $t_1 \in [t_0,t_f)$ such that $\bar{\mathbf{x}}(t_1,\hat{\mathbf{p}}) = \mathbf{x}(t_1,\hat{\mathbf{p}})$, there exists $t_4 \in (t_1,t_f]$, $L > 0$, and a continuous function $\bar{\mathbf{y}} : [t_1,t_4] \times P \to \mathbb{R}^{n_y}$ such that*

$$(\bar{\mathbf{x}}(t,\hat{\mathbf{p}}),\bar{\mathbf{y}}(t,\hat{\mathbf{p}})) \in D_x \times D_y, \tag{21}$$

$$\mathbf{g}(t,\hat{\mathbf{p}},\bar{\mathbf{x}}(t,\hat{\mathbf{p}}),\bar{\mathbf{y}}(t,\hat{\mathbf{p}})) = \mathbf{0}, \tag{22}$$

$$||\mathbf{y}(t,\hat{\mathbf{p}}) - \bar{\mathbf{y}}(t,\hat{\mathbf{p}})||_\infty \leq L||\mathbf{x}(t,\hat{\mathbf{p}}) - \bar{\mathbf{x}}(t,\hat{\mathbf{p}})||_\infty, \tag{23}$$

$$||\dot{\mathbf{x}}(t,\hat{\mathbf{p}}) - \mathbf{f}(t,\hat{\mathbf{p}},\bar{\mathbf{x}}(t,\hat{\mathbf{p}}),\bar{\mathbf{y}}(t,\hat{\mathbf{p}}))||_\infty \leq L||\mathbf{x}(t,\hat{\mathbf{p}}) - \bar{\mathbf{x}}(t,\hat{\mathbf{p}})||_\infty, \tag{24}$$

*for all $t \in [t_1,t_4]$.*

*Proof* Since $(\mathbf{x},\mathbf{y})$ is regular, Theorem 2.2 may be applied to conclude that their exists an open ball around $(t_1,\hat{\mathbf{p}},\mathbf{x}(t_1,\hat{\mathbf{p}}))$, $V_1 \subset D_t \times D_p \times D_x$, and a function $\mathbf{h} \in C^1(V_1, D_y)$ such that $\mathbf{y}(t_1,\hat{\mathbf{p}}) = \mathbf{h}(t_1,\hat{\mathbf{p}},\mathbf{x}(t_1,\hat{\mathbf{p}}))$ and

$$\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{h}(t,\mathbf{p},\mathbf{z}_x)) = \mathbf{0}, \quad \forall (t,\mathbf{p},\mathbf{z}_x) \in V_1. \tag{25}$$

Moreover, Lemma A.2 shows that there exists an open ball around $(t_1,\hat{\mathbf{p}})$, $U_1 \subset D_t \times D_p$, such that $(t,\mathbf{p},\mathbf{x}(t,\mathbf{p})) \in V_1$ and $\mathbf{y}(t,\mathbf{p}) = \mathbf{h}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))$, $\forall (t,\mathbf{p}) \in U_1 \cap (I \times P)$. Since $\overline{\mathbf{x}}(\cdot,\hat{\mathbf{p}})$ is continuous and $(t_1,\hat{\mathbf{p}},\overline{\mathbf{x}}(t_1,\hat{\mathbf{p}})) = (t_1,\hat{\mathbf{p}},\mathbf{x}(t_1,\hat{\mathbf{p}})) \in V_1$, $U_1$ may be chosen small enough that in addition $(t,\mathbf{p},\overline{\mathbf{x}}(t,\hat{\mathbf{p}})) \in V_1$, $\forall (t,\mathbf{p}) \in U_1 \cap (I \times P)$. Choosing $t_4 > t_1$ such that $[t_1,t_4] \times \{\hat{\mathbf{p}}\} \subset U_1 \cap (I \times P)$, define $\overline{\mathbf{y}}(t,\hat{\mathbf{p}}) \equiv \mathbf{h}(t,\hat{\mathbf{p}},\overline{\mathbf{x}}(t,\hat{\mathbf{p}}))$, $\forall t \in [t_1,t_4]$. Equation (21) now follows since $\mathbf{h}$ maps into $D_y$, and (22) follows from (25).

Since both $\mathbf{f}$ and $\mathbf{h}$ are continuously differentiable, the mappings

$$(t,\mathbf{p},\mathbf{z}_x) \mapsto \mathbf{h}(t,\mathbf{p},\mathbf{z}_x),$$
$$(t,\mathbf{p},\mathbf{z}_x) \mapsto \mathbf{f}(t,\mathbf{p},\mathbf{z}_x,\mathbf{h}(t,\mathbf{p},\mathbf{z}_x)),$$

are Lipschitz on any compact $K \subset V_1$ by Lemma 2.1. Let $K \equiv \{(t,\mathbf{p},\mathbf{z}_x) \in V_1 : t \in [t_1,t_4],\ \mathbf{p} = \hat{\mathbf{p}},\ \mathbf{z}_x = \mathbf{x}(t,\hat{\mathbf{p}}) \text{ or } \mathbf{z}_x = \overline{\mathbf{x}}(t,\hat{\mathbf{p}})\}$. Letting $L$ be the maximum of the corresponding Lipschitz constants, we arrive at (23) and (24). □

**Lemma 5.2** *Let $I = [t_0,t_f] \subset \mathbb{R}$. Given any $\epsilon, L > 0$, there exists $\rho \in C^1(I,\mathbb{R})$ non-decreasing and satisfying*

$$0 < \rho(t) \le \epsilon \quad \textit{and} \quad \rho'(t) > L\rho(t), \quad \forall t \in I. \tag{26}$$

*Proof* Choosing any $\gamma > 0$, the required properties are easily checked for $\rho(t) = \epsilon e^{(L+\gamma)(t-t_f)}$. □

**Lemma 5.3** *Let $I = [t_0,t_f] \subset D_t$ and $P \subset D_p$ be intervals, let $(\mathbf{x},\mathbf{y})$ be a regular solution of* (1) *on $I \times P$, and let $\mathbf{v},\mathbf{w} : I \to \mathbb{R}^{n_x}$ be continuous and satisfy*

(EX): $\mathbf{v}(t) \le \mathbf{w}(t)$, $\forall t \in I$.
(IC): $\mathbf{v}(t_0) \le \mathbf{x}_0(\mathbf{p}) \le \mathbf{w}(t_0)$, $\forall \mathbf{p} \in P$.

*Suppose $\exists (t,\hat{\mathbf{p}}) \in I \times P$ such that $\mathbf{x}(t,\hat{\mathbf{p}}) \notin [\mathbf{v}(t),\mathbf{w}(t)]$ and define*

$$t_1 \equiv \inf\{t \in I : \mathbf{x}(t,\hat{\mathbf{p}}) \notin [\mathbf{v}(t),\mathbf{w}(t)]\}. \tag{27}$$

*Then $t_0 \le t_1 < t_f$ and*

$$\mathbf{x}(t,\hat{\mathbf{p}}) \in [\mathbf{v}(t),\mathbf{w}(t)], \quad \forall t \in [t_0,t_1]. \tag{28}$$

*Moreover, given any $t_4 \in (t_1,t_f]$ and any $\epsilon, L > 0$, there exists an index $i \in \{1,\ldots,n_x\}$, a non-decreasing function $\rho \in C^1([t_1,t_4],\mathbb{R})$ satisfying* (26) *on $[t_1,t_4]$, and numbers $t_2,t_3 \in [t_1,t_4]$ with $t_2 < t_3$ such that*

$$\mathbf{v}(t) - \mathbf{1}\rho(t) < \mathbf{x}(t,\hat{\mathbf{p}}) < \mathbf{w}(t) + \mathbf{1}\rho(t), \quad \forall t \in [t_2,t_3) \tag{29}$$

*and*

$$x_i(t_2,\hat{\mathbf{p}}) = v_i(t_2), \quad x_i(t_3,\hat{\mathbf{p}}) = v_i(t_3) - \rho(t_3), \quad \textit{and} \quad x_i(t,\hat{\mathbf{p}}) < v_i(t), \tag{30}$$

$$\left( \textit{or } x_i(t_2,\hat{\mathbf{p}}) = w_i(t_2), \quad x_i(t_3,\hat{\mathbf{p}}) = w_i(t_3) + \rho(t_3), \quad \textit{and} \quad x_i(t,\hat{\mathbf{p}}) > w_i(t), \right) \tag{31}$$

*for all $t \in (t_2,t_3)$.*

*Proof* By the definition of the infimum, we have $\mathbf{x}(t,\hat{\mathbf{p}}) \in [\mathbf{v}(t),\mathbf{w}(t)]$ for all $t \in I$ such that $t < t_1$. If $t_1 > t_0$, then continuity ensures that this inclusion also holds at $t_1$, so that (28) holds. If $t_1 = t_0$, then (28) holds by Hypothesis (IC). By the assumption that $\mathbf{x}(t,\hat{\mathbf{p}}) \notin [\mathbf{v}(t),\mathbf{w}(t)]$ for some $t \in I$, it follows that $t_1 < t_f$.

Choose any $\epsilon, L > 0$ and any $t_4 \in (t_1,t_f]$ and define

$$m \equiv \max_{t\in[t_1,t_4]} ||\mathbf{x}(t,\hat{\mathbf{p}}) - \text{mid}\,(\mathbf{x}(t,\hat{\mathbf{p}}),\mathbf{v}(t),\mathbf{w}(t))||_\infty . \tag{32}$$

There must exist $t \in (t_1,t_4)$ such that $\mathbf{x}(t,\hat{\mathbf{p}}) \notin [\mathbf{v}(t),\mathbf{w}(t)]$, since otherwise $t_1$ would not satisfy (27). It follows that $m > 0$. Applying Lemma 5.2, we now choose $\rho \in C^1([t_1,t_4],\mathbb{R})$ such that

$$0 < \rho(t) \leq \min\left(\frac{m}{2},\epsilon\right), \quad \text{and} \quad \rho'(t) > L\rho(t), \quad \forall t \in [t_1,t_4]. \tag{33}$$

Now define

$$t_3 \equiv \inf\{t \in [t_1,t_4] : \mathbf{x}(t,\hat{\mathbf{p}}) \notin \text{int}\,([\mathbf{v}(t) - \mathbf{1}\rho(t), \mathbf{w}(t) + \mathbf{1}\rho(t)])\}. \tag{34}$$

Because $\rho < m$, this set in nonempty, and $t_3 > t_1$ by (28) and positivity of $\rho$. Because $t_3$ is a lower bound, (29) holds. Because $t_3$ is the greatest lower bound, either $x_i(t_3,\hat{\mathbf{p}}) = v_i(t_3) - \rho(t_3)$ or $x_i(t_3,\hat{\mathbf{p}}) = w_i(t_3) + \rho(t_3)$ for some index $i$. Suppose the former (the proof in the latter case is analogous) and define

$$t_2 \equiv \sup\{t \in [t_1,t_3] : x_i(t,\hat{\mathbf{p}}) \geq v_i(t)\}. \tag{35}$$

By (28), this set is nonempty, and the fact that $x_i(t_3,\hat{\mathbf{p}}) = v_i(t_3) - \rho(t_3)$ ensures that $t_1 \leq t_2 < t_3$. Because $t_2$ is an upper bound, $x_i(t) < v_i(t)$, $\forall t \in (t_2,t_3)$, and $x_i(t_2) = v_i(t_2)$ holds because it is the least upper bound. □

Theorem 5.1 below is the first of the three bounding results proven in this section. Its statement requires the following definition.

**Definition 5.1** **($\mathcal{B}_i^{L/U}$)** Let $\mathcal{B}_i^L, \mathcal{B}_i^U : \mathbb{IR}^{n_x} \to \mathbb{IR}^{n_x}$ be defined by $\mathcal{B}_i^L([\mathbf{v},\mathbf{w}]) = \{\mathbf{z} \in [\mathbf{v},\mathbf{w}] : z_i = v_i\}$ and $\mathcal{B}_i^U([\mathbf{v},\mathbf{w}]) = \{\mathbf{z} \in [\mathbf{v},\mathbf{w}] : z_i = w_i\}$, for every $i = 1,\ldots,n_x$.

Note that the computation of $\mathcal{B}_i^{L/U}$ is trivial. For example, $\mathcal{B}_i^L(\mathbf{v},\mathbf{w}) = [\mathbf{v},\mathbf{w}']$, where $w'_i = v_i$ and $w'_j = w_j$, $\forall j \neq i$.

**Theorem 5.1** *Let $I = [t_0,t_f] \subset D_t$ and $P \subset D_p$ be intervals and let $\mathbf{v},\mathbf{w} : I \to \mathbb{R}^{n_x}$ be absolutely continuous functions satisfying*

(EX): $\mathbf{v}(t) \leq \mathbf{w}(t)$, $\forall t \in I$.
(IC): $\mathbf{v}(t_0) \leq \mathbf{x}_0(\mathbf{p}) \leq \mathbf{w}(t_0)$, $\forall \mathbf{p} \in P$.
(RHS): *For a.e. $t \in I$ and each index $i$,*
1. $\dot{v}_i(t) \leq f_i(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)$ *for all* $(\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \in P \times D_x \times D_y$ *such that* $\mathbf{z}_x \in \mathcal{B}_i^L([\mathbf{v}(t),\mathbf{w}(t)])$ *and* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) = \mathbf{0}$,
2. $\dot{w}_i(t) \geq f_i(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)$ *for all* $(\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \in P \times D_x \times D_y$ *such that* $\mathbf{z}_x \in \mathcal{B}_i^U([\mathbf{v}(t),\mathbf{w}(t)])$ *and* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) = \mathbf{0}$.

*Then every regular solution of* (1) *on $I \times P$ satisfies* $\mathbf{x}(t,\mathbf{p}) \in [\mathbf{v}(t),\mathbf{w}(t)]$, $\forall (t,\mathbf{p}) \in I \times P$.

*Proof* Let $(\mathbf{x},\mathbf{y})$ be any regular solution of (1) on $I \times P$. Choose any $\hat{\mathbf{p}} \in P$ and suppose that there exists $t \in I$ such that $\mathbf{x}(t,\hat{\mathbf{p}}) \notin [\mathbf{v}(t),\mathbf{w}(t)]$. It will be shown that this results in a contradiction.

Define $t_1$ as in (27) and define $\overline{\mathbf{x}}$ as in (20). Noting that the hypotheses of Lemma 5.3 are satisfied, (28) implies that $\overline{\mathbf{x}}(t_1,\hat{\mathbf{p}}) = \mathbf{x}(t_1,\hat{\mathbf{p}})$. Then, the hypotheses of Lemma 5.1 are verified, so that there exists $t_4 \in (t_1,t_f]$, $L > 0$ and $\overline{\mathbf{y}}$ satisfying (21)-(24). Applying Lemma 5.3 with $t_4$, $L$ and arbitrary $\epsilon > 0$ yields an index $i \in \{1,\ldots,n_x\}$, a non-decreasing function $\rho \in C^1([t_1,t_4],\mathbb{R})$ satisfying (26) on $[t_1,t_4]$, and numbers $t_2,t_3 \in [t_1,t_4]$ with $t_2 < t_3$ such that (29) and (30) hold (the proof is analogous if instead (31) holds).

It will now be shown that $\dot{v}_i(t) - \rho'(t) \le \dot{x}_i(t,\hat{\mathbf{p}})$ for a.e. $t \in [t_2,t_3]$. Choose any $t \in (t_2,t_3)$. By (30) and Hypothesis (EX), we have $x_i(t,\hat{\mathbf{p}}) < v_i(t) \le w_i(t)$. By definition, this implies that $\overline{\mathbf{x}}(t,\hat{\mathbf{p}}) \in \mathcal{B}_i^L([\mathbf{v}(t),\mathbf{w}(t)])$. Then, by (21) and (22), the point $(\hat{\mathbf{p}},\overline{\mathbf{x}}(t,\hat{\mathbf{p}}),\overline{\mathbf{y}}(t,\hat{\mathbf{p}}))$ satisfies all of the of conditions of Hypothesis (RHS).1. Combining this with (24) gives

$$\dot{v}_i(t) \le f_i(t,\hat{\mathbf{p}},\overline{\mathbf{x}}(t,\hat{\mathbf{p}}),\overline{\mathbf{y}}(t,\hat{\mathbf{p}})) \le \dot{x}_i(t,\hat{\mathbf{p}}) + L\|\mathbf{x}(t,\hat{\mathbf{p}}) - \overline{\mathbf{x}}(t,\hat{\mathbf{p}})\|_\infty, \tag{36}$$

for a.e. $t \in [t_2,t_3]$. By (29), $\|\mathbf{x}(t,\hat{\mathbf{p}}) - \overline{\mathbf{x}}(t,\hat{\mathbf{p}})\|_\infty$ is bounded by $\rho(t)$ for all $t \in [t_2,t_3)$. Then, since $\rho'(t) > L\rho(t)$ for a.e. $t \in [t_1,t_4]$,

$$\dot{v}_i(t) - \rho'(t) \le \dot{x}_i(t,\hat{\mathbf{p}}) + L\rho(t) - \rho'(t) < \dot{x}_i(t,\hat{\mathbf{p}}), \tag{37}$$

for a.e. $t \in [t_2,t_3]$.

Applying Theorem 2.1, the function $v_i - \rho - x_i(\cdot,\hat{\mathbf{p}})$ is non-increasing on $(t_2,t_3)$, so that in particular,

$$v_i(t_3) - \rho(t_3) - x_i(t_3,\hat{\mathbf{p}}) \le v_i(t_2) - \rho(t_2) - x_i(t_2,\hat{\mathbf{p}}). \tag{38}$$

Using (30), this implies that $0 \le -\rho(t_2)$, which is a contradiction because $\rho(t) > 0$ for all $t \in [t_2,t_3]$. Thus, we must have $\mathbf{x}(t,\hat{\mathbf{p}}) \in [\mathbf{v}(t),\mathbf{w}(t)]$, $\forall t \in I$. In fact, since $\hat{\mathbf{p}} \in P$ was chosen arbitrarily, we have $\mathbf{x}(t,\mathbf{p}) \in [\mathbf{v}(t),\mathbf{w}(t)]$, $\forall (t,\mathbf{p}) \in I \times P$. □

Theorem 5.1 is very similar to existing results for bounding the solutions of explicit ODEs [40,7,33]. In [7] it was shown that interval arithmetic can be used to derive an auxiliary system of ODEs whose solutions satisfy conditions analogous to (IC) and (RHS) in Theorem 5.1, and these ODEs can be solved efficiently using a state-of-the-art numerical integrator to provide bounds. We present similar approaches for DAEs in Part 2 of this article. However, there is a problem with using Theorem 5.1 directly. Using interval methods to satisfy (RHS) would require some procedure for computing bounds on the zeros of $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\cdot)$ with $(t,\mathbf{p},\mathbf{z}_x)$ restricted to a given interval. Using the interval Hansen-Sengupta method, it is only possible to refine such an enclosure when provided with a guaranteed *a priori* enclosure.

A further complication is that Theorem 5.1 produces bounds that enclose *all* regular solutions of (1) on $I \times P$. However, in applications it is very likely that there will be a particular solution of interest, specified by a consistent initial condition $\mathbf{y}(t_0,\hat{\mathbf{p}})$ for some $\hat{\mathbf{p}} \in P$ (see Corollary 3.1). Theorem 5.1 provides no mechanism for restricting $\mathbf{v}$ and $\mathbf{w}$ based on this information because (RHS) requires that $\dot{v}_i$ and $\dot{w}_i$ bound

$f_i(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)$ for *all* $\mathbf{z}_y$ satisfying $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)=\mathbf{0}$. The following theorem shows that both of these problems can be avoided by modifying (RHS) in the case where intervals satisfying the conditions of Theorem 4.2 are available.

**Theorem 5.2** *Let* $(I,P,Z_x,Z_y,Z'_y)\in\mathbb{I}D_t\times\mathbb{I}D_p\times\mathbb{I}D_x\times\mathbb{I}D_y\times\mathbb{I}D_y$, $I=[t_0,t_f]$ *and* $Z'_y\subset Z_y$, *and let* $(\mathbf{x},\mathbf{y})\in C^1(I\times P,Z_x)\times C^1(I\times P,Z'_y)$ *be a regular solution of* (1) *on* $I\times P$. *Suppose further that* $\exists\mathbf{H}\in C^1(I\times P\times Z_x,Z'_y)$ *such that, for every* $(t,\mathbf{p},\mathbf{z}_x)\in I\times P\times Z_x$, $\mathbf{z}_y=\mathbf{H}(t,\mathbf{p},\mathbf{z}_x)$ *is the unique element of* $Z_y$ *satisfying* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)=\mathbf{0}$. *Let* $\mathbf{v},\mathbf{w}:I\to\mathbb{R}^{n_x}$ *be absolutely continuous functions satisfying*

(EX): $\mathbf{v}(t)\le\mathbf{w}(t)$ *and* $Z_x\cap[\mathbf{v}(t),\mathbf{w}(t)]\neq\emptyset$, $\forall t\in I$.
(IC): $\mathbf{v}(t_0)\le\mathbf{x}_0(\mathbf{p})\le\mathbf{w}(t_0)$, $\forall\mathbf{p}\in P$.
(RHS): *For a.e.* $t\in I$ *and each index* $i$,

1. $\dot{v}_i(t)\le f_i(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)$ *for all* $(\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)\in P\times Z_x\times Z'_y$ *such that* $\mathbf{z}_x\in\mathcal{B}_i^L(Z_x\cap[\mathbf{v}(t),\mathbf{w}(t)])$ *and* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)=\mathbf{0}$,
2. $\dot{w}_i(t)\ge f_i(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)$ *for all* $(\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)\in P\times Z_x\times Z'_y$ *such that* $\mathbf{z}_x\in\mathcal{B}_i^U(Z_x\cap[\mathbf{v}(t),\mathbf{w}(t)])$ *and* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)=\mathbf{0}$.

*Then* $\mathbf{x}(t,\mathbf{p})\in[\mathbf{v}(t),\mathbf{w}(t)]$ *for all* $(t,\mathbf{p})\in I\times P$.

*Proof* Choose any $\hat{\mathbf{p}}\in P$ and suppose that there exists $t\in I$ such that $\mathbf{x}(t,\hat{\mathbf{p}})\notin[\mathbf{v}(t),\mathbf{w}(t)]$. It will be shown that this results in a contradiction.

Define $\bar{\mathbf{x}}(t,\hat{\mathbf{p}})$ as in (20). Clearly, $\bar{\mathbf{x}}(t,\hat{\mathbf{p}})\in[\mathbf{v}(t),\mathbf{w}(t)]$, $\forall t\in I$. Let $[\mathbf{z}_x^L,\mathbf{z}_x^U]\equiv Z_x$. Since $x_j(t,\hat{\mathbf{p}})\in[z_{x,j}^L,z_{x,j}^U]$ by definition, it follows that $\bar{x}_j(t,\hat{\mathbf{p}})\in[z_{x,j}^L,z_{x,j}^U]$ for any index $j$ such that $x_j(t,\hat{\mathbf{p}})=\bar{x}_j(t,\hat{\mathbf{p}})$. Alternatively, for any $j$ such that $x_j(t,\hat{\mathbf{p}})\neq\bar{x}_j(t,\hat{\mathbf{p}})$, we have $x_j(t,\hat{\mathbf{p}})<v_j(t)$ (or $x_j(t,\hat{\mathbf{p}})>w_j(t)$), which, combined with the fact that $Z_x\cap[\mathbf{v}(t),\mathbf{w}(t)]$ is nonempty by hypothesis, gives

$$z_{x,j}^L\le x_j(t,\hat{\mathbf{p}})<v_j(t)=\mathrm{mid}(v_j(t),w_j(t),x_j(t,\hat{\mathbf{p}}))=\bar{x}_j(t,\hat{\mathbf{p}})\le z_{x,j}^U \tag{39}$$

$$\Big(\text{or}\quad z_{x,j}^U\ge x_j(t,\hat{\mathbf{p}})>w_j(t)=\mathrm{mid}(v_j(t),w_j(t),x_j(t,\hat{\mathbf{p}}))=\bar{x}_j(t,\hat{\mathbf{p}})\ge z_{x,j}^L\Big). \tag{40}$$

Therefore $\bar{\mathbf{x}}(t,\hat{\mathbf{p}})\in Z_x$.

Define $t_1$ as in (27), define $t_4\equiv t_f$, and define $\bar{\mathbf{y}}(t,\hat{\mathbf{p}})\equiv\mathbf{H}(t,\hat{\mathbf{p}},\bar{\mathbf{x}}(t,\hat{\mathbf{p}}))$, $\forall t\in I$. By the definition of $\mathbf{H}$, it follows that $\bar{\mathbf{y}}(t,\hat{\mathbf{p}})\in Z'_y$ for all $t\in[t_1,t_4]$ and (22) holds. Moreover, it can be shown that (24) holds by noting that the function

$$(t,\mathbf{p},\mathbf{z}_x)\mapsto\mathbf{f}(t,\mathbf{p},\mathbf{z}_x,\mathbf{H}(t,\mathbf{p},\mathbf{z}_x)),$$

is Lipschitz on compact subsets of $I\times P\times Z_x$, exactly as in Lemma 5.1. Applying Lemma 5.3 with $t_4$, $L$ and arbitrary $\epsilon>0$ yields an index $i\in\{1,\ldots,n_x\}$, a nondecreasing function $\rho\in C^1([t_1,t_4],\mathbb{R})$ satisfying (26) on $[t_1,t_4]$, and numbers $t_2,t_3\in[t_1,t_4]$ with $t_2<t_3$ such that (29) and (30) hold (the proof is analogous if instead (31) holds).

It will now be shown that (36) holds for a.e. $t\in[t_2,t_3]$. Choose any $t\in(t_2,t_3)$. It was argued above that $\bar{\mathbf{x}}(t,\hat{\mathbf{p}})\in Z_x\cap[\mathbf{v}(t),\mathbf{w}(t)]$ and $\bar{\mathbf{y}}(t,\hat{\mathbf{p}})\in Z'_y$. By (30) and Hypothesis (EX), we have $z_{x,i}^L\le x_i(t,\hat{\mathbf{p}})<v_i(t)=\mathrm{mid}(v_i(t),w_i(t),x_i(t,\hat{\mathbf{p}}))=\bar{x}_i(t,\hat{\mathbf{p}})$, and therefore $\bar{\mathbf{x}}(t,\hat{\mathbf{p}})\in\mathcal{B}_i^L(Z_x\cap[\mathbf{v}(t),\mathbf{w}(t)])$. Then, by (22), the point $(\hat{\mathbf{p}},\bar{\mathbf{x}}(t,\hat{\mathbf{p}}),\bar{\mathbf{y}}(t,\hat{\mathbf{p}}))$ satisfies all of the conditions of Hypothesis (RHS).1. Combining this with (24) proves

(36), and the remainder of the proof follows exactly as is the proof of Theorem 5.1. □

The final result below shows that the complications with Theorem 5.1 can also be avoided without having to first satisfy the conditions of Theorem 4.2, as in Theorem 5.2. Instead, we require satisfaction of (13) pointwise along the bounding trajectories $\mathbf{v}$ and $\mathbf{w}$, as in the following Hypothesis.

**Hypothesis 5.1** *Let* $(I,P) \in \mathbb{I}D_t \times \mathbb{I}D_p$, $\mathbf{C} : I \to \mathbb{R}^{n_y \times n_y}$ *and* $\widetilde{\mathbf{z}}_y : I \to \mathbb{R}^{n_y}$. *Suppose that* $\mathbf{z}_y^L, \mathbf{z}_y^U : I \to \mathbb{R}^{n_y}$ *and* $\mathbf{v}, \mathbf{w} : I \to \mathbb{R}^{n_x}$ *are continuous and satisfy*

(EX): $\mathbf{v}(t) \le \mathbf{w}(t)$ *and* $\mathbf{z}_y^L(t) \le \mathbf{z}_y^U(t)$, $\forall t \in I$.
(ALG): *For all* $t \in I$,

$$([\mathbf{v}(t), \mathbf{w}(t)], Z_y(t)) \in \mathbb{I}D_x \times \mathbb{I}D_y, \tag{41}$$

$$\widetilde{\mathbf{z}}_y(t) \in \mathrm{int}(Z_y(t)), \tag{42}$$

$$\emptyset \neq Z_y'(t) \equiv \mathcal{H}([t,t], P, [\mathbf{v}(t), \mathbf{w}(t)], Z_y(t), \widetilde{\mathbf{z}}_y(t), \mathbf{C}(t)) \subset \mathrm{int}(Z_y(t)), \tag{43}$$

*where* $Z_y(t) \equiv [\mathbf{z}_y^L(t), \mathbf{z}_y^U(t)]$ *and* $\mathcal{H}$ *is defined as in Corollary 4.1.*

**Lemma 5.4** *Suppose Hypothesis 5.1 holds and define*

$$V \equiv \{(t, \mathbf{p}, \mathbf{z}_x) \in I \times P \times D_x : \mathbf{z}_x \in [\mathbf{v}(t), \mathbf{w}(t)]\}. \tag{44}$$

*There exists* $\mathbf{H} \in C^1(V, D_y)$ *such that, for every* $(t, \mathbf{p}, \mathbf{z}_x) \in V$, $\mathbf{z}_y = \mathbf{H}(t, \mathbf{p}, \mathbf{z}_x)$ *is an element of* $Z_y'(t)$ *and satisfies* $\mathbf{g}(t, \mathbf{p}, \mathbf{z}_x, \mathbf{z}_y) = \mathbf{0}$ *uniquely among elements of* $Z_y(t)$.

*Proof* Choose any $t \in I$ and define $V_t \equiv [t,t] \times P \times [\mathbf{v}(t), \mathbf{w}(t)]$. By Hypothesis 5.1 and Conclusion 3 of Corollary 4.1, there exists $\mathbf{H}_t \in C^1(V_t, Z_y'(t))$ such that, for every $(t, \mathbf{p}, \mathbf{z}_x) \in V_t$, $\mathbf{z}_y = \mathbf{H}_t(t, \mathbf{p}, \mathbf{z}_x)$ is the unique element of $Z_y(t)$ satisfying $\mathbf{g}(t, \mathbf{p}, \mathbf{z}_x, \mathbf{z}_y) = \mathbf{0}$. Define $\mathbf{H} : V \to D_y$ by $\mathbf{H}(t, \mathbf{p}, \mathbf{z}_x) = \mathbf{H}_t(t, \mathbf{p}, \mathbf{z}_x)$. By the properties of each $H_t$ above, it only remains to show that $\mathbf{H} \in C^1(V, D_y)$.

By Lemma 23.1 in [20], it suffices to show that, for every $(\hat{t}, \hat{\mathbf{p}}, \hat{\mathbf{z}}_x) \in V$, there exists an open ball $\hat{U}$ and a function $\hat{\mathbf{h}} \in C^1(\hat{U}, D_y)$ that agrees with $\mathbf{H}$ on $\hat{U} \cap V$. Choose any such point and let $\hat{\mathbf{z}}_y = \mathbf{H}(\hat{t}, \hat{\mathbf{p}}, \hat{\mathbf{z}}_x)$. Applying Theorem 2.2 at the point $(\hat{t}, \hat{\mathbf{p}}, \hat{\mathbf{z}}_x, \hat{\mathbf{z}}_y)$ gives an open ball around $(\hat{t}, \hat{\mathbf{p}}, \hat{\mathbf{z}}_x)$, $\hat{V} \subset D_t \times D_p \times D_x$, an open ball around $\hat{\mathbf{z}}_y$, $\hat{Q} \subset D_y$, and $\hat{\mathbf{h}} \in C^1(\hat{V}, \hat{Q})$ such that $\hat{\mathbf{h}}(\hat{t}, \hat{\mathbf{p}}, \hat{\mathbf{z}}_x) = \hat{\mathbf{z}}_y$ and, for every $(t, \mathbf{p}, \mathbf{z}_x) \in \hat{V}$, $\mathbf{z}_y = \hat{\mathbf{h}}(t, \mathbf{p}, \mathbf{z}_x)$ is the unique element of $\hat{Q}$ satisfying $\mathbf{g}(t, \mathbf{p}, \mathbf{z}_x) = \mathbf{0}$. Noting that $\hat{\mathbf{z}}_y = \mathbf{H}(\hat{t}, \hat{\mathbf{p}}, \hat{\mathbf{z}}_x)$ is in $Z_y'(\hat{t})$, and hence in $\mathrm{int}(Z_y(\hat{t}))$ by (43), choose an open ball $\hat{Q}'$ around $\hat{\mathbf{z}}_y$ such that its closure is contained in $\mathrm{int}(Z_y(\hat{t}))$. By continuity of $\mathbf{z}_y^L$ and $\mathbf{z}_y^U$, $\exists \delta > 0$ such that $\hat{Q}' \subset \mathrm{int}(Z_y(t))$, for all $t \in I$ with $|t - \hat{t}| < \delta$. By continuity of $\hat{\mathbf{h}}$, there exists an open ball around $(\hat{t}, \hat{\mathbf{p}}, \hat{\mathbf{z}}_x)$, $\hat{U} \subset \hat{V}$, so small that any $(t, \mathbf{p}, \mathbf{z}_x) \in \hat{U} \cap V$ has $|t - \hat{t}| < \delta$ and $\hat{\mathbf{h}}(t, \mathbf{p}, \mathbf{z}_x) \in \hat{Q}'$. Then, for any $(t, \mathbf{p}, \mathbf{z}_x) \in \hat{U} \cap V$, both $\hat{\mathbf{h}}(t, \mathbf{p}, \mathbf{z}_x)$ and $\mathbf{H}(t, \mathbf{p}, \mathbf{z}_x)$ are zeros of $\mathbf{g}(t, \mathbf{p}, \mathbf{z}_x, \cdot)$ in $Z_y(t)$, and hence $\hat{\mathbf{h}}(t, \mathbf{p}, \mathbf{z}_x) = \mathbf{H}(t, \mathbf{p}, \mathbf{z}_x)$. □

**Lemma 5.5** *Suppose Hypothesis 5.1 holds and let* $(\mathbf{x}, \mathbf{y})$ *be a solution of* (1) *on* $I \times P$. *For any* $I' \equiv [t', t''] \subset I$ *and* $\mathbf{p}' \in P$, *the following implication holds:*

$$\left.\begin{array}{r} \mathbf{x}(t,\mathbf{p}) \in [\mathbf{v}(t), \mathbf{w}(t)], \quad \forall (t,\mathbf{p}) \in I' \times P \\ \mathbf{y}(t', \mathbf{p}') \in Z_y(t') \end{array}\right\} \Longrightarrow \begin{array}{l} \mathbf{y}(t,\mathbf{p}) \in Z_y'(t), \\ \forall (t,\mathbf{p}) \in I' \times P \end{array} \tag{45}$$

*Proof* First, it is shown that the implication

$$(\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) \in [\mathbf{v}(t),\mathbf{w}(t)]\times Z_y(t) \implies \mathbf{y}(t,\mathbf{p}) \in Z'_y(t) \tag{46}$$

holds for any $(t,\mathbf{p}) \in I\times P$. Let $V$ and $\mathbf{H}$ be as in Lemma 5.4 and suppose that the hypothesis of (46) holds. By definition $\mathbf{H}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))$ is the unique zero of $\mathbf{g}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\cdot)$ in $Z_y(t)$. But $\mathbf{y}(t,\mathbf{p})$ is a zero of $\mathbf{g}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\cdot)$ in $Z_y(t)$, and hence $\mathbf{y}(t,\mathbf{p}) = \mathbf{H}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))$. Noting that $\mathbf{H}$ maps into $Z'_y(t)$, (46) is established.

Under the hypotheses of (45), (46) implies that $\mathbf{y}(t',\mathbf{p}') \in Z'_y(t')$. If the conclusion of (45) fails, then there must exist $(t_2,\mathbf{p}_2) \in (t',t'']\times P$ such that $\mathbf{y}(t_2,\mathbf{p}_2) \notin Z'_y(t_2)$. Furthermore, this point must satisfy $\mathbf{y}(t_2,\mathbf{p}_2) \notin Z_y(t_2)$, since otherwise (46) provides a contradiction. Continuity of $\mathbf{y}$, $\mathbf{z}^L_y$ and $\mathbf{z}^U_y$ then imply that $\exists(t_1,\mathbf{p}_1) \in (t',t'']\times P$ such that $\mathbf{y}(t_1,\mathbf{p}_1)$ is an element of the boundary of $Z_y(t_1)$, and hence of $Z_y(t_1)$, but not an element of $Z'_y(t_1) \subset \mathrm{int}(Z_y(t_1))$. Again, (46) provides a contradiction. □

**Theorem 5.3** *Suppose Hypothesis 5.1 holds. Additionally, let* $\mathbf{v},\mathbf{w}$ *be absolutely continuous and satisfy*

(IC): $\mathbf{v}(t_0) \le \mathbf{x}_0(\mathbf{p}) \le \mathbf{w}(t_0)$, $\forall \mathbf{p} \in P$.

(RHS): *For a.e.* $t \in I$ *and each index* $i$,

1. $\dot{v}_i(t) \le f_i(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)$ *for all* $(\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \in P\times D_x\times Z'_y(t)$ *such that* $\mathbf{z}_x \in \mathcal{B}^L_i([\mathbf{v}(t),\mathbf{w}(t)])$ *and* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) = \mathbf{0}$,
2. $\dot{w}_i(t) \ge f_i(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y)$ *for all* $(\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) \in P\times D_x\times Z'_y(t)$ *such that* $\mathbf{z}_x \in \mathcal{B}^U_i([\mathbf{v}(t),\mathbf{w}(t)])$ *and* $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{z}_y) = \mathbf{0}$.

*Then every regular solution of* (1) *on* $I\times P$ *with* $\mathbf{y}(t_0,\widetilde{\mathbf{p}}) \in Z_y(t_0)$ *for at least one* $\widetilde{\mathbf{p}} \in P$ *must satisfy* $(\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})) \in [\mathbf{v}(t),\mathbf{w}(t)]\times Z'_y(t)$ *for all* $(t,\mathbf{p}) \in I\times P$.

*Proof* Let $(\mathbf{x},\mathbf{y})$ be a regular solution of (1) on $I\times P$ satisfying $\mathbf{y}(t_0,\widetilde{\mathbf{p}}) \in Z_y(t_0)$ for some $\widetilde{\mathbf{p}} \in P$. Choose any $\hat{\mathbf{p}} \in P$ and suppose that there exists $t \in I$ such that $\mathbf{x}(t,\hat{\mathbf{p}}) \notin [\mathbf{v}(t),\mathbf{w}(t)]$. It will be shown that this results in a contradiction.

Define $t_1$ as in (27). Noting that the hypotheses of Lemma 5.3 are satisfied, (28) holds and (45) implies that $\mathbf{y}(t,\hat{\mathbf{p}}) \in Z'_y(t)$, $\forall t \in [t_0,t_1]$. Define $\overline{\mathbf{x}}$ as in Lemma 5.1. Noting that $\overline{\mathbf{x}}(t_1,\hat{\mathbf{p}}) = \mathbf{x}(t_1,\hat{\mathbf{p}})$ by (28), Lemma 5.1 furnishes $t_4 \in (t_1,t_f]$, $L > 0$ and $\overline{\mathbf{y}}$ satisfying (21)-(24). By (23) and (43), $\overline{\mathbf{y}}(t_1,\hat{\mathbf{p}}) = \mathbf{y}(t_1,\hat{\mathbf{p}}) \in \mathrm{int}(Z_y(t_1))$. By continuity of $\overline{\mathbf{y}}$, $\mathbf{z}^L_y$, $\mathbf{z}^U_y$, it is possible to restrict $t_4$ so that

$$\overline{\mathbf{y}}(t,\hat{\mathbf{p}}) \in Z_y(t), \quad \forall t \in [t_1,t_4]. \tag{47}$$

We now apply Lemma 5.3 with $t_4$, $L$ and arbitrary $\epsilon > 0$. This yields an index $i \in \{1,\ldots,n_x\}$, a non-decreasing function $\rho \in C^1([t_1,t_4],\mathbb{R})$ satisfying (26) on $[t_1,t_4]$, and numbers $t_2,t_3 \in [t_1,t_4]$ with $t_2 < t_3$ such that (29) and (30) hold (the proof is analogous if instead (31) holds).

It will now be shown that (36) holds for a.e. $t \in [t_2,t_3]$. Choose any $t \in (t_2,t_3)$. By (30) and Hypothesis 5.1 (EX), we have $x_i(t,\hat{\mathbf{p}}) < v_i(t) \le w_i(t)$. By definition, this implies that $\overline{\mathbf{x}}(t,\hat{\mathbf{p}}) \in \mathcal{B}^L_i([\mathbf{v}(t),\mathbf{w}(t)])$. Since $\overline{\mathbf{x}}(t,\hat{\mathbf{p}}) \in [\mathbf{v}(t),\mathbf{w}(t)]$ and $\overline{\mathbf{y}}(t,\hat{\mathbf{p}})$ is a zero of $\mathbf{g}(t,\hat{\mathbf{p}},\overline{\mathbf{x}}(t,\hat{\mathbf{p}}),\cdot)$ by (22), Equation (47) and Corollary 4.1 show that $\overline{\mathbf{y}}(t,\hat{\mathbf{p}}) \in Z'_y(t)$. Then, by (21) and (22), the point $(\hat{\mathbf{p}},\overline{\mathbf{x}}(t,\hat{\mathbf{p}}),\overline{\mathbf{y}}(t,\hat{\mathbf{p}}))$ satisfies all of the conditions of (RHS).1. Combining this with (24) proves (36) and, exactly as is the proof of Theorem 5.1, we conclude that $\mathbf{x}(t,\mathbf{p}) \in [\mathbf{v}(t),\mathbf{w}(t)]$, $\forall(t,\mathbf{p}) \in I\times P$. The theorem now follows from (45). □

## 6 Conclusions

We have presented a detailed analysis characterizing interval enclosures of the solutions of semi-explicit, index-one DAEs subject to uncertain initial conditions and parameters. The primary contributions are (1) a set of conditions guaranteeing existence and uniqueness of a solution and providing a crude enclosure, and (2) three theorems giving sufficient conditions for some functions to describe bounds on one or all solutions pointwise in the independent variable. What remains is to develop methods for satisfying these conditions computationally, thus leading to efficient, constructive procedures for computing bounds. We take up this task in Part 2.

## A Uniqueness Proofs

**Lemma A.1** *Let $E \subset \mathbb{R}^n$ be connected and let $\psi : E \to \mathbb{R}$ be continuous. If the set $\{\boldsymbol{\xi} \in E : \psi(\boldsymbol{\xi}) = 0\}$ is nonempty and open with respect to $E$, then $\psi(\boldsymbol{\xi}) = 0$, $\forall \boldsymbol{\xi} \in E$.*

*Proof* Let $E_1 = \{\boldsymbol{\xi} \in E : \psi(\boldsymbol{\xi}) = 0\}$ and $E_2 = \{\boldsymbol{\xi} \in E : \psi(\boldsymbol{\xi}) \neq 0\}$, and note that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. Since $E$ is connected, it cannot be written as the disjoint union of two nonempty open (w.r.t. $E$) sets. But $E_1$ is nonempty and open w.r.t. $E$ by hypothesis, and $E_2$ is open w.r.t. $E$ because it is the inverse image of an open set under a continuous mapping on $E$. Hence, $E_2 = \emptyset$ and $E_1 = E$. □

**Lemma A.2** *Let $(\mathbf{x},\mathbf{y}) \in C^1(I \times P, D_x) \times C^1(I \times P, D_y)$ and $(\mathbf{x}^*,\mathbf{y}^*) \in C^1(\widetilde{I} \times \widetilde{P}, D_x) \times C^1(\widetilde{I} \times \widetilde{P}, D_y)$ be solutions of* (1a) *on $I \times P$ and $\widetilde{I} \times \widetilde{P}$, respectively, and suppose that $(\mathbf{x},\mathbf{y})$ is regular. Then*

1. *For any $(t',\mathbf{p}') \in I \times P$, there exists an open ball around $(t',\mathbf{p}')$, $U' \subset D_t \times D_p$, an open ball around $(t',\mathbf{p}',\mathbf{x}(t',\mathbf{p}'))$, $V' \subset D_t \times D_p \times D_x$, an open ball around $\mathbf{y}(t',\mathbf{p}')$, $Q' \subset D_y$, and a function $\mathbf{h} \in C^1(V',Q')$ satisfying $(t,\mathbf{p},\mathbf{x}(t,\mathbf{p})) \in V'$ and $\mathbf{y}(t,\mathbf{p}) = \mathbf{h}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p})) \in Q'$, $\forall (t,\mathbf{p}) \in U' \cap (I \times P)$.*
2. *If $\hat{P} \subset P \cap \widetilde{P}$ is connected and $\exists (t',\hat{\mathbf{p}}) \in (I \cap \widetilde{I}) \times \hat{P}$ such that $\mathbf{x}(t',\mathbf{p}) = \mathbf{x}^*(t',\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$, and $\mathbf{y}(t',\hat{\mathbf{p}}) = \mathbf{y}^*(t',\hat{\mathbf{p}})$, then $\mathbf{y}(t',\mathbf{p}) = \mathbf{y}^*(t',\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$.*

*Proof* Choose any $(t',\mathbf{p}') \in I \times P$. Since $(\mathbf{x},\mathbf{y})$ is a regular solution of (1a) on $I \times P$, $(t',\mathbf{p}',\mathbf{x}(t',\mathbf{p}'),\mathbf{y}(t',\mathbf{p}')) \in \mathcal{G} \cap \mathcal{G}_R$. Then, by Theorem 2.2, there exists an open ball around $(t',\mathbf{p}',\mathbf{x}(t',\mathbf{p}'))$, $V' \subset D_t \times D_p \times D_x$, an open ball around $\mathbf{y}(t',\mathbf{p}')$, $Q' \subset D_y$, and a function $\mathbf{h} \in C^1(V',Q')$ such that $\mathbf{h}(t',\mathbf{p}',\mathbf{x}(t',\mathbf{p}')) = \mathbf{y}(t',\mathbf{p}')$ and, for each $(t,\mathbf{p},\mathbf{z}_x) \in V'$, $\mathbf{h}(t,\mathbf{p},\mathbf{z}_x)$ is the unique element of $Q'$ satisfying $\mathbf{g}(t,\mathbf{p},\mathbf{z}_x,\mathbf{h}(t,\mathbf{p},\mathbf{z}_x)) = \mathbf{0}$. Now, by continuity, there exists an open ball $U'$ around the point $(t',\mathbf{p}')$ small enough that $(t,\mathbf{p},\mathbf{x}(t,\mathbf{p})) \in V'$ for every $(t,\mathbf{p}) \in U' \cap (I \times P)$, and it follows that

$$\mathbf{g}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{h}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))) = \mathbf{0}, \quad \forall (t,\mathbf{p}) \in U' \cap (I \times P). \tag{48}$$

Again by continuity, it is possible to choose $U'$ small enough that $\mathbf{y}(t,\mathbf{p}) \in Q'$ for all $(t,\mathbf{p}) \in U' \cap (I \times P)$, which implies, by the uniqueness property of $\mathbf{h}$ in $Q'$, that

$$\mathbf{y}(t,\mathbf{p}) = \mathbf{h}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p})), \quad \forall (t,\mathbf{p}) \in U' \cap (I \times P). \tag{49}$$

This establishes the first conclusion of the lemma.

To prove the second conclusion, choose any $\hat{P}$, $\hat{\mathbf{p}}$ and $t'$ as in the hypothesis of the lemma and define

$$R \equiv \{\mathbf{p} \in \hat{P} : \|\mathbf{y}(t',\mathbf{p}) - \mathbf{y}^*(t',\mathbf{p})\| = 0\}. \tag{50}$$

By hypothesis, $\hat{\mathbf{p}} \in R$ so that $R$ is nonempty. It will be shown than $R$ is open with respect to $\hat{P}$. Choose any $\mathbf{p}' \in R$ and, corresponding to the point $(t',\mathbf{p}')$, let $U'$, $V'$, $Q'$ and $\mathbf{h}$ be as in the first conclusion of the lemma. By hypothesis, $(t',\mathbf{p}',\mathbf{x}^*(t',\mathbf{p}')) = (t',\mathbf{p}',\mathbf{x}(t',\mathbf{p}')) \in V'$, and by the definition of $R$, $\mathbf{y}^*(t',\mathbf{p}') = \mathbf{y}(t',\mathbf{p}') \in Q'$, so continuity implies that we may choose an open all around $\mathbf{p}'$, $J_{\mathbf{p}'}$, small enough that $J_{\mathbf{p}'} \times \{t'\} \subset U'$, and $(t',\mathbf{p},\mathbf{x}^*(t',\mathbf{p})) \in V'$ and $\mathbf{y}^*(t',\mathbf{p}) \in Q'$, for all $\mathbf{p} \in J_{\mathbf{p}'} \cap \widetilde{P}$. Then the first conclusion of the theorem gives

$$\mathbf{y}(t',\mathbf{p}) = \mathbf{h}(t',\mathbf{p},\mathbf{x}(t',\mathbf{p})), \quad \forall \mathbf{p} \in J_{\mathbf{p}'} \cap \hat{P}, \tag{51}$$

and an identical argument shows that

$$\mathbf{y}^*(t',\mathbf{p}) = \mathbf{h}(t',\mathbf{p},\mathbf{x}^*(t',\mathbf{p})), \quad \forall \mathbf{p} \in J_{\mathbf{p}'} \cap \hat{P}. \tag{52}$$

But $\mathbf{x}^*(t',\mathbf{p}) = \mathbf{x}(t',\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$ by hypothesis, so this implies that $\mathbf{y}^*(t',\mathbf{p}) = \mathbf{y}(t',\mathbf{p})$, $\forall \mathbf{p} \in J_{\mathbf{p}'} \cap \hat{P}$. Thus $R$ is open with respect to $\hat{P}$. Now, since $\hat{P}$ is connected by hypothesis and $R$ is nonempty and open with respect to $\hat{P}$, Lemma A.1 shows that $R = \hat{P}$; i.e. $\mathbf{y}^*(t',\mathbf{p}) = \mathbf{y}(t',\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$. □

**Lemma A.3** *Let* $(\mathbf{x},\mathbf{y}) \in C^1(I \times P, D_x) \times C^1(I \times P, D_y)$ *and* $(\mathbf{x}^*,\mathbf{y}^*) \in C^1(\widetilde{I} \times \widetilde{P}, D_x) \times C^1(\widetilde{I} \times \widetilde{P}, D_y)$ *be solutions of* (1a) *on* $I \times P$ *and* $\widetilde{I} \times \widetilde{P}$, *respectively, and suppose that* $(\mathbf{x},\mathbf{y})$ *is regular. If* $\hat{P} \subset P \cap \widetilde{P}$ *is connected and compact and* $\exists(\hat{t},\hat{\mathbf{p}}) \in (I \cap \widetilde{I}) \times \hat{P}$ *such that* $\mathbf{x}(\hat{t},\mathbf{p}) = \mathbf{x}^*(\hat{t},\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$, *and* $\mathbf{y}(\hat{t},\hat{\mathbf{p}}) = \mathbf{y}^*(\hat{t},\hat{\mathbf{p}})$, *then* $\mathbf{x}(t,\mathbf{p}) = \mathbf{x}^*(t,\mathbf{p})$ *and* $\mathbf{y}(t,\mathbf{p}) = \mathbf{y}^*(t,\mathbf{p})$, $\forall (t,\mathbf{p}) \in (I \cap \widetilde{I}) \times \hat{P}$.

*Proof* Choose any $\hat{P}$, $\hat{\mathbf{p}}$ and $\hat{t}$ as in the hypothesis of the lemma and define

$$R \equiv \{t \in I \cap \widetilde{I} : \max_{\mathbf{p}\in\hat{P}} (\|\mathbf{x}(t,\mathbf{p}) - \mathbf{x}^*(t,\mathbf{p})\|) + \|\mathbf{y}(t,\hat{\mathbf{p}}) - \mathbf{y}^*(t,\hat{\mathbf{p}})\| = 0\}. \tag{53}$$

$R$ is nonempty since it contains $\hat{t}$. It will be shown that $R$ is open with respect to $I \cap \widetilde{I}$. Choose any $t' \in R$. Applying the second conclusion of Lemma A.2, we have $\mathbf{y}^*(t',\mathbf{p}) = \mathbf{y}(t',\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$. Choose any $\mathbf{p}' \in \hat{P}$ and, corresponding to the point $(t',\mathbf{p}')$, let $U'$, $V'$, $Q'$ and $\mathbf{h}$ be as in the first conclusion of Lemma A.2. By the definition of $R$, $(t',\mathbf{p}',\mathbf{x}^*(t',\mathbf{p}')) = (t',\mathbf{p}',\mathbf{x}(t',\mathbf{p}')) \in V'$ and, by the argument above, $\mathbf{y}^*(t',\mathbf{p}') = \mathbf{y}(t',\mathbf{p}') \in Q'$. Then continuity implies that there exists an open ball around $t'$, $J_{t'}$, and an open ball around $\mathbf{p}'$, $J_{\mathbf{p}'}$, such that $J_{t'} \times J_{\mathbf{p}'} \subset U'$, and $(t,\mathbf{p},\mathbf{x}^*(t,\mathbf{p})) \in V'$ and $\mathbf{y}^*(t,\mathbf{p}) \in Q'$, for all $(t,\mathbf{p}) \in (J_{t'} \times J_{\mathbf{p}'}) \cap (\widetilde{I} \times \widetilde{P})$. From Lemma A.2, we have

$$\mathbf{y}(t,\mathbf{p}) = \mathbf{h}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p})), \quad \forall (t,\mathbf{p}) \in (J_{t'} \times J_{\mathbf{p}'}) \cap (I \times \hat{P}), \tag{54}$$

and an identical argument using the uniqueness property of $\mathbf{h}$ in $Q'$ shows that

$$\mathbf{y}^*(t,\mathbf{p}) = \mathbf{h}(t,\mathbf{p},\mathbf{x}^*(t,\mathbf{p})), \quad \forall (t,\mathbf{p}) \in (J_{t'} \times J_{\mathbf{p}'}) \cap (\widetilde{I} \times \hat{P}). \tag{55}$$

Then, by definition,

$$\dot{\mathbf{x}}(t,\mathbf{p}) = \mathbf{f}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{h}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}))), \quad \forall (t,\mathbf{p}) \in (J_{t'} \times J_{\mathbf{p}'}) \cap (I \times \hat{P}), \tag{56}$$

$$\dot{\mathbf{x}}^*(t,\mathbf{p}) = \mathbf{f}(t,\mathbf{p},\mathbf{x}^*(t,\mathbf{p}),\mathbf{h}(t,\mathbf{p},\mathbf{x}^*(t,\mathbf{p}))), \quad \forall (t,\mathbf{p}) \in (J_{t'} \times J_{\mathbf{p}'}) \cap (\widetilde{I} \times \hat{P}). \tag{57}$$

But $\mathbf{f}$ and $\mathbf{h}$ are continuously differentiable and hence the mapping $(t,\mathbf{p},\mathbf{z}_x) \mapsto \mathbf{f}(t,\mathbf{p},\mathbf{h}(t,\mathbf{p},\mathbf{z}_x))$ is Lipschitz on $V'$ by Lemma 2.1. The definition of $R$ gives $\mathbf{x}(t',\mathbf{p}) = \mathbf{x}^*(t',\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$, so a standard application of Gronwall's inequality shows that $\mathbf{x}(t,\mathbf{p}) = \mathbf{x}^*(t,\mathbf{p})$, $\forall (t,\mathbf{p}) \in (J_{t'} \times J_{\mathbf{p}'}) \cap ((I \cap \widetilde{I}) \times \hat{P})$. Furthermore, this implies that $\mathbf{y}(t,\mathbf{p}) = \mathbf{h}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p})) = \mathbf{h}(t,\mathbf{p},\mathbf{x}^*(t,\mathbf{p})) = \mathbf{y}^*(t,\mathbf{p})$, $\forall (t,\mathbf{p}) \in (J_{t'} \times J_{\mathbf{p}'}) \cap ((I \cap \widetilde{I}) \times \hat{P})$.

Now, since $\mathbf{p}' \in \hat{P}$ was chosen arbitrarily, the preceding construction applies to every $\mathbf{p} \in \hat{P}$. Thus, to every $\mathbf{q} \in \hat{P}$, there corresponds an open ball around $t'$, $J_{t'}(\mathbf{q})$, and an open ball around $\mathbf{q}$, $J_{\mathbf{q}}$, such that $(\mathbf{x},\mathbf{y})(t,\mathbf{p}) = (\mathbf{x}^*,\mathbf{y}^*)(t,\mathbf{p})$, $\forall (t,\mathbf{p}) \in (J_{t'}(\mathbf{q}) \times J_{\mathbf{q}}) \cap ((I \cap \widetilde{I}) \times \hat{P})$. Noting that the $J_{\mathbf{q}}$ constructed in this way form an open cover of $\hat{P}$, compactness of $\hat{P}$ implies that there exist finitely many elements of $\hat{P}$, $\mathbf{q}_1,\ldots,\mathbf{q}_n$, such that $\hat{P}$ is covered by $J_{\mathbf{q}_1} \cup \ldots \cup J_{\mathbf{q}_n}$. Let $J^*_{t'} \equiv J_{t'}(\mathbf{q}_1) \cap \ldots \cap J_{t'}(\mathbf{q}_n)$. Then, for every $\mathbf{p} \in \hat{P}$, there exists $i \in \{1,\ldots,n\}$ such that $\mathbf{p} \in J_{\mathbf{q}_i}$, which implies that $(\mathbf{x},\mathbf{y})(t,\mathbf{p}) = (\mathbf{x}^*,\mathbf{y}^*)(t,\mathbf{p})$, $\forall t \in J^*_{t'} \cap (I \cap \widetilde{I})$. Therefore, $J^*_{t'} \cap (I \cap \widetilde{I})$ is contained in $R$, so that $t'$ is an interior point of $R$ when viewed as a subset of $I \cap \widetilde{I}$, and since $t' \in R$ was chosen arbitrarily, $R$ is open with respect to $I \cap \widetilde{I}$. Since $I \cap \widetilde{I}$ is connected and $R$ is nonempty and open with respect to $I \cap \widetilde{I}$, Lemma A.1 shows that $R = I \cap \widetilde{I}$. But by definition, this implies that $\mathbf{x}(t,\mathbf{p}) = \mathbf{x}^*(t,\mathbf{p})$ and $\mathbf{y}(t,\hat{\mathbf{p}}) = \mathbf{y}^*(t,\hat{\mathbf{p}})$, $\forall (t,\mathbf{p}) \in (I \cap \widetilde{I}) \times \hat{P}$. Finally, the second conclusion of Lemma A.2 implies that $\mathbf{y}(t,\mathbf{p}) = \mathbf{y}^*(t,\mathbf{p})$, $\forall (t,\mathbf{p}) \in (I \cap \widetilde{I}) \times \hat{P}$. □

**Theorem A.1** *Let* $(\mathbf{x},\mathbf{y}) \in C^1(I \times P, D_x) \times C^1(I \times P, D_y)$ *and* $(\mathbf{x}^*,\mathbf{y}^*) \in C^1(\widetilde{I} \times \widetilde{P}, D_x) \times C^1(\widetilde{I} \times \widetilde{P}, D_y)$ *be solutions of* (1a) *on* $I \times P$ *and* $\widetilde{I} \times \widetilde{P}$, *respectively, and suppose that* $(\mathbf{x},\mathbf{y})$ *is regular. If* $\hat{P} \subset P \cap \widetilde{P}$ *is connected and* $\exists(\hat{t},\hat{\mathbf{p}}) \in (I \cap \widetilde{I}) \times \hat{P}$ *such that* $\mathbf{x}(\hat{t},\mathbf{p}) = \mathbf{x}^*(\hat{t},\mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$, *and* $\mathbf{y}(\hat{t},\hat{\mathbf{p}}) = \mathbf{y}^*(\hat{t},\hat{\mathbf{p}})$, *then* $\mathbf{x}(t,\mathbf{p}) = \mathbf{x}^*(t,\mathbf{p})$ *and* $\mathbf{y}(t,\mathbf{p}) = \mathbf{y}^*(t,\mathbf{p})$, $\forall (t,\mathbf{p}) \in (I \cap \widetilde{I}) \times \hat{P}$.

*Proof* Choose any $\mathbf{p} \in \hat{P}$. Clearly, $\{\mathbf{p}\} \subset P \cap \widetilde{P}$ is compact and connected, and Lemma A.2 guarantees that $\mathbf{y}(\hat{t}, \mathbf{p}) = \mathbf{y}^*(\hat{t}, \mathbf{p})$. Then Lemma A.3 shows that $\mathbf{x}(t, \mathbf{p}) = \mathbf{x}^*(t, \mathbf{p})$ and $\mathbf{y}(t, \mathbf{p}) = \mathbf{y}^*(t, \mathbf{p})$, $\forall t \in I \cap \widetilde{I}$. □

Corollary 3.1 is a simple consequence of these developments. By the definition of a solution of (1), we have $\mathbf{x}(t_0, \mathbf{p}) = \mathbf{x}^*(t_0, \mathbf{p})$, $\forall \mathbf{p} \in \hat{P}$, and $\mathbf{y}(t_0, \hat{\mathbf{p}}) = \mathbf{y}^*(t_0, \hat{\mathbf{p}})$ by hypothesis. Since $\hat{P}$ is connected, the result follows from Theorem A.1.

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