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*Semiclassical States Associated with
Isotropic Submanifolds of Phase Space*

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Semiclassical States Associated with Isotropic Submanifolds of Phase Space

Dedicated to Louis Boutet de Monvel

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Abstract. We define classes of quantum states associated with isotropic submanifolds of cotangent bundles. The classes are stable under the action of semiclassical pseudo-differential operators and covariant under the action of semiclassical Fourier integral operators. We develop a symbol calculus for them; the symbols are symplectic spinors. We outline various applications.

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1. Introduction

Among the contributions to mathematics that Boutet de Monvel is most remembered for is his work on Hermite distributions and Toeplitz operators, and the purpose of this paper is to give a semi-classical account of this theory, an account that is largely inspired by Boutet’s paper [5] on partial differential equations with multiple characteristics (in which he introduces Hermite distributions for the first time), his paper [7] with Sjöstrand (in which the classical theory of the Bergman projector for bounded domains in \mathbb{C}^n is shown to have an elegant and succinct microlocal description as a Toeplitz operator), and the monograph [6] (co-written with one of the authors of this paper) in which the theory of Toeplitz operators is developed ab ovo and the spectral properties of these operators are investigated in detail.

In what follows, M will be a smooth manifold and T^*M its cotangent bundle. We recall that if one is given a Lagrangian submanifold, Λ of T^*M , one can associate with Λ a space, $I(M, \Lambda)$, of oscillatory functions having the property that their semi-classical wave front sets are contained in Λ ; and one can define for these functions a symbol calculus which has good functorial properties with respect to composition by semi-classical pseudodifferential operators. (For more details, see, for instance chapter 8 of the reference [12].) Our goal below will be to develop an analog of this theory for isotropic submanifold of T^*M similar to the isotropic analog of the classical theory of Lagrangian distributions: the theory of Hermite distributions developed by Boutet and his collaborators in the references cited above. (The salient property of Hermite distribution is that their microsupports are contained in an isotropic submanifold, Σ , of T^*M , and this will be the case for our “semi-classical oscillatory functions of isotropic type” as well.)

We will begin in Section 2 by defining these functions in the special case for which Σ is a vector subspace of the zero section in $T^*\mathbb{R}^n$ and show that in this “model” case these functions have good compositional properties with respect to semi-classical pseudodifferential and Fourier integral operators, have a well-defined symbol calculus and satisfy analogs of the first- and second-order transport equations described in [10], §10. Then in Section 3, we will extend this theory to manifold setting and, in particular, show how to describe intrinsically the manifold versions of the results in Section 2.

Finally in Section 4, we will briefly discuss some applications of this theory to partial differential equations and several complex variable theory (applications which we intend to discuss in more detail in a projected sequel to this paper.)

2. Local Theory

2.1. THE MODEL CLASS

Let us fix a splitting $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$ of Euclidean space, together with a splitting of the standard coordinates $x = (t, u)$. The dual coordinates in $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ will be denoted $\xi = (\tau, \mu)$. We denote:

$$Y = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n, \quad \Sigma_0 = \{(t, u = 0; \xi = 0)\} \subset T^*\mathbb{R}^n. \tag{2.1}$$

Σ_0 is an isotropic submanifold of $T^*\mathbb{R}^n$, as it is contained in the zero section.

DEFINITION 2.1. Let r be a half-integer. The class $I^r(\Sigma_0)$ consists of all smooth, \hbar -dependent functions of the form

$$\Upsilon(t, u, \hbar) = a(t, \hbar^{-1/2}u, \hbar) : \mathbb{R}^n \times (0, \hbar_0) \rightarrow \mathbb{C}, \tag{2.2}$$

where $a(t, u, \hbar)$ has an asymptotic expansion as $\hbar \rightarrow 0$ of the form

$$a(t, u, \hbar) \sim \hbar^r \sum_{j=0}^{\infty} a_j(t, u) \hbar^{j/2}, \tag{2.3}$$

where $\forall j, a_j(t, u)$ is a Schwartz function in the u variable with estimates locally uniform in t . More precisely, we assume that for all j ,

$$\begin{aligned} \forall \alpha, \beta, N, \forall K \subset \mathbb{R}^k \text{ compact, } \exists C \text{ such that } \forall (t, u) \in K \times \mathbb{R}^l, \\ |\partial_t^\alpha \partial_u^\beta a_j(t, u)| \leq C (1 + \|u\|)^{-N}, \end{aligned} \tag{2.4}$$

and the expansion (2.3) means that $\forall \alpha, \beta, N, \forall K \subset \mathbb{R}^k$ compact, $\exists C$ such that $\forall (t, u) \in K \times \mathbb{R}^l$,

$$\left| \partial_t^\alpha \partial_u^\beta \left(a(t, u) - \sum_{j=0}^N \hbar^{r+j/2} a_j(t, u) \right) \right| \leq C \hbar^{r+N+1}. \tag{2.5}$$

The following is not hard to prove:

LEMMA 2.2. *The semiclassical wave-front set of $a \in I^r(\Sigma_0)$ is contained in Σ_0 .*

DEFINITION 2.3. *Let $\Upsilon \in I^r(\Sigma_0)$ be as in the previous definition. Then its rough symbol is the function*

$$\begin{aligned} \sigma_\Upsilon : \Sigma_0 &\rightarrow \mathcal{S}(\mathbb{R}^l) \\ (t, 0; 0) &\mapsto a_0(t, \cdot), \end{aligned} \tag{2.6}$$

where $\mathcal{S}(\mathbb{R}^l)$ denotes the class of Schwartz functions on \mathbb{R}^l .

Remark 2.4. We make the following remarks about the symbol, anticipating the general definition: At every point $s \in \Sigma_0$, the symplectic normal space

$$\mathcal{N}_s := (T_s \Sigma_0)^\circ / T_s \Sigma_0 \tag{2.7}$$

can be canonically identified with the (u, μ) symplectic vector space. This vector space inherits a polarization $L_s \subset \mathcal{N}_s$ (a Lagrangian subspace) from the vertical polarization of $T^*\mathbb{R}^n$, namely $L_s = \{u = 0\}$. At every point $s \in \Sigma_0$, $\sigma_\Upsilon(s)$ is to be thought of as a Schwartz function in the quotient $\mathcal{N}_s/L_s \cong \mathbb{R}^l$. To obtain an invariant version of the symbol, in the manifold case, we will have to “decorate” the rough symbol with half-forms, so that the invariant symbol will be an object of the form

$$(t, 0; 0) \mapsto a_0(t, \cdot) (\wedge^{1/2} dt) (\wedge^{1/2} du), \tag{2.8}$$

where we henceforth let

$$\wedge^{1/2} dt = (dt_1 \wedge \dots \wedge dt_k)^{1/2} \quad \text{and} \quad \wedge^{1/2} du = (du_1 \wedge \dots \wedge du_l)^{1/2}. \tag{2.9}$$

2.2. INVARIANCE UNDER Ψ DOS AND THE LOCAL TRANSPORT EQUATIONS

It is clear from the definition that the class $I^r(\Sigma_0)$ is invariant under the action of differential operators that are supported near Y (or whose coefficients have at most polynomial growth in u). We now prove:

THEOREM 2.5. *Let $\Upsilon \in I^r(\Sigma_0)$ and P be a semiclassical pseudodifferential operator of degree d on \mathbb{R}^n whose Schwartz kernel is compactly supported in the u variables. Then,*

$$P(\Upsilon) \in I^{r+d}(\Sigma_0) \quad \text{and} \quad \sigma_{P(\Upsilon)}(s) = \mathfrak{p}(s)\sigma_\Upsilon(s), \tag{2.10}$$

where \mathfrak{p} is the symbol of P .

Proof. For simplicity and the purpose of proving this theorem as well as proving the next two theorems, we write

$$\Upsilon(t, u, \hbar) = \hbar^r a \left(t, \frac{u}{\sqrt{\hbar}}, \hbar \right) = \hbar^r a_0 \left(t, \frac{u}{\sqrt{\hbar}} \right) + \hbar^{r+\frac{1}{2}} a_1 \left(t, \frac{u}{\sqrt{\hbar}} \right) + \hbar^{r+1} a_2 \left(t, \frac{u}{\sqrt{\hbar}} \right)$$

and we assume that the symbol of P is

$$\hbar^d \mathfrak{p}(t, u, \tau, \mu) = \hbar^d p_0(t, u, \tau, \mu) + \hbar^{d+1} p_1(t, u, \tau, \mu).$$

Then,

$$\begin{aligned}
 P(\Upsilon)(t, u) &= \frac{1}{(2\pi\hbar)^n} \iint e^{\frac{i}{\hbar}[(t-\tilde{t})\cdot\tau+(y-\tilde{u})\cdot\mu]} \hbar^d p(t, u, \tau, \mu, \hbar) \hbar^r a\left(\tilde{t}, \frac{\tilde{u}}{\sqrt{\hbar}}, \hbar\right) d\tilde{t}d\tilde{u}d\tau d\mu \\
 &= \frac{\hbar^{d+r}}{(2\pi\hbar)^n} \int e^{\frac{i}{\hbar}(t\cdot\tau+u\cdot\mu)} p(t, u, \tau, \mu, \hbar) \hbar^{\frac{1}{2}} \hat{a}\left(\frac{\tau}{\hbar}, \frac{\mu}{\sqrt{\hbar}}, \hbar\right) d\tau d\mu \\
 &= \frac{\hbar^{d+r}}{(2\pi\hbar)^n} \hbar^{\frac{1}{2}} \hbar^{k+\frac{1}{2}} \int e^{i(t\cdot\tau+\frac{u}{\sqrt{\hbar}}\cdot\mu)} p(t, u, \hbar\tau, \sqrt{\hbar}\mu, \hbar) \hat{a}(\tau, \mu, \hbar) d\tau d\mu \\
 &= \hbar^{d+r} b\left(t, \frac{u}{\sqrt{\hbar}}, \hbar\right),
 \end{aligned}$$

where \hat{a} is the Fourier transform of a , and b is the function

$$b(t, u, \hbar) = \frac{1}{(2\pi)^n} \int e^{i(t\cdot\tau+u\cdot\mu)} p(t, \sqrt{\hbar}u, \hbar\tau, \sqrt{\hbar}\mu, \hbar) \hat{a}(\tau, \mu, \hbar) d\tau d\mu. \tag{2.11}$$

In particular, $b(t, u, 0) = p_0(t, 0, 0, 0)a_0(t, u)$. The conclusion follows. □

If $\mathfrak{p}|_{\Sigma_0} \equiv 0$, then $P(\Upsilon) \in I^{r+d+1/2}(\Sigma_0)$, and one can ask what its symbol is.

THEOREM 2.6. *In the situation of Theorem 2.5, assume that $\mathfrak{p}|_{\Sigma_0} \equiv 0$. Then, $P(\Upsilon) \in I^{r+d+1/2}(\Sigma_0)$ and*

$$\sigma_{P(\Upsilon)}(t, 0; 0)(u) = \left(\sum_{j=1}^l u_j \frac{\partial \mathfrak{p}}{\partial u_j}(t, 0; 0) \right) a_0(t, u) + \frac{1}{i} \sum_{j=1}^l \frac{\partial \mathfrak{p}}{\partial \mu_j}(t, 0; 0) \frac{\partial a_0}{\partial u_j}(t, u). \tag{2.12}$$

Proof. If $p_0(t, 0, 0, 0) = 0$, then

$$p(t, \sqrt{\hbar}u, \hbar\tau, \sqrt{\hbar}\mu, \hbar) = \hbar^{1/2} \sum_{j=1}^l \left(u_j \frac{\partial p_0}{\partial u_j} + \mu_j \frac{\partial p_0}{\partial \mu_j} \right) + O(\hbar).$$

Therefore, (2.11) becomes

$$\begin{aligned}
 b(t, u, \hbar) &= \hbar^{1/2} \int e^{i(t\cdot\tau+u\cdot\mu)} \sum_{j=1}^l \left(u_j \frac{\partial p_0}{\partial u_j} + \mu_j \frac{\partial p_0}{\partial \mu_j} \right) \hat{a}_0(\tau, \mu) d\tau d\mu + O(\hbar) \\
 &= \hbar^{1/2} \int e^{i(t\cdot\tau+u\cdot\mu)} \sum_{j=1}^l \left(u_j \frac{\partial p_0}{\partial u_j} \hat{a}_0(\tau, \mu) + \frac{1}{i} \frac{\partial p_0}{\partial \mu_j} \widehat{\frac{\partial a_0}{\partial u_j}}(\tau, \mu) \right) d\tau d\mu + O(\hbar),
 \end{aligned}$$

where we have used that $\mu_j \hat{a}_0(\tau, \mu) = \frac{1}{i} \widehat{\frac{\partial a_0}{\partial u_j}}(\tau, \mu)$. The functions $\frac{\partial p_0}{\partial u_j}, \frac{\partial p_0}{\partial \mu_j}$ are all evaluated at the point $(t, 0, 0, 0)$. The conclusion follows. □

Remark 2.7. The right-hand side of (2.12) has a good interpretation that carries over to the general case. Since Σ_0 is isotropic, $\mathfrak{p}|_{\Sigma_0} \equiv 0$ implies that the Hamilton

vector field of \mathfrak{p} at $s \in \Sigma_0$, $\Xi_{\mathfrak{p}}(s)$, belongs to the annihilator $(T_s \Sigma_0)^\circ$. It projects to the symplectic normal space, \mathcal{N}_s , and therefore defines a Lie algebra element of the Heisenberg group of \mathcal{N}_s . (2.12) is just the action of this Lie algebra element on $\sigma_\Upsilon(s)$.

One can go still further, under the following assumption:

$$(*) \quad \mathfrak{p}|_{\Sigma_0} \equiv 0 \quad \text{and} \quad \Xi_{\mathfrak{p}} \text{ is tangent to } \Sigma_0.$$

THEOREM 2.8. *In the situation of Theorem 2.5, assume further that (*) holds. Then, $P(\Upsilon) \in I^{r+d+1}(\Sigma_0)$ and its symbol is*

$$\begin{aligned} \sigma_{P(\Upsilon)}(s)(u) &= p_1(s)a_0(t, u) + \frac{1}{\sqrt{-1}} \sum \frac{\partial p_0}{\partial \tau_j}(s) \frac{\partial a_0}{\partial t_j}(t, u) \\ &\quad + \frac{1}{2} \sum \left(\frac{\partial^2 p_0}{\partial u_i \partial u_j}(s) a_0(t, u) u_i u_j + \frac{2}{\sqrt{-1}} \frac{\partial^2 p_0}{\partial u_i \partial \mu_j}(s) \frac{\partial a_0}{\partial u_j}(t, u) u_i \right. \\ &\quad \left. - \frac{\partial^2 p_0}{\partial \mu_i \partial \mu_j}(s) \frac{\partial^2 a_0}{\partial u_i \partial u_j}(t, u) \right). \end{aligned} \tag{2.13}$$

Proof. The fact that Ξ_{p_0} is tangent to Σ_0 implies that at $s = (t, 0, 0, 0)$,

$$\frac{\partial p_0}{\partial u_j}(s) = \frac{\partial p_0}{\partial \mu_j}(s) = 0, \quad 1 \leq j \leq l.$$

So modulo terms of $o(\hbar)$, one has

$$\begin{aligned} p(t, \sqrt{\hbar}u, \hbar\tau, \sqrt{\hbar}\mu) &= \hbar p_1(s) + \hbar \sum \frac{\partial p_0}{\partial \tau_j}(s) \tau_j \\ &\quad + \frac{\hbar}{2} \sum \left(\frac{\partial^2 p_0}{\partial u_i \partial u_j}(s) u_i u_j + \frac{\partial^2 p_0}{\partial \mu_i \partial \mu_j}(s) \mu_i \mu_j + 2 \frac{\partial^2 p_0}{\partial u_i \partial \mu_j}(s) u_i \mu_j \right). \end{aligned}$$

Substituting this into (2.11), the conclusion follows. □

Remark 2.9. Once again, the right-hand side of (2.13) has an interesting interpretation see Theorem 3.12. For now, we simply point out that the second line in (2.13) is the action of the infinitesimal metaplectic representation of the Hessian of p_0 with respect to the (u, μ) variables, on the function $a_0(t, \cdot)$.

2.3. INVARIANCE UNDER FIOS PRESERVING Σ_0

In this section, we prove that the model classes $I^\bullet(\Sigma_0)$ are invariant under the action of zeroth-order semiclassical Fourier integral operators associated with (not necessarily homogeneous) canonical transformations $f : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ that preserve

Σ_0 as a set. By transplantation, this will allow us to define isotropic states on manifolds. In the next section, we will study how the (rough) symbols transform under the action of FIOs.

We state our main theorem:

THEOREM 2.10. *If $\gamma : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ is a symplectomorphism mapping Σ_0 into Σ_0 and F_γ , a semiclassical Fourier integral operator quantizing γ , then F maps $I(\Sigma_0)$ into $I(\Sigma_0)$.*

Note that, since we already know that the classes $I^\bullet(\Sigma_0)$ are invariant under Ψ DOs, without loss of generality, we will only consider FIOs preserving Σ_0 and having an amplitude identically equal to one.

To prove Theorem 2.10, we shall first study a number of special cases and then see that the general case is a combination of these special cases.

2.3.1. *Invariance Under FIOs Associated with Lifted Diffeomorphisms*

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism which maps Y onto itself. Then, f lifts to a symplectomorphism

$$\gamma_f : (x, \xi) \mapsto (y, \eta), \quad y = f(x), \xi = df_x^* \eta \tag{2.14}$$

which maps Σ_0 onto itself and γ_f is quantized by the pull-back map,

$$f^* : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n).$$

We first observe

LEMMA 2.11. *The pull-back operator f^* maps $I^r(\Sigma_0)$ onto itself.*

Proof. If we write $f(t, u) = (\tilde{t}, \tilde{u}) = (f_1(t, u), f_2(t, u))$, then the condition $f(Y) \subset Y$ implies that $f_2(t, u) = g_2(t, u)u$. As a consequence,

$$f^* \Upsilon(t, u, \hbar) = a \left(f_1(t, u), \frac{f_2(t, u)}{\sqrt{\hbar}}, \hbar \right) = a \left(f_1(t, u), g_2(t, u) \frac{u}{\sqrt{\hbar}}, \hbar \right)$$

is an element in $I^r(\Sigma_0)$. □

2.3.2. *Invariance Under FIOs Associated with Symplectomorphisms Fixing the zero Section*

Next, we will consider semiclassical FIOs for which the underlying canonical transformation $\gamma : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ maps the zero section onto itself identically. Since the zero section is carried out to $T_0^*\mathbb{R}^n$ by the symplectomorphism $(x, \xi) \mapsto (-\xi, x)$, it suffices to characterize all symplectomorphisms γ which carry the zero section in $T^*\mathbb{R}^n$ identically to $T_0^*\mathbb{R}^n$.

LEMMA 2.12. *If $\gamma : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ is a symplectomorphism mapping the cotangent space $T_0^*\mathbb{R}^n = \mathbb{R}^n$ identically onto the zero section, \mathbb{R}^n in $T^*\mathbb{R}^n$, then it has to be horizontal, i.e., defined by a generating function, $\phi(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, with the property*

$$\gamma(x, \xi) = (y, \eta) \quad \text{iff} \quad \xi = -\frac{\partial\phi}{\partial x}, \quad \eta = \frac{\partial\phi}{\partial y}.$$

Moreover, ϕ has to be of the form

$$\phi(x, y, \hbar) = x \cdot y + \sum \varphi_{i,j}(x, y, \hbar) y_i y_j$$

and the Fourier integral operator quantizing γ has to have a Schwartz kernel of the form

$$e^{\frac{i}{\hbar}(x \cdot y + \sum \varphi_{i,j}(x, y, \hbar) y_i y_j)} a(x, y, \hbar). \tag{2.15}$$

Proof. ‘‘Horizontal’’ is equivalent to the condition that the graph, Γ , of γ is nowhere vertical, or alternatively that for any pair $p, q \in M = T^*\mathbb{R}^n$ with $q = \gamma(p)$, $d\gamma_p : T_p M \rightarrow T_q M$ does not map vectors $v \neq 0$ tangent to the cotangent fiber of M at p onto vectors, $w = d\gamma_p(v)$ tangent to the cotangent fiber of M at q . However in a neighborhood of the zero section in M , this is ruled out by the condition that γ maps the zero section \mathbb{R}^n of M onto $T_0^*\mathbb{R}^n$.

Now, assume that Γ is horizontal, with generating function ϕ . Let γ be the corresponding symplectomorphism. We would like to find the conditions so that γ maps the zero section onto $T_0^*\mathbb{R}^n$. Clearly, this is the case if and only if for all $x \in \mathbb{R}^n$,

$$\frac{\partial}{\partial x} \phi(x, 0) = 0, \tag{2.16}$$

or equivalently if $\phi(x, 0)$ is a constant function of x , and without loss of generality we can assume that this constant function is zero, i.e.,

$$\phi(x, y) = \sum y_i \phi_i(x) + \sum y_i y_j \phi_{i,j}(x, y). \tag{2.17}$$

Moreover, for $\phi(x, y)$ to be a generating function of a symplectomorphism, we need to require that the matrix

$$\left[\frac{\partial^2 \phi}{\partial x_i \partial y_j}(x, y) \right] \tag{2.18}$$

is everywhere of rank n , and hence in particular that $\left[\frac{\partial \phi_i}{\partial x_j}(x) \right]$ is invertible. Thus modulo a change of variables (which, as we have seen, will not change the class $I(\Sigma)$), we can assume that

$$\phi(x, y) = \sum x_i y_i + \sum y_i y_j \phi_{i,j}(x, y, \hbar). \tag{2.19}$$

Hence, the F.I.O. quantizing γ is the operator

$$A_\gamma f(y) = \int e^{\frac{i}{\hbar}(x \cdot y + \sum y_i y_j \phi_{ij}(x, y, \hbar))} a(x, y, \hbar) f(x) dx. \quad (2.20)$$

□

To obtain the general transformation preserving the zero section, it suffices to pre-compose the γ s alluded above with the transformation $J : (x, \xi) \mapsto (-\xi, x)$, which is associated with the semiclassical Fourier transform. So, we need

LEMMA 2.13. *Let $\Sigma_1 = \{(x=0; \tau, \mu=0)\} \subset T_0^* \mathbb{R}^n$ and let $I^r(\Sigma_1)$ denote the image of $I^r(\Sigma_0)$ under the semiclassical Fourier transform. Then the elements of $I^r(\Sigma_1)$ are precisely the functions of the form*

$$b\left(\hbar^{-1}t, \hbar^{-1/2}u, \hbar\right) \quad (2.21)$$

where b has an asymptotic expansion as before.

Proof. Let $\Upsilon(t, u, \hbar) = a(t, \hbar^{-1/2}u, \hbar)$ be an element in $I^r(\Sigma_0)$. Then its semiclassical Fourier transform is

$$\begin{aligned} \mathcal{F}_\hbar \Upsilon(\tau, \mu) &= \int e^{-\frac{i}{\hbar}(t \cdot \tau + u \cdot \mu)} a\left(t, \frac{u}{\sqrt{\hbar}}, \hbar\right) dt du \\ &= \int e^{-i\left(t \cdot \frac{\tau}{\hbar} + \frac{u}{\sqrt{\hbar}} \cdot \frac{\mu}{\sqrt{\hbar}}\right)} a\left(t, \frac{u}{\sqrt{\hbar}}, \hbar\right) dt du, \end{aligned}$$

which can be written in the form $b(\hbar^{-1}\tau, \hbar^{-1/2}\mu, \hbar)$, where

$$b(\tau, \mu, \hbar) = \int e^{-i\left(t \cdot \tau + \frac{u}{\sqrt{\hbar}} \cdot \mu\right)} a\left(t, \frac{u}{\sqrt{\hbar}}, \hbar\right) dt du$$

is the Fourier transform of a , and thus has an asymptotic expansion as in (2.3). □

Now, suppose $b\left(\frac{t}{\hbar}, \frac{u}{\sqrt{\hbar}}, \hbar\right)$ is an isotropic function on \mathbb{R}^n with microsupport on the subset $\mu=0$ of $T_0^* \mathbb{R}^n$ and framed by $T_0^* \mathbb{R}^n$. Let $(Fb)(t, u, \hbar)$ be equal to

$$\begin{aligned} &\int e^{\frac{i}{\hbar}(t \cdot \tilde{t} + u \cdot \tilde{u} + \sum \tilde{t}_i \tilde{t}_j \phi_{ij}(t, u, \tilde{t}, \tilde{u}, \hbar) + \sum \tilde{u}_i \tilde{u}_j \phi_{ij}(t, u, \tilde{t}, \tilde{u}, \hbar) + \sum \tilde{t}_i \tilde{u}_j \psi_{ij}(t, u, \tilde{t}, \tilde{u}, \hbar))} \\ &a\left(t, u, \tilde{t}, \tilde{u}, \hbar\right) b\left(\frac{\tilde{t}}{\hbar}, \frac{\tilde{u}}{\sqrt{\hbar}}, \hbar\right) d\tilde{t} d\tilde{u}. \end{aligned} \quad (2.22)$$

Replacing in this integral \tilde{t}_i by $\hbar \tilde{t}_i$ and \tilde{u}_i by $\sqrt{\hbar} \tilde{u}_i$, it becomes $\hbar^{k+\frac{1}{2}} g\left(t, \frac{u}{\sqrt{\hbar}}, \hbar\right)$, where

$$\begin{aligned} g(t, u, \hbar) &= \int e^{i\left((t, u) \cdot (\tilde{t}, \tilde{u}) + \sum \tilde{u}_i \tilde{u}_j \tilde{\phi}_{ij} + \hbar \sum \tilde{t}_i \tilde{t}_j \tilde{\phi}_{ij} + \sqrt{\hbar} \sum \tilde{t}_i \tilde{u}_j \tilde{\psi}_{ij}\right)} \\ &a\left(t, u, \hbar \tilde{t}, \tilde{u}, \hbar\right) b\left(\tilde{t}, \tilde{u}, \hbar\right) d\tilde{t} d\tilde{u}, \end{aligned} \quad (2.23)$$

with

$$\tilde{\phi}_{ij}(t, u, \tilde{t}, \tilde{u}, \hbar) = \phi_{ij}\left(t, \sqrt{\hbar}u, \hbar\tilde{t}, \sqrt{\hbar}\tilde{u}, \hbar\right)$$

and likewise for $\tilde{\phi}_{ij}$ and $\tilde{\psi}_{ij}$. Note that in particular,

$$g(t, u, 0) = a(t, u, 0, 0, 0) \int e^{i((t,u)\cdot(\tilde{t},\tilde{u}) + \sum \tilde{u}_i \tilde{u}_j \phi_{ij}(t,0))} b(\tilde{t}, \tilde{u}, 0) d\tilde{t}d\tilde{u}. \tag{2.24}$$

Let us next compose the operator (2.22) with the semiclassical Fourier transform

$$\Upsilon = b\left(\tilde{t}, \frac{\tilde{u}}{\sqrt{\hbar}}\right) \mapsto \hat{b}\left(\frac{\tilde{t}}{\hbar}, \frac{\tilde{u}}{\sqrt{\hbar}}\right). \tag{2.25}$$

This gives us an operator

$$\Upsilon \mapsto \int e^{\frac{i}{\hbar}((t,u)\cdot(\tilde{t},\tilde{u}) + \sum \tilde{t}_i \tilde{t}_j \phi_{ij} + \sum \tilde{u}_i \tilde{u}_j \phi_{ij} + \sum \tilde{t}_i \tilde{u}_j \psi_{ij})} a(t, u, \tilde{t}, \tilde{u}, \hbar) \hat{b}\left(\frac{\tilde{t}}{\hbar}, \frac{\tilde{u}}{\sqrt{\hbar}}\right) d\tilde{t}d\tilde{u}, \tag{2.26}$$

and by the Lemma 2.12 above this operator is the quantization of a symplectomorphism $\gamma : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ mapping the zero section identically onto itself. Moreover, the operator (2.26) maps the space of functions $I^\bullet(\Sigma_0)$ onto itself when Σ_0 is the subset $\tilde{u} = 0$ of the zero section of $T^*\mathbb{R}^n$ (by the calculation we have just made). Also, note that if we set $\hbar = 0$, we get by the formulae (2.23) and (2.24) the expression

$$\hbar^{k+\frac{l}{2}} g(t, u, 0) = a(t, u, 0, 0, 0) \int e^{i(t,u)\cdot(\tilde{t},\tilde{u})} e^{i \sum \tilde{u}_i \tilde{u}_j \phi_{ij}(t,0)} \hat{b}(\tilde{t}, \tilde{u}, 0) d\tilde{t}d\tilde{u} \tag{2.27}$$

for the leading term in (2.26). In other words for t fixed, the function

$$\sigma_t(g)(u) := g(t, u, 0)$$

is the function

$$\sigma_t(a)(F_2^{-1} e^{i \sum \lambda_{ij}^t u_i u_j} F_2) \sigma_t(b), \quad \sigma_t(f) = f(t, u, 0), \tag{2.28}$$

where F_2 is the Fourier transform $h(u) \mapsto \int e^{-iu\tilde{u}} h(u) du$ and λ_{ij}^t is the constant $\phi_{ij}(t, 0)$.

2.3.3. Invariance Under FIOs Associated with Fibre-Preserving Symplectomorphisms

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ function with the property

$$\phi(t, 0) = \frac{\partial \phi}{\partial x}(t, 0) = 0. \tag{2.29}$$

Then the symplectomorphism

$$\gamma_\phi : (x, \xi) \mapsto (x, \xi + d\phi(x)) \tag{2.30}$$

preserves Σ_0 and maps the zero section in $T^*\mathbb{R}^n$ onto the Lagrangian submanifold

$$\Lambda_\phi = \{(x, d\phi(x)) : x \in \mathbb{R}^n\}. \tag{2.31}$$

Moreover, γ_ϕ has a natural quantization, the semi-classical Fourier integral operator

$$T_\phi f(x) = e^{\frac{i\phi}{\hbar}} f(x). \tag{2.32}$$

Claim: This operator preserves $I^\bullet(\Sigma_0)$.

Proof. Given $\Upsilon = a(t, \frac{u}{\sqrt{\hbar}}, \hbar) \in I(\Sigma_0)$, let $\Upsilon_1 = e^{\frac{i\phi}{\hbar}} \Upsilon$. By (2.29),

$$\phi(t, u) = \sum \frac{u_r}{\hbar^{1/2}} \frac{u_s}{\hbar^{1/2}} \psi_{r,s} \left(t, \hbar^{1/2} \frac{u}{\hbar^{1/2}} \right) = \psi \left(t, \frac{u}{\hbar^{1/2}}, \hbar \right);$$

hence,

$$\Upsilon_1 = \Upsilon_2 \left(t, \frac{u}{\hbar^{1/2}}, \hbar \right) \tag{2.33}$$

where $\Upsilon_2(t, u, \hbar) = e^{i \sum u_r u_s \psi_{r,s}(t, \hbar^{1/2} u)} a(t, u, \hbar)$ and hence is in $I(\Sigma_0)$. □

We also note for future reference that

$$\Upsilon_1(t, u, 0) = e^{i \sum u_r u_s \psi_{r,s}(t, 0)} a(t, u, 0). \tag{2.34}$$

2.3.4. Invariance Under Partial Fourier Transforms

Let $T^*\mathbb{R}^n = T^*\mathbb{R}^k \times T^*\mathbb{R}^l$ and let $\gamma : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ be the symplectomorphism which is equal to the identity on $T^*\mathbb{R}^k$ and the map, $(u, \eta) \mapsto (-\eta, u)$ on $T^*\mathbb{R}^l$. This symplectomorphism maps the zero section, Σ_0 in $T^*\mathbb{R}^k$ identically onto itself and maps the zero section of $T^*\mathbb{R}^n$ onto the conormal bundle of Σ_0 in $T^*\mathbb{R}^n$. Moreover, its quantization is the semiclassical Fourier transform

$$f(t, u) \mapsto \int e^{\frac{i u \mu}{\hbar}} f(t, \tilde{u}) d\tilde{u} \tag{2.35}$$

in the variable u with t held fixed. In particular, it maps $\Upsilon(t, u, \hbar) = a(t, \frac{u}{\sqrt{\hbar}}, \hbar)$ in $I^0(\Sigma_0)$ onto

$$\Upsilon_1(t, u, \hbar) = \hbar^{\frac{l}{2}} \hat{a} \left(t, \frac{u}{\sqrt{\hbar}}, \hbar \right) \tag{2.36}$$

in $I^{\frac{l}{2}}(\Sigma_0)$ where $\hat{a}(t, u, \hbar)$ is the classical Fourier transform of a in the variable u with \hbar and t held fixed.

We note for future reference that if $\sigma_t(\Upsilon)(u) = a(t, u, 0)$ and $\sigma_t(\Upsilon_1)(u) = \hat{a}(t, u, 0)$, then

$$\sigma_t(\Upsilon_1) = \int e^{iu \cdot \mu} \sigma_t(\mu) d\mu. \tag{2.37}$$

2.3.5. *Decomposition of Symplectomorphisms Preserving Σ_0*

We first describe the linear symplectomorphisms which preserves Σ_0 . If $A : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ is a linear symplectomorphism and is the identity on Σ_0 , then it is the identity on $T^*\mathbb{R}^n / \Sigma_0^o$ and hence is basically a linear symplectomorphism of $\Sigma_0^o / \Sigma_0 = T^*\mathbb{R}^l$. As above, let $(u_1, \dots, u_l, \mu_1, \dots, \mu_l)$ be cotangent coordinates on $T^*\mathbb{R}^l$. Then by a standard theorem in symplectic linear algebra, the group of linear symplectomorphisms of $T^*\mathbb{R}^l$ is generated by linear mappings of type I, II and III, i.e., linear maps of the form

$$\begin{pmatrix} B & 0 \\ 0 & (B^t)^{-1} \end{pmatrix}, \tag{2.38}$$

$$\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}, \tag{2.39}$$

and

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.40}$$

We will prove that

THEOREM 2.14. *If $\gamma : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ is a symplectomorphism which maps Σ_0 onto Σ_0 and is the identity on Σ_0 , then it is a composition*

$$\gamma = A\gamma_\phi\gamma_f\gamma_0, \tag{2.41}$$

where γ_0 is a symplectomorphism whose restriction to the zero section of $T^*\mathbb{R}^n$ is the identity, γ_f is the symplectomorphism (2.14), γ_ϕ is the symplectomorphism (2.30) and A a linear symplectomorphism whose restriction to Σ_0 is the identity.

Proof. Let $M = T^*\mathbb{R}^n$ and let $(TM)_{\text{vert}}$ be the vertical subbundle of TM , i.e., for each $p \in M$ the vectors in $(T_pM)_{\text{vert}}$ are vectors which are tangent at p to the fiber at p of the projection, $T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$. Given $A \in Sp(\mathbb{R}^{2l})$, we get from $A^{-1}\gamma$, by restriction to the zero section, \mathbb{R}^n , of M a map, $f_A : \mathbb{R}^n \rightarrow M$, a map, $df_A : T\mathbb{R}^n \rightarrow TM$, and as we vary A , a map

$$df : Sp(\mathbb{R}^{2l}) \times T\mathbb{R}^n \rightarrow TM. \tag{2.42}$$

It is easy to see that this map is transversal to $(TM)_{\text{vert}}$, and hence, by Thom transversality, there exists an $A \in Sp(\mathbb{R}^{2l})$ such that df_A is transversal to $(TM)_{\text{vert}}$,

i.e., the graph of $f_A : \mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ is horizontal. In particular, its image is a horizontal Lagrangian submanifold of M of the form $\text{Image}(\gamma_\phi|_{\mathbb{R}^n})$, where γ_ϕ is the symplectomorphism (2.30). Thus,

$$\gamma = A\gamma_\phi\gamma_f\gamma_0, \tag{2.43}$$

where γ_f is the symplectomorphism (2.14), γ_ϕ the symplectomorphism (2.30) and γ_0 a symplectomorphism which is the identity on the zero section of $T^*\mathbb{R}^n$. \square

Finally, we note that if A is a linear symplectomorphism of the form (2.38), it is a symplectomorphism of type γ_f ; if it is of the form (2.39), it is of type γ_ϕ ; and if it is of the form (2.40), its quantization is the partial Fourier transform (2.35).

2.3.6. Proof of Theorem 2.10

By Theorem 2.14, it suffices to prove this for the quantization (2.32) of γ_ϕ , the quantization f^* of γ_f , the quantization (2.35) of the linear symplectomorphism (2.40) and all quantizations of γ_0 . However, for the quantizations of γ_ϕ , γ_f and the symplectomorphism (2.40) that we have just alluded to, this assertion follows from Lemma 2.12. \square

2.4. TRANSFORMATION OF THE SYMBOLS UNDER FIOS PRESERVING Σ_0

Given an element, $\Upsilon = \hbar^l a(t, \frac{u}{\sqrt{\hbar}}, \hbar)$ of $I^l(\Sigma_0)$ we have defined its symbol

$$\sigma(\Upsilon)_t \in \mathcal{S}(\mathbb{R}^l)$$

at $t \in \Sigma_0$ to be the Schwartz function

$$\sigma(\Upsilon)(t) = a_t(u) := a(t, u, 0).$$

Next, we will discuss the “symbolic calculus” of these symbols, i.e., describe how they transform under composition by the FIOs in theorem 2.10. In view of the factorization, $\gamma = A\gamma_\phi\gamma_f\gamma_0$ in Theorem 2.14, it suffices to describe how they transform for the FIOs (2.26), (2.32) and (2.35) and for the pull-back map

$$f^* : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$$

quantizing γ_f . However, for the FIO (2.26), we showed that symbols transform by the formula (2.28), for the FIO (2.32) by the formula (2.34) and for the FIO (2.35) by the formula (2.37). Moreover, in addition it is easy to check that for the pull-back map f^* , symbols transform by the recipe

$$\sigma_t(f^*\Upsilon)(u) = \sigma_{f(t)}(\Upsilon)((df_t)^{-1}(u)). \tag{2.44}$$

These formulas, by the way, have a good abstract interpretation which we will discuss later in this paper. Namely, if $\gamma : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ is a symplectomorphism whose

restriction to Σ_0 is the identity, then for $t \in \Sigma_0$ the linear map $(d\gamma)_t$ restricts to a linear symplectomorphism of the symplectic normal bundle to Σ_0 at t , or, in other words, an element $(L_\gamma)_t$ of $Sp(\mathbb{R}^{2l})$; and for the Fourier integral operator, F_γ , quantizing γ in each of the cases above,

$$\sigma_t(F\Upsilon) = (L_\gamma)_t^\# \sigma_t(\Upsilon), \tag{2.45}$$

where $(L_\gamma)_t^\#$ is the metaplectic representation of $(L_\gamma)_t$ on the Schwartz space $\mathcal{S}(\mathbb{R}^l)$.

3. Global Theory

We begin with the global definition on manifolds. We will keep the notation of §2 for the model spaces $I^r(\Sigma_0)$.

DEFINITION 3.1. Let M be a smooth manifold of dimension n , and $\Sigma \subset T^*M$ an isotropic submanifold of its cotangent bundle of codimension $n + l$. Let r be a half-integer. Then the space $I^r(\Sigma)$ is defined as the set of all \hbar -dependent functions, $\Upsilon : M \times (0, \hbar_0) \rightarrow \mathbb{C}$, with wave-front set contained in Σ , such that there exists a microlocal partition of unity $\{\chi_\ell\}$ of a neighborhood of Σ , and zeroth-order semiclassical Fourier integral operators F_ℓ , from some open sets $\mathcal{U}_\ell \subset \mathbb{R}^n$ to M , such that

$$\chi_\ell(\Upsilon) = F_\ell(\Upsilon_\ell) + O(\hbar^\infty), \tag{3.1}$$

where $\Upsilon_\ell \in I^r(\Sigma_0)$ is supported in \mathcal{U}_ℓ and F_ℓ is associated with a canonical transformation mapping $\Sigma_0 \cap T^*\mathcal{U}_\ell$ diffeomorphically onto a relative open set in Σ .

The rest of this section is devoted to the global definition of the symbol of an element in $I^r(\Sigma)$, and to the symbol calculus under the action of pseudodifferential operators, including the transport equations.

3.1. THE METAPLECTIC REPRESENTATION À LA BLATTNER–KOSTANT–STERNBERG

A special case of the work of Blattner, Kostant and Sternberg on geometric quantization and representation theory is a construction of the metaplectic representation that we now review. For the original exposition, see [1] and [2]. This material will play a crucial role in the definition of the intrinsic symbol of an isotropic state.

We begin by recalling that one can quantize a symplectic vector space (V, ω) by choosing a metaplectic structure on it and a Lagrangian subspace $L \subset V$. The result is a Hilbert space, \mathcal{H}_L . Its elements are sections of the (trivial) pre-quantum bundle of V tensored with half-forms transverse to the translates of L , which are covariantly constant and square-integrable over the quotient V/L . The point we want to underline here is that the construction is covariant with respect to metaplectic linear maps. More precisely, if V' is another metaplectic vector space,

$L' \subset V'$ a Lagrangian submanifold, and $g: V \rightarrow V'$ a metaplectic isomorphism mapping L to L' , then the inverse of the natural pull-back operator induces a unitary operator

$$U_L^g: \mathcal{H}_L \rightarrow \mathcal{H}_{L'}. \tag{3.2}$$

In particular, if $g \in \text{Mp}(V, \omega)$ (the metaplectic automorphisms of (V, ω)), then the action of g on V induces a unitary operator

$$U_L^g: \mathcal{H}_L \rightarrow \mathcal{H}_{g(L)}. \tag{3.3}$$

It is evident that one has the cocycle condition

$$U_{g(L)}^{g'} \circ U_L^g = U_L^{g'g}. \tag{3.4}$$

With this natural construction at hand, the key ingredient needed in the construction of the metaplectic representation is the BKS pairing: Given two lagrangian subspaces $L, L' \subset V$, there is a sesquilinear pairing

$$(\cdot, \cdot): \mathcal{H}_L \times \mathcal{H}_{L'} \rightarrow \mathbb{C} \tag{3.5}$$

which in fact corresponds to a unitary operator

$$V_{L',L}: \mathcal{H}_L \rightarrow \mathcal{H}_{L'} \tag{3.6}$$

in the sense that

$$\forall \psi \in \mathcal{H}_L, \psi' \in \mathcal{H}_{L'} \quad (\psi, \psi') = \langle V_{L',L}(\psi), \psi' \rangle_{\mathcal{H}_{L'}}. \tag{3.7}$$

These unitary operators also satisfy a cocycle condition:

$$V_{L'',L'} \circ V_{L',L} = V_{L'',L}. \tag{3.8}$$

In addition, the pairing and the action of Mp satisfy a naturality condition: Given $L, L' \subset V$ Lagrangians, and $g \in \text{Mp}$, it is clear from the definitions that

$$\forall \psi \in \mathcal{H}_L, \psi' \in \mathcal{H}_{L'} \quad (U_L^g(\psi), U_{L'}^g(\psi')) = (\psi, \psi'), \tag{3.9}$$

and from this it follows that the following diagram commutes as:

$$\begin{array}{ccc} \mathcal{H}_L & \xrightarrow{V_{L',L}} & \mathcal{H}_{L'} \\ U_L^g \downarrow & & \downarrow U_{L'}^g \\ \mathcal{H}_{g(L)} & \xrightarrow{V_{g(L'),g(L)}} & \mathcal{H}_{g(L')} \end{array} \tag{3.10}$$

These two constructions together give the metaplectic representation:

DEFINITION 3.2. For every $g \in \text{Mp}(V, \omega)$ and a Lagrangian subspace $L \subset V$, define

$$\text{Mp}_L(g): \mathcal{H}_L \rightarrow \mathcal{H}_L, \quad \text{Mp}_L(g) = V_{L,g(L)} \circ U_L^g. \tag{3.11}$$

LEMMA 3.3. *With the previous notations, one has: $\text{Mp}_L(g') \circ \text{Mp}_L(g) = \text{Mp}_L(g'g)$.*

Proof. (Sketch.) Consider the commutative diagram:

$$\begin{array}{ccc}
 \mathcal{H}_{g(L)} & \xrightarrow{V_{L,g(L)}} & \mathcal{H}_L \\
 U_{g(L)}^{g'} \downarrow & & \downarrow U_L^{g'} \\
 \mathcal{H}_{g'g(L)} & \xrightarrow{V_{g'(L),g'g(L)}} & \mathcal{H}_{g'(L)}
 \end{array} \tag{3.12}$$

and use it to flip the middle U and V in the composition $\text{Mp}_L(g') \circ \text{Mp}_L(g)$. Then use the cocycle conditions. □

It is known that Mp_L is the metaplectic representation.

Note that if $L, \Lambda \subset V$ are transverse Lagrangians, then there is an isomorphism

$$\mathcal{H}_L \cong L^2(\Lambda), \tag{3.13}$$

where the right-hand side is the Hilbert space of half forms on Λ with respect to the metalinear structure on Λ inherited from the metaplectic structure on V . Thus, we obtain the metaplectic representation on $L^2(\Lambda)$, arising from (3.13) and the representation Mp_L of $\text{Mp}(V)$ on \mathcal{H}_L . However, if L' is another Lagrangian subspace transverse to Λ , then $\text{Mp}_{L'}$ induces a *different* (though of course isomorphic) metaplectic representation of the same group, $\text{Mp}(V)$, on $L^2(\Lambda)$.

We can use the previous results to define *an abstract Hilbert space associated with the metaplectic vector space V* , as follows:

DEFINITION 3.4. If V is a metaplectic vector space, define

$$\mathcal{H}^V = \{(L, \psi); L \subset V \text{ Lagrangian subspace and } \psi \in \mathcal{H}_L\} / \sim, \tag{3.14}$$

where

$$(L, \psi) \sim (L', \psi') \iff \psi' = V_{L',L}(\psi), \tag{3.15}$$

with the norm

$$\|[L, \psi]\| = \|\psi\|_{\mathcal{H}_L}. \tag{3.16}$$

We will denote by

$$\mathcal{S}^V \subset \mathcal{H}^V \tag{3.17}$$

the image of the space of smooth vectors of any \mathcal{H}_L (under the Heisenberg representation).

One can easily check the following:

LEMMA 3.5. *Let V be a metaplectic vector space, $\Psi = [(L, \psi)] \in \mathcal{H}_V$ and $g \in \text{Mp}(V)$. Then the element*

$$\text{Mp}(g)(\Psi) := [(L, \text{Mp}_L(\psi))] \in \mathcal{H}_V \tag{3.18}$$

is well defined, and $g \mapsto \text{Mp}(g)$ is the metaplectic representation of $\text{Mp}(V)$ on the abstract space \mathcal{H}_V .

By a slight generalization of (3.12), we get that a metaplectic map $f : V \rightarrow V'$ induces a unitary operator

$$U^f : \mathcal{H}^V \rightarrow \mathcal{H}^{V'} \tag{3.19}$$

by choosing any $L \subset V$ and considering $U_L^f : \mathcal{H}_L^V \rightarrow \mathcal{H}_{L'}^{V'}$ with $L' = f(L)$.

In what follows, we will also need the representation of the Heisenberg group of V on the abstract Hilbert space \mathcal{H}^V . It is known that, for each polarization $L \subset V$, there is a unique (up to isomorphism) representation of the Heisenberg group on \mathcal{H}^V such that the Lie algebra element $(0, 1) \in V \oplus \mathbb{R}$ acts as multiplication by $\sqrt{-1}$. It is also known that the metaplectic representation intertwines these representations (for different choices of L); therefore, the Heisenberg representation is well defined on the abstract Hilbert space \mathcal{H}^V .

3.2. SYMBOLS OF ISOTROPIC STATES

Let M be a smooth manifold of dimension n , and $\Sigma \subset T^*M$ an isotropic submanifold of codimension $n+l$. We will denote by $\mathcal{N}^\Sigma \rightarrow \Sigma$ the vector bundle (of rank $2l$) whose fiber at $s \in \Sigma$ is the symplectic normal vector space

$$\mathcal{N}_s^\Sigma := (T_s \Sigma)^\circ / T_s \Sigma. \tag{3.20}$$

To have a global notion of the symbol of an element $\Upsilon \in I^r(\Sigma)$, we need to assume that $\mathcal{N}^\Sigma \rightarrow \Sigma$ has a metaplectic structure and choose one such structure. We will proceed henceforth under this assumption.

We pick once and for all a metaplectic structure on \mathbb{R}^{2n} , which induces a metaplectic structure on the symplectic normals $\mathcal{N}_s^{\Sigma_0}$.

If V is a metaplectic vector space, $\wedge^{1/2}V$ will denote the one-dimensional space of half-forms on V , that is, functions ψ on the space of metaplectic frames m of V that transform according to the rule $\psi(g \cdot m) = \det^{1/2}(g)\psi(m)$, for all metaplectic linear maps g .

We first define the symbol of a model state:

DEFINITION 3.6. Let $\Upsilon \in I^r(\Sigma_0)$ be given by Equation (2.2). Then:

(1) The *model symbol* of Υ at $s = (t, 0; 0, 0) \in \Sigma_0$ is

$$\tilde{\sigma}_\Upsilon(s) = a_0(t, \cdot) (dt_1 \wedge \cdots \wedge dt_k)^{1/2} (du_1 \wedge \cdots \wedge du_l)^{1/2} \in \wedge^{1/2} \mathbb{R}^k \otimes \mathcal{S}(\mathbb{R}^l), \tag{3.21}$$

where $\mathcal{S}(\mathbb{R}^l)$ now denotes the space of Schwartz half-forms on \mathbb{R}^l .

(2) Note that, in a canonical way, for any $s \in \Sigma_0$,

$$\mathcal{N}_s^{\Sigma_0} \cong \mathbb{R}^{2l}. \tag{3.22}$$

In particular, there is a canonical polarization (lagrangian subspace) in all normal spaces, $L_0 \subset \mathcal{N}_s^{\Sigma_0}$, arising from the vertical polarization of $T^*\mathbb{R}^{2n}$, which gives us an isomorphism

$$\mathcal{H}_{L_0}^{\mathcal{N}_s^{\Sigma_0}} = L^2(\mathbb{R}^l), \tag{3.23}$$

with $L^2(\mathbb{R}^l)$ denoting now the space of square integrable half-forms on \mathbb{R}^l .

We define the *symbol* of Υ at s to be the element

$$\sigma_\Upsilon(s) \in \mathcal{S}(\mathcal{N}_s^{\Sigma_0}) \otimes \wedge^{1/2} T_{(t,0)} \Sigma_0, \tag{3.24}$$

represented by $\tilde{\sigma}_\Upsilon(s)$, in the abstract space of smooth vectors of the quantization of the symplectic normal.

In the manifold case, the symbols will be “transplanted” from the model case by Fourier integral operators. That this is possible follows from the following Lemma, which in fact we have already proved in Section 2:

LEMMA 3.7. *Let $\Upsilon \in I'(\Sigma_0)$ and F be a Fourier integral operator from \mathbb{R}^n to itself associated with a transformation $f : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ that preserves Σ_0 (set-wise). Let $s \in \Sigma$, and let*

$$\varphi : \mathcal{N}_s^{\Sigma_0} \rightarrow \mathcal{N}_{f(s)}^{\Sigma_0} \tag{3.25}$$

be the symplectomorphism induced by the differential df_s . Assume that it lifts to an Mp map, so that we have a metaplectic operator

$$\mathbf{Mp}(\varphi) : L^2(\mathbb{R}^l) \rightarrow L^2(\mathbb{R}^l). \tag{3.26}$$

Then,

$$\tilde{\sigma}_{F(\Upsilon)}(f(s)) = \mathbf{Mp}(\varphi)(\tilde{\sigma}_\Upsilon(s)) \otimes \nu, \tag{3.27}$$

where ν is the image of the half-form factor of the symbol of Υ times the symbol of F , under the canonical map

$$\forall \sigma \in \Sigma_0 \quad \wedge^{1/2} T_s \Sigma_0 \otimes \wedge^{1/2} T\Gamma_{(f(s),s)} \rightarrow \wedge^{1/2} T_{f(s)} \Sigma_0, \tag{3.28}$$

where Γ is the graph of f .

For completeness, we mention that (3.28) arises, as in the composition of Fourier integral operators and Lagrangian distributions, from the short exact sequence

$$0 \rightarrow T_{f(s)}\Sigma_0 \rightarrow T\Gamma_{(f(s),s)} \oplus T_s\Sigma_0 \rightarrow \mathbb{R}^{2n},$$

where the first map is $v \mapsto ((v, df^{-1}(v)), df^{-1}(v))$ and the second is $((v, w), w_1) \mapsto w - w_1$; see [11] and §6 of [6] (though the present case is much simpler, because f is a transformation).

We now pass to the manifold case. We refer to §5 of [6] for general background of metaplectic structures on cotangent bundles, and how they arise from metalinear structures on the base.

Let $\mathcal{U} \subset \mathbb{R}^n$ open and $f : T^*\mathcal{U} \rightarrow T^*M$ be a symplectic embedding mapping $T^*\mathcal{U} \cap \Sigma_0$ onto a relatively open set of Σ . Let us further assume that f is an Mp map, in the sense that the maps induced by the differential df :

$$\varphi_{s_0} : \mathcal{N}_{s_0}^{\Sigma_0} \rightarrow \mathcal{N}_s^\Sigma, \quad s = f(s_0) \tag{3.29}$$

are metaplectic maps, $\forall s_0 \in T^*\mathcal{U} \cap \Sigma_0$.

COROLLARY 3.8. *Let $F_j : C^\infty(\mathcal{U}_j) \rightarrow C^\infty(M)$, with $j = 1, 2$, be FIOs associated with canonical embeddings $f_j : T^*\mathcal{U}_j \rightarrow T^*M$, as above. Let $\Upsilon_j \in I^r(\Sigma_0)$ have support in \mathcal{U}_j , and assume that*

$$F_1(\Upsilon_1) = F_2(\Upsilon_2) \quad \text{mod } I^{r+1/2}(\Sigma). \tag{3.30}$$

Let $s_j \in T^\mathcal{U}_j \cap \Sigma_0$ be such that $f_1(s_1) = s = f_2(s_2)$, and let $\varphi_j : \mathcal{N}_{s_j}^{\Sigma_0} \rightarrow \mathcal{N}_s^\Sigma$ be the corresponding (metaplectic) maps (3.29).*

Then, with the notation (3.19),

$$U^{\varphi_1}(\sigma_{\Upsilon_1}(s_1)) = U^{\varphi_2}(\sigma_{\Upsilon_2}(s_2)). \tag{3.31}$$

Proof. Let us define g to have a commutative diagram of Mp maps:

$$\begin{array}{ccc} & \mathcal{N}_s^\Sigma & \\ \varphi_1 \nearrow & & \nwarrow \varphi_2 \\ \mathcal{N}_{s_1}^{\Sigma_0} = \mathbb{R}^{2k} & \xrightarrow{g} & \mathbb{R}^{2k} = \mathcal{N}_{s_2}^{\Sigma_0} \end{array} \tag{3.32}$$

φ_j maps the vertical polarization L_0 to a polarization L_j . We need to show that the images of $\tilde{\sigma}_j$ under

$$U_{L_0}^{\varphi_j} : L^2(\mathbb{R}^k) \rightarrow \mathcal{H}_{L_j} \tag{3.33}$$

with $j = 1, 2$ represent the same element of the abstract Hilbert space of \mathcal{N}_s^Σ . (Recall that $L_0 \subset \Sigma_0$ denotes the vertical polarization.)

Let us denote by \mathcal{H}_{L_j} the quantization of \mathcal{N}_s^Σ with respect to the polarization L_j . Since $\varphi_1 = \varphi_2 \circ g$,

$$U_{L_0}^{\varphi_1} = U_{g(L_0)}^{\varphi_2} \circ U_{L_0}^g, \tag{3.34}$$

which we rewrite as

$$U_{L_0}^{\varphi_1} = \left[U_{g(L_0)}^{\varphi_2} \circ V_{g(L_0), L_0} \right] \circ \left[V_{L_0, g(L_0)} \circ U_{L_0}^g \right] = \left[U_{g(L_0)}^{\varphi_2} \circ V_{g(L_0), L_0} \right] \circ \text{Mp}(g)_{L_0}. \tag{3.35}$$

Apply both sides to the Schwartz function $\tilde{\sigma}_{\Upsilon_1}(s)$. We make two replacements on the right-hand side of the resulting equality: first, by Lemma 3.7, $\text{Mp}(g)_{L_0}(\tilde{\sigma}_{\Upsilon_1}(s)) = \tilde{\sigma}_{\Upsilon_2}(s)$; second, by naturality,

$$U_{g(L_0)}^{\varphi_2} \circ V_{g(L_0), L_0} = V_{L_1, L_2} \circ U_{L_0}^{\varphi_2}. \tag{3.36}$$

We conclude that

$$U_{L_0}^{\varphi_1}(\tilde{\sigma}_{\Upsilon_1}(s)) = V_{L_1, L_2} \left(U_{L_0}^{\varphi_2}(\tilde{\sigma}_{\Upsilon_2}(s)) \right). \tag{3.37}$$

But this shows that $U_{L_0}^{\varphi_1}(\tilde{\sigma}_{\Upsilon_1}(s))$ and $U_{L_0}^{\varphi_2}(\tilde{\sigma}_{\Upsilon_2}(s))$ represent the same element in the abstract quantization of \mathcal{N}_s^Σ . □

This Corollary allows us to make the following conclusions:

DEFINITION 3.9. Let $\Upsilon \in I^r(\Sigma)$ be given by $\Upsilon = F(\Upsilon_0)$, where $\Upsilon_0 \in I^r(\Sigma_0)$ and F is a zeroth-order FIO associated with a canonical transformation f , as in Definition 3.1. Then the symbol of Υ at $s \in \Sigma$ is the element

$$\sigma_\Upsilon(s) \in \mathcal{H}^{\mathcal{N}_s^\Sigma} \otimes \wedge^{1/2} T_s \Sigma^{1/2}, \tag{3.38}$$

which is the image of the symbol of Σ_0 at $s_0 := f^{-1}(s)$ under the map φ induced by df_s , tensored with the image of the symbol of F and the half-form part of the symbol of Υ_0 , under the generalization of (3.28)

$$\forall \sigma \in \Sigma_0 \quad \wedge^{1/2} T_s \Sigma_0 \otimes \wedge^{1/2} T\Gamma_{(f(s), s)} \rightarrow \wedge^{1/2} T_{f(s)} \Sigma. \tag{3.39}$$

We extend this definition to a general $\Upsilon \in I^r(\Sigma)$ by linearity.

Note that the symbol of $\Upsilon \in I^r(\Sigma)$ can be regarded as a section of an infinite-rank bundle over Σ , with fibers $\mathcal{H}^{\mathcal{N}_s^\Sigma} \otimes \wedge^{1/2} T_s \Sigma^{1/2}$. These are the *symplectic spinors* of [10].

3.3. THE SYMBOL CALCULUS

In this section, we re-interpret the results of Section 2.2 in the language of the global symbol.

THEOREM 3.10. *Let A be a semiclassical Ψ DO of order m on M , and $\Upsilon \in I^r(\Sigma)$. Then, $A(\Upsilon) \in I^{r+m}(\Sigma)$, and its symbol is simply the pointwise product $(\alpha|_{\Sigma})\sigma_{\Upsilon}$, where α is the principal symbol of A .*

In the remainder of this section, we assume that

$$\alpha|_{\Sigma} \equiv 0. \tag{3.40}$$

Let Ξ denote the Hamilton vector field of α . Since Σ is isotropic,

$$\forall s \in \Sigma \quad \Xi_s \in (T_s \Sigma)^{\circ}, \tag{3.41}$$

and therefore Ξ_s projects to a vector $\xi_s \in \mathcal{N}_s^{\Sigma}$. It therefore defines an element $(\xi_s, 0)$ in the Lie algebra $\mathcal{N}_s^{\Sigma} \oplus \mathbb{R}$ of the Heisenberg group of \mathcal{N}_s^{Σ} , which we continue to denote by ξ_s .

3.3.1. Transport Equations

THEOREM 3.11. (1st transport equation) *In the situation of Theorem 3.10, assume (3.40). Then, $A(\Upsilon) \in I^{r+m+1/2}(\Sigma)$, and its symbol is*

$$\sigma_{A(\Upsilon)}(s) = d\rho(\xi_s)(\sigma_{\Upsilon}(s)), \tag{3.42}$$

where $d\rho$ is the infinitesimal Heisenberg representation of the Heisenberg group of \mathcal{N}_s^{Σ} .

Next, we consider the case when, in addition to (3.40), one has:

$$\forall s \in \Sigma \quad \Xi_s \in T_s \Sigma, \quad \text{that is to say} \quad \xi_s = 0. \tag{3.43}$$

In that case, the right-hand side of (3.42) is zero, and $A(\Upsilon) \in I^{r+m+1}(\Sigma)$.

To compute its symbol, let us introduce the flow $\phi_r : \Sigma \rightarrow \Sigma$ of the restriction of ξ to Σ . Since ϕ_r is the restriction of a Hamiltonian flow on T^*M , it has a natural lift Φ_r to the symplectic normal bundle

$$\begin{array}{ccc} \mathcal{N}^{\Sigma} & \xrightarrow{\Phi_r} & \mathcal{N}^{\Sigma} \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\phi_r} & \Sigma, \end{array} \tag{3.44}$$

which is a symplectomorphism fiber-wise. Since Φ_0 is the identity, Φ_r has a natural lift to the Mp structure of the symplectic normal. Therefore, we get unitary operators:

$$U_s^r : \mathcal{H}^{\mathcal{N}_s^{\Sigma}} \longrightarrow \mathcal{H}^{\mathcal{N}_{\phi_r(s)}^{\Sigma}}. \tag{3.45}$$

THEOREM 3.12. (2nd transport equation) *In the situation of Theorem 3.10, assume (3.40) and (3.43). Then, $A(\Upsilon) \in I^{r+m+1}(\Sigma)$, and its symbol at $s \in \Sigma$ is*

$$\sigma_{A(\Upsilon)}(s) = \frac{1}{\sqrt{-1}} \frac{d}{dr} U_s^{-r} (\sigma_\Upsilon(\phi_r(s)))|_{r=0} + \sigma_A^{sub} \sigma_\Upsilon, \tag{3.46}$$

where σ_A^{sub} denotes the subprincipal symbol of A .

We can think of (3.46) as a Lie derivative on the infinite-rank bundle over Σ with fibers $\mathcal{H}^{N_s^\Sigma}$. The Lie derivative exists because of the existence of the natural lifts of ϕ_r to the bundle automorphisms U^r , as explained above.

Proof. By the manifest covariance of (3.46) with respect to invertible FIOs, it suffices to prove it in the model case. Consider again $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$, with coordinates $x = (t, u)$, and the model isotropic $\Sigma_0 = \{(t, 0; 0)\} \subset T^*\mathbb{R}^n$, and fix $s = (t_0, 0; 0) \in \Sigma_0$. Consider also a simple model isotropic state, $\Upsilon(t, u, \hbar) = a(t, u\hbar^{-1/2})$. Let us identify all Hilbert spaces $\mathcal{H}^{N_s^\Sigma}$ with $L^2(\mathbb{R}^l)$. Then, by the discussion of Section 3.1, we have:

$$U_s^{-r} (\sigma_\Upsilon(\phi_r(s))) = Mp(g_r)(a(T(r, s), \cdot)) \phi_r^*((\wedge^{1/2} dt)(\wedge^{1/2} du)), \tag{3.47}$$

where g_r is the linear symplectic transformation induced by $d\phi_r$ in the symplectic normal, and $T(r, s)$ is the value of the t coordinate at $\phi_r(s)$. By Leibniz' rule, upon differentiation of (3.47) we obtain the sum of three terms:

$$I = mp(\chi)(a(t_0, \cdot)) (\wedge^{1/2} dt)(\wedge^{1/2} du), \tag{3.48}$$

where $\chi = \frac{d}{dr} dg_r|_{r=0}$, and mp is the infinitesimal metaplectic representation;

$$II = \frac{d}{dr} a(T(r, s), \cdot)|_{r=0} (\wedge^{1/2} dt)(\wedge^{1/2} du); \tag{3.49}$$

and

$$III = a(t_0, \cdot) \frac{1}{2} \left(\sum \frac{\partial^2 H}{\partial t_i \partial \tau_i} + \frac{\partial^2 H}{\partial u_j \partial \mu_j} \right) (\wedge^{1/2} dt)(\wedge^{1/2} du), \tag{3.50}$$

where H is the principal symbol of A . (The calculation of the Lie derivative $\frac{d}{dr} \phi_r^*((dt_1 \wedge \dots \wedge dt_k)^{1/2} (du_1 \wedge \dots \wedge du_l)^{1/2})|_{t=0}$, yielding (3.50), is exactly as in the proof of Proposition 1.3.1 in [12].) Let us now omit the half-form factors in I, II and III, and show that $\frac{1}{\sqrt{-1}}(I + II + III) + \sigma_A^{sub} a(t_0, \cdot)$ equals (2.13) (with the notational change $p_0 = H$). The term $\frac{1}{\sqrt{-1}}I$ gives exactly the second line of (2.13). Further, given the assumption that the Hamilton field of H is tangent to Σ_0 ,

$$\frac{1}{\sqrt{-1}} II = \frac{1}{\sqrt{-1}} \sum \frac{\partial H}{\partial \tau_j}(s) \frac{\partial a}{\partial t_j}(t, u),$$

so all that remains to be verified is that

$$\frac{1}{\sqrt{-1}} \frac{1}{2} \left(\sum \frac{\partial^2 H}{\partial \tau_i \partial \tau_i} + \frac{\partial^2 H}{\partial u_j \partial \mu_j} \right) + \sigma_A^{\text{sub}} = p_1(s).$$

But this is equivalent to the expression for σ_A^{sub} in coordinates (see e.g., §1.3.4 in [12]). □

3.3.2. Isotropic regularity

We conclude this section with the observation that our spaces of isotropic states satisfy a certain *isotropic regularity* condition.

Let $\Sigma \subset T^*M$ be an isotropic submanifold. We let

$$\mathcal{P}(\Sigma) = \{f \in C^\infty(T^*M); f|_\Sigma \equiv 0 \text{ and } \Xi_f \text{ is tangent to } \Sigma\}. \tag{3.51}$$

LEMMA 3.13. $\mathcal{P}(\Sigma)$ is a Poisson subalgebra of $C^\infty(T^*M)$ and an ideal as well.

Proof. Let $f, g \in \mathcal{P}(\Sigma)$. We need to show that $\{f, g\} \in \mathcal{P}(\Sigma)$. Since $\{f, g\} = \mathcal{L}_{\Xi_f} g$, Ξ_f is tangent to Σ and g is constant on Σ , $\{f, g\}|_\Sigma = 0$. Also, $\Xi_{\{f, g\}} = [\Xi_f, \Xi_g]$, so this is tangent to Σ if both Ξ_f, Ξ_g are.

To prove the second part, let $f \in \mathcal{P}(\Sigma)$ and $g \in C^\infty(T^*M)$. Clearly, fg vanishes on Σ , and, since at any point on Σ $\Xi_{fg} = g\Xi_f$, Ξ_{fg} is tangent to Σ . □

The following is immediate from the symbol calculus:

COROLLARY 3.14. *Given an isotropic $\Sigma \subset T^*M$, the spaces $I^r(\Sigma)$ are stable under the action of arbitrary compositions*

$$P_1 \circ \dots \circ P_N,$$

where P_1, \dots, P_N are first-order semiclassical pseudodifferential operators whose symbols are all in $\mathcal{P}(\Sigma)$.

Thus, elements in $I^r(\Sigma)$ “do not get worse” upon application of any number of first-order operators, provided their symbols are in $\mathcal{P}(\Sigma)$. We conjecture that indeed the spaces $I^r(\Sigma)$ can be characterized by such an isotropic regularity condition, in a similar way to Hörmander’s characterization of Lagrangian distributions in [9] (but in the semiclassical setting).

3.4. NORM ESTIMATES

This short section is devoted to the proof of the following:

THEOREM 3.15. *Let $\Sigma \subset T^*X$ be an isotropic of dimension $n - l$, $n = \dim(X)$, and $\Upsilon \in I^r(\Sigma)$ with compact support. Then,*

$$\|\Upsilon\|^2 = \hbar^{2r+l/2} \int_{\Sigma} |\sigma_{\Upsilon}|^2 + O(\hbar^{2r+l/2+1/2}), \tag{3.52}$$

where $|\sigma_{\Upsilon}|^2$ is the top-degree form on Σ obtained by integrating, at each point $s \in \Sigma$, the norm squared of the Schwartz function $\sigma_{\Upsilon}(s)$ (times the square of the half-form factor along Σ).

Proof. It suffices to verify the formula in the model case, that is, when

$$\Upsilon = a(t, \hbar^{-1/2}u, \hbar) : \mathbb{R}^n \times (0, \hbar_0) \rightarrow \mathbb{C},$$

where $a(t, u, \hbar) \sim \hbar^r \sum_{j=0}^{\infty} a_j(t, u) \hbar^{j/2}$ and we further assume that Υ is compactly supported in the t variables. The proof is then elementary, reducing to the calculation of the leading term

$$\hbar^{2r} \iint |a_0(t, \hbar^{-1/2}u)|^2 du dt = \hbar^{2r+l/2} \iint |a_0(t, v)|^2 dv dt.$$

□

3.5. AN ALTERNATE APPROACH

We will conclude this description of isotropic states by showing that there is an alternate description of these states involving the Hermite distributions of Boutet de Monvel [5]. This alternate description follows [14].

3.5.1. Local Theory

Let $U \subset \mathbb{R}^n$ be an open set. Consider $T^*(U \times \mathbb{R})$ with coordinates $(x, \theta; \xi, \kappa)$, and let

$$T^*(U \times \mathbb{R})_+ = \{(x, \theta; \xi, \kappa); \kappa > 0\}.$$

We begin by noticing that if $\mathfrak{h} \in C_0^{-\infty}(U \times \mathbb{R})$ is a distribution whose wave-front set is contained in $T^*(U \times \mathbb{R})_+$, then the partial Fourier transform

$$\widehat{\mathfrak{h}}(x, \hbar) := \int e^{-i\hbar^{-1}\theta} \mathfrak{h}(x, \theta) d\theta \tag{3.53}$$

is a smooth function of x for each \hbar , since the wave-front set of \mathfrak{h} does not contain covectors conormal to the fibers of the projection $U \times \mathbb{R} \rightarrow U$.

DEFINITION 3.16. Let r be a half-integer and $\widetilde{\Sigma} \subset T^*(U \times \mathbb{R})_+$ a conic isotropic submanifold. We then define $\tilde{I}^r(\widetilde{\Sigma})$ to be the class of \hbar -dependent functions on U given by the partial Fourier transform (3.53), as \mathfrak{h} ranges over the space $I^{-r+\frac{n}{2}}(U \times \mathbb{R}, \widetilde{\Sigma})$ of Hermite distributions in the sense of Boutet de Monvel [6].

The alternative approach of this section is embodied by the following:

THEOREM 3.17. *Let $\Sigma \subset T^*U$ be a connected isotropic submanifold (not necessarily conic), and assume that there is a conic isotropic submanifold, $\tilde{\Sigma} \subset T^*(U \times \mathbb{R})_+$, such that*

$$\Sigma = \{(x, p) \in T^*U; \exists \theta \in \mathbb{R} (x, \theta; p, \kappa = 1) \in \tilde{\Sigma}\}.$$

Then there exists $\theta_0 \in \mathbb{R}$ such that

$$\tilde{I}^r(\tilde{\Sigma}) = e^{i\hbar^{-1}\theta_0} I^r(\Sigma).$$

Remark 3.18. The hypothesis of the theorem can be seen to be equivalent to the existence of a function $\psi: \Sigma \rightarrow \mathbb{R}$ such that

$$d\psi = \iota^*(pd_x), \tag{3.54}$$

where $\iota: \Sigma \hookrightarrow T^*U$ is the inclusion and pd_x the tautological one form. It therefore is always satisfied (micro) locally. To such a function, one associates the isotropic $\tilde{\Sigma} \subset T^*(U \times \mathbb{R})$ given by

$$\tilde{\Sigma} = \{(x, \theta; \xi, \kappa); \theta = \psi(x, p), \xi = \kappa p, (x, p) \in \Sigma\}.$$

The overall phase factor $e^{i\hbar^{-1}\theta_0}$ reflects the fact that the function ψ is only defined up to a constant, if Σ is connected.

Remark 3.19. To make the statement in the theorem clear, the conclusion is that there is a θ_0 such that for any Hermite distribution $\mathfrak{h} \in I^{-r+\frac{n}{2}}(U \times \mathbb{R}, \tilde{\Sigma})$ there is a corresponding $\Upsilon \in I^r(\Sigma)$ such that $\widehat{\mathfrak{h}} = e^{i\hbar^{-1}\theta_0} \Upsilon$. Moreover, there is a simple correspondence between the symbols of \mathfrak{h} and Υ , which roughly speaking says that under the identification of Σ with the subset $\tilde{\Sigma} \cap \{\kappa = 1\}$, the symbol of Υ is the restriction of the symbol of \mathfrak{h} to the set $\kappa = 1$ (the symplectic normal spaces of Σ and of $\tilde{\Sigma}$ are naturally isomorphic at corresponding points).

The proof of Theorem 3.17 is given in the remainder of this section.

PROPOSITION 3.20. *If Σ_0 is the model isotropic, then for a suitable choice of $\tilde{\Sigma}_0$ one has $\tilde{I}^r(\tilde{\Sigma}_0) = I^r(\Sigma_0)$.*

Proof. Recall that the model isotropic $\Sigma_0 \subset T^*\mathbb{R}^{k+l}$ is

$$\Sigma_0 = \{(t, u = 0; \xi = 0)\},$$

where the coordinates on $T^*\mathbb{R}^{k+l}$ are $(t, u; \xi = (\tau, \mu))$. A canonical “lift” to $T^*(\mathbb{R}^{k+l} \times \mathbb{R})$ is

$$\tilde{\Sigma}_0 = \{(t, u = 0, \theta = 0; \xi = 0, \kappa > 0)\}.$$

To simplify the notation, we will take without loss of generality $r = 0$.

To show that $I^0(\Sigma_0) \subset \tilde{I}^0(\tilde{\Sigma}_0)$, let $\Upsilon(t, u, \hbar) = a(t, \hbar^{-1/2}u, \hbar) \in I^0(\Sigma_0)$. Recall that $a(t, u, \hbar) \sim \sum_{j=0}^\infty a_j(t, u) \hbar^{j/2}$, where the a_j are Schwartz in the variables u . It will be enough to show that $\Upsilon_j \in \tilde{I}^j(\tilde{\Sigma}_0)$, where

$$\Upsilon_j(t, u, \hbar) = a(t, \hbar^{-1/2}u) \hbar^{j/2}.$$

Let

$$\mathfrak{h}_j(t, u, \theta) = \int_0^\infty e^{i\kappa\theta} a_j(t, \kappa^{1/2}u) \kappa^{-j/2} d\kappa.$$

Let \hat{a}_j be the Fourier transform of a_j in the u variables, so that

$$a_j(t, u) = \int e^{i\zeta \cdot u} \hat{a}_j(t, \zeta) d\zeta.$$

Then,

$$\mathfrak{h}_j(t, u, \theta) = \int_{\kappa>0} e^{i(\kappa\theta + \kappa^{1/2}\zeta \cdot u)} \hat{a}_j(t, \zeta) \kappa^{-j/2} d\kappa d\zeta.$$

If we let $\eta = \sqrt{\kappa} \zeta$, by substitution we get

$$\mathfrak{h}_j(t, u, \theta) = \int_{\kappa>0} e^{i(\kappa\theta + \eta \cdot u)} \hat{a}_j(t, \eta/\sqrt{\kappa}) \kappa^{-1/2} d\eta d\kappa.$$

But this expression shows exactly that \mathfrak{h}_j is a Hermite distribution in the stated class.

Now, we prove that $\tilde{I}^0(\tilde{\Sigma}_0) \subset I^0(\Sigma_0)$. Let \mathfrak{h} be a Hermite distribution associated with $\tilde{\Sigma}_0$. By the discussion on symbols, \mathfrak{h} induces a symbol in the same symbol space as those of elements in $I^0(\Sigma_0)$. Let $\Upsilon_0 \in I^0(\Sigma_0)$ be any Hermite state with the same symbol as the one induced by \mathfrak{h} . Then, from the previous part of the proof, $\Upsilon_0 \in \tilde{I}^0(\tilde{\Sigma}_0)$ and $\hat{\mathfrak{h}} - \Upsilon_0 \in \tilde{I}^{1/2}(\tilde{\Sigma}_0)$. Repeat the argument inductively, to obtain $\Upsilon_\infty \in \tilde{I}^0(\tilde{\Sigma}_0)$ such that $\hat{\mathfrak{h}} - \Upsilon_\infty \in \tilde{I}^\infty(\tilde{\Sigma}_0)$. □

PROPOSITION 3.21. *The classes $\tilde{I}^r(\tilde{\Sigma})$ are equivariant under the action of semiclassical FIOs on $C^\infty(U)$.*

Proof. By [14], semiclassical FIOs on $C^\infty(U)$ correspond to Hörmander’s FIOS on $C^\infty(U \times \mathbb{R})$ that commute with the \mathbb{R} action on $C^\infty(U \times \mathbb{R})$, and the classes of Hermite distributions are invariant under FIOs; see [6] §3. □

3.5.2. Global Theory

Let M be a manifold and $\Sigma \subset T^*M$ a isotropic submanifold. Let $\iota: \Sigma \hookrightarrow T^*M$ be the inclusion, and denote by pdx the tautological one form of T^*M . The isotropic condition on Σ is that ι^*pdx is closed. It is not generally true that ι^*pdx is exact,

as assumed in the previous section, but in some cases it is true that Σ satisfies the Bohr–Sommerfeld condition, namely, that $\exists f : \Sigma \rightarrow S^1$ smooth, such that

$$i^* p dx = \sqrt{-1} d \log(f). \tag{3.55}$$

We can then lift Σ to a homogeneous submanifold of the cotangent bundle of $M \times S^1$:

$$\tilde{\Sigma} := \left\{ (x, f(x, p); \kappa p, \kappa) \in T^*(M \times S^1); (x, p) \in \Sigma, \kappa \in \mathbb{R}^+ \right\}. \tag{3.56}$$

It is not difficult to check that $\tilde{\Sigma}$ is an isotropic submanifold of $T^*(M \times S^1)$. Conversely, a conic isotropic submanifold $\tilde{\Sigma} \subset T^*(M \times S^1)_+$, where

$$T^*(M \times S^1)_+ := \{(t, u, \theta; \tau, \eta, \kappa) \in T^*(M \times S^1); \kappa > 0\},$$

gives rise to an isotropic $\Sigma \subset T^*M$ by the process of reduction:

$$\Sigma = (\tilde{\Sigma} \cap \{\kappa = 1\}) / S^1.$$

The results of the previous section imply:

THEOREM 3.22. *Let $\Upsilon \in I(M \times S^1, \tilde{\Sigma})$ be a Hermite distribution in the sense of Boutet de Monvel [6] and let*

$$\Upsilon(x, \theta) = \sum_m \Upsilon_m(x) e^{im\theta}$$

be its Fourier series. Then the family $\{\Upsilon_m\}$ is an isotropic state associated with Σ , in the sense of Definition 3.1, after the substitution $\hbar = 1/m$.

4. Applications

In this section, we will briefly describe some applications of the theorems above.

4.1. PROPAGATION OF COHERENT STATES

We begin with:

DEFINITION 4.1. Let X be a manifold, and $p_0 = (x_0, \xi_0) \in T^*X$. By a *coherent state* centered at p_0 , we will mean any element $\Upsilon \in I^0(\{p_0\})$, that is, any isotropic state associated with $\Sigma = \{p_0\}$.

Now, let P be a self-adjoint semiclassical pseudo-differential operator of order zero on X . As an important example, if X is a Riemannian manifold and $V : X \rightarrow \mathbb{R}$ a C^∞ function, we can take for P the Schrödinger operator

$$P(\psi) = \hbar^2 \Delta \psi + V \psi \tag{4.1}$$

with Δ the Laplace–Beltrami operator. Let us denote by $H(x, \xi) : T^*X \rightarrow \mathbb{R}$ the symbol of P (so that in the example $H(x, \xi) = \|\xi\|^2 + v(x)$), and let us assume that H is proper. For each function $\psi_0 \in C^\infty(X)$, let $\psi(x, t)$ be the solution of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = P(\psi) \tag{4.2}$$

with initial condition $\psi(x, 0) = \psi_0$. Then, for each t , the map $\psi_0 \mapsto \psi(x, t)$ is a semiclassical Fourier integral operator

$$F_t : C^\infty(X) \rightarrow C^\infty(X) \tag{4.3}$$

associated with the graph of the time t map

$$\phi_t : T^*X \rightarrow T^*X$$

of the Hamilton flow of H .

The first result on propagation of coherent states is the following:

THEOREM 4.2. *Let Υ be a coherent state centered at $p_0 \in T^*X$. Then, for each $t \in \mathbb{R}$, $F_t(\Upsilon)$ is a coherent state, namely $F_t(\Upsilon) \in I^0(\{\phi_t(p_0)\})$. Moreover, the symbol of $F_t(\Upsilon)$ is the result of applying $Mp(g)$ to the symbol of Υ , where*

$$g = d(\phi_t)_{p_0} : T_{p_0}T^*X \rightarrow T_{\phi_t(p_0)}T^*X.$$

The proof is immediate, by the global theory of Section 3.2, in particular the Definition 3.9 of the symbol on manifolds. (Note that the symbols of coherent states do not have a half-form component, since the isotropic is just a point.)

We now show that if $F : C^\infty(X) \rightarrow C^\infty(X \times \mathbb{R})$ is the operator

$$F(\psi_0) = \psi(x, t)$$

and Υ is a coherent state, then $F(\Upsilon)$ is still an isotropic state on $X \times \mathbb{R}$. This is *not* immediate from the results of Section 3, because the operator F is a semi-classical FIO associated with a canonical relation that is not a transformation.

In fact we will prove something slightly more general. Let X and Y be manifolds, and let $\Gamma \subset T^*X \times T^*Y$ be a canonical relation (not necessarily the graph of a symplectomorphism; in particular, we do not assume that X and Y have the same dimension). Let $F : C^\infty(Y) \rightarrow C^\infty(X)$ be a semiclassical Fourier integral operator quantizing Γ . We will prove:

THEOREM 4.3. *Let $p_0 = (y_0, \eta_0) \in T^*Y$ be a regular value of the projection $\pi : \Gamma \rightarrow T^*Y$. Then, $\pi^{-1}(p_0) = \Sigma$ is an isotropic submanifold of T^*X . Moreover, if $\psi_{\hbar} \in C^\infty(Y)$ is a coherent state centered at p_0 , then*

$$F(\psi_{\hbar}) \in I(\Sigma). \tag{4.4}$$

Proof. By a partition of unity argument, we can assume that the Schwartz kernel of F is supported on an open set $U \times V$, where U and V are coordinate patches in Y and X , and hence we can assume without loss of generality that $Y = \mathbb{R}^m$ and $X = \mathbb{R}^n$. Less obviously, we can also assume that Γ is a horizontal submanifold of $T^*(X \times Y)$, i.e., that its projection onto $X \times Y$ is a bijection. To see this we note:

LEMMA 4.4. *There exist linear symplectomorphisms, $A : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ and $B : T^*\mathbb{R}^m \rightarrow T^*\mathbb{R}^m$, such that AFB is horizontal. Hence, “Theorem 4.3 for AFB ” implies Theorem 4.3 for F ”.*

(We will omit the proof of this since it is, more or less verbatim, the same as the proof of Theorem 2.14 in the main lemma segment of Section 2.)

Thus, we are reduced to proving Theorem 4.3 for FIOs of the form

$$Fu(x) = \int a(x, y, \hbar) e^{\frac{i\phi(x,y)}{\hbar}} u(y) dy, \tag{4.5}$$

where ϕ is the defining function of Γ , i.e.,

$$(x, \xi, y, \eta) \in \Gamma \Leftrightarrow \xi = \frac{\partial\phi}{\partial x} \quad \text{and} \quad \eta = -\frac{\partial\phi}{\partial y}. \tag{4.6}$$

We can also assume without loss of generality that $p_0 = (y_0, \eta_0) = (0, 0)$. Hence, the transversality condition in Theorem 4.3 asserts that the equations

$$\frac{\partial\phi}{\partial y}(0, 0) = 0, \quad \xi = \frac{\partial\phi}{\partial x}(x, 0) \tag{4.7}$$

are a non-degenerate system of defining equations for Σ . In particular,

$$d \frac{\partial\phi}{\partial y_i}(x, 0), \quad i = 1, \dots, m \tag{4.8}$$

are linearly independent and, hence by a change of coordinates, we can assume

$$\frac{\partial\phi}{\partial y_i}(x, 0) = x_i, \quad i = 1, \dots, m \tag{4.9}$$

and

$$\phi(x, y) = \phi_0(x) + \sum_{i=1}^m x_i y_i + \sum a_{i,j}(x, y) y_i y_j. \tag{4.10}$$

Thus if ψ_{p_0} is the coherent state

$$\psi \left(\frac{y}{\hbar^{1/2}} \right), \quad \psi \in \mathcal{S}(\mathbb{R}^m), \tag{4.11}$$

$F\psi_{p_0}$ is equal to

$$e^{\frac{i\phi_0(x)}{\hbar}} \tilde{\psi} \left(t, \frac{u}{\hbar^{1/2}}, \hbar \right), \tag{4.12}$$

where $t = (x_{m+1}, \dots, x_n)$, $u = (x_1, \dots, x_m)$ and

$$\tilde{\psi}(t, u, \hbar) = \hbar^{n/2} \int e^{iu \cdot \mu} f_a(t, u, \mu, \hbar) d\mu \tag{4.13}$$

with $f_a(t, u, y, \hbar)$ given by

$$a(t, \hbar^{1/2}u, \hbar^{1/2}\mu, \hbar) e^{i \sum a_{jk}(t, u, \hbar^{1/2}\mu) \mu_j \mu_k} \psi(\mu). \tag{4.14}$$

Thus in particular, by (4.13), $\tilde{\psi}(t, u, \hbar)$ rapidly decreases as a function of u and hence (4.12) is an isotropic state with microsupport on the set $u=0$ and $\xi = \frac{\partial \phi_0}{\partial x}$. Note, however, that by (4.7), this set is just the isotropic subset, $\Sigma = \pi^{-1}(p_0)$ of T^*X . □

COROLLARY 4.5. *Let P be a semi-classical self-adjoint pseudodifferential operator on X with proper symbol, and*

$$F : C^\infty(X) \rightarrow C^\infty(X \times \mathbb{R}) \tag{4.15}$$

*the fundamental solution of the Schrödinger equation (4.2). Let Υ be a coherent state centered at $p_0 = (x_0, \xi_0) \in T^*X$. Then, $F(\Upsilon) \in I^0(\Sigma)$, where*

$$\Sigma = \{(x, \xi; t, \tau), \quad (x, \xi) = \phi_t(x_0, \xi_0), \tau = p(x_0, \xi_0)\}. \tag{4.16}$$

Proof. This follows from the previous theorem and the fact that the canonical relation, Γ , of the operator F is defined by the condition:

$$((x, \xi), (y, \eta), (t, \tau)) \in \Gamma, \tag{4.17}$$

if and only if

$$(y, \eta) = \phi_t(x, \xi) \quad \text{and} \quad \tau = p(x, \xi) = p(y, \eta). \tag{4.18}$$

□

Remark 4.6. Keeping the notation of the previous corollary, suppose that the trajectory, $\gamma \subset T^*X$, of p_0 is periodic of period $T > 0$. Let $\rho \in C^\infty(\mathbb{R})$ be T -periodic, and consider the push-forward

$$u = \int_0^T F(\Upsilon) \rho(t) dt,$$

where Υ is a coherent state centered at p_0 . We claim that one can show that $u \in I^{1/2}(\gamma)$. Moreover, if $\lambda = H(p_0)$, then $(P - \lambda)u \in I^1(\gamma)$, because the symbol of $P - \lambda$ is zero on γ (see Theorem 3.10). But note that, since the Hamilton field of H is tangent to γ , by the first transport equation (Theorem 3.11), we in fact have that $(P - \lambda)u \in I^{3/2}(\gamma)$.

In suitable situations, one can construct quasi-modes $u \in I(\gamma)$ by symbolic methods, that is, non-trivial isotropic states satisfying $(P - \lambda)u \in I^r(\gamma)$ for all $r > 0$. By the discussion above, the first obstruction is that the symbol of u should satisfy the characteristic equation (equal zero) at some point on γ . Then, by invariance with respect to the bicharacteristic flow, the symbol of u satisfies the second transport equation at all points of this bicharacteristic.

4.2. A RESULT ON THE PSEUDOSPECTRUM

The following theorem has a symbolic proof, and as an immediate corollary we obtain a result on the pseudospectrum of a non-self-adjoint pseudodifferential operator. The latter result was first proved in [8] by other methods.

THEOREM 4.7. *Let A be a semiclassical pseudodifferential operator on a manifold M with principal symbol $H : T^*M \rightarrow \mathbb{C}$. Let $p \in T^*M$ be such that*

$$\{\Re(H), \Im(H)\}(p) < 0. \tag{4.19}$$

Then, there exists $\Upsilon \in I^0(\{p\})$ with non-zero symbol such that

$$(A - \lambda I)(\Upsilon) = O(\hbar^\infty), \tag{4.20}$$

where $\lambda = H(p)$ and the asymptotics are in the C^∞ topology.

Proof. If we let $P = A - \lambda I$, then for any $\Upsilon \in I^0(\{p\})$ $P(\Upsilon) \in I^{1/2}(\{p\})$, because the symbol of P vanishes at p . By Theorem 3.11, the symbol of $P(\Upsilon)$ is

$$\sigma_{P(\Upsilon)} = d\rho(\xi)(\sigma_\Upsilon), \tag{4.21}$$

where $\xi \in T_p(T^*M)$ is the Hamilton field of H evaluated at p , and $d\rho$ is the infinitesimal Schrödinger representation of the Heisenberg group of $V := T_p(T^*M)$.

Claim: Under the assumption (4.19), the operator $d\rho(\xi) : \mathcal{S}(V) \rightarrow \mathcal{S}(V)$ is onto and has a non-trivial kernel. Here, $\mathcal{S}(V)$ is the space of smooth vectors for the metaplectic representation of V (Schwartz functions).

This claim is precisely Lemma 3.1 of [4], but we sketch the simple argument: It suffices to prove the statement in a model of the metaplectic representation, say $\mathcal{S}(V) = \mathcal{S}(\mathbb{R}^n)$ with the usual Schrödinger representation, after choosing a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ on $V \cong \mathbb{R}^{2n}$, so that $\xi = \epsilon e_1 + i f_1$. The condition on the sign of the Poisson bracket corresponds to: $\epsilon > 0$, and in this model

$$d\rho(\xi) = \mathcal{L} = \frac{\partial}{\partial x_1} + \epsilon x_1. \tag{4.22}$$

The kernel of this operator contains Schwartz functions (e.g., $e^{-(\epsilon x_1^2 + x_2^2 + \dots + x_n^2)/2}$), and by variation of parameters one can check that the solution of the ODE $\mathcal{L}u = f$, where f is Schwartz.

With this claim at hand, choose the symbol of Υ so that (4.21) is zero, and denote by σ_1 the symbol of $P(\Upsilon) \in I^1(\{p\})$.

Next, let us look for $\gamma_1 \in I^{1/2}(\{p\})$ so that

$$P(\Upsilon + \gamma_1) \in I^{3/2}(\{p\}). \tag{4.23}$$

For any $\gamma_1 \in I^{1/2}(\{p\})$, one has $P(\gamma_1) \in I^1(\{p\})$ and (4.23) will hold if γ_1 satisfies $\sigma_{P(\gamma_1)} = -\sigma_1$, which, once again by the first transport equation, amounts to

$$d\rho(\xi)(\sigma_{\gamma_1}) = -\sigma_1. \tag{4.24}$$

By the previous claim, there is a solution to this problem, and we take γ_1 to have it as its symbol. This constructs $\Upsilon_1 = \Upsilon + \gamma_1$ such that $P(\Upsilon_1) \in I^{3/2}(\{p\})$.

Now continue this process to all orders, at each step solving an inhomogeneous equation of the form (4.24). □

In the situation of the previous theorem, one can conclude that λ is in the *semi-classical pseudospectrum* of A , since

$$\lim_{\hbar \rightarrow 0} \frac{\|(A - \lambda I)(\Upsilon)\|}{\|\Upsilon\|} = 0. \tag{4.25}$$

This result on the pseudospectrum of A was previously proved by Dencker, Sjöstrand and Zworski in [8].

4.3. COMPLEX ANALYTIC EXAMPLES OF ISOTROPIC STATES

Here, we briefly indicate how to construct many examples of isotropic states in the complex analytic category.

Consider a compact complex manifold Z , a holomorphic line bundle, $\mathbb{L} \rightarrow Z$, and a Hermitian inner product, $\langle \cdot, \cdot \rangle$, on \mathbb{L} which is positive definite in the sense that the curvature form associated with the intrinsic metric connection on \mathbb{L} is a Kähler form. Let \mathbb{L}^* be the dual line bundle to \mathbb{L} . Then,

$$D(\mathbb{L}^*) = \{(z, v) \in \mathbb{L}^*, \langle v, v \rangle_z^* < 1\}$$

is a strictly pseudoconvex domain. Let $X = \partial D$ be equipped with the volume form $\alpha \wedge (d\alpha)^{n-1}$, where α is the connection form. X is a circle bundle over Z . We let

$$\Pi : L^2(X) \rightarrow H^2(X) \tag{4.26}$$

be the Szegő projector. (Here, $H^2(X)$ is the space of boundary values of holomorphic functions on D .) This projector was studied in, for instance, [6] and [7]. It is

known that the Schwartz kernel of Π is an Hermite distribution associated with the conic isotropic

$$\{(x, x; r\alpha_x, -r\alpha_x) \in T^*X \times T^*X; x \in X, r > 0\}. \tag{4.27}$$

Now, let u be a Hermite (or Lagrangian) distribution on X . If the isotropic submanifold of u satisfies a “clean intersection condition” with respect to (4.27), then $\Pi(u)$ is a Hermite distribution on X , with respect to a conic isotropic

$$\tilde{\Sigma} \subset \{(x, r\alpha_x); x \in X, r > 0\}.$$

(For details on this composition theorem, see §7 in [6].) Furthermore, $\Pi(u)$ is in the generalized Hardy space, and we can decompose it as

$$\Pi(u) = \sum_{m=1}^{\infty} u_m,$$

with respect to the circle bundle action on X . Specifically, for each m , $u_m \in \mathcal{H}_m$, where \mathcal{H}_m is the space of functions in $H^2(X)$ which transform under the action of S^1 by the character $e^{im\theta}$. (Note, by the way, that for each m , u_m can be interpreted as a holomorphic section of $\mathbb{L}^m \rightarrow Z$.)

The results of Section 3.5.2 immediately imply:

COROLLARY 4.8. *Let $U \subset Z$ be an open set such that the bundle $\pi : X \rightarrow Z$ is trivial over U , and fix a trivialization $\pi^{-1}(U) \cong U \times S^1$. In this trivialization, let*

$$u_m(z, \theta) = \Upsilon_m(z) e^{im\theta}.$$

Then the sequence $\{\Upsilon_m\}$ is an isotropic semiclassical state on U , where $\hbar = 1/m$.

Borthwick, Paul and Uribe considered in [3] the case when u is Lagrangian and gave some applications. We hope to provide details and applications of the case when u is Hermite in a future paper.

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