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Faraday Pilot-Wave Hydrodynamics: Modelling and Computation

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A millimetric droplet bouncing on the surface of a vibrating fluid bath can self-propel by virtue of a resonant interaction with its own wave field (Couder et al. 2005a; Protière et al. 2006). This system represents the first known example of a pilot-wave system of the form envisaged by Louis de Broglie in his double-solution pilot-wave theory (de Broglie 1930, 1956, 1987). We here develop a numerical model of pilot-wave hydrodynamics by coupling recent models of the droplet’s bouncing dynamics (Moláček & Bush 2013a,b) with a more realistic model of weakly viscous wave generation and evolution. (Lamb 1932; Dias *et al.* 2008). The resulting model is the first to capture a number of features reported in experiment, including the rapid transient wave generated during impact, the Doppler effect, and walker-walker interactions.

1. Introduction

Louis de Broglie (1926, 1930, 1987) proposed that microscopic particles such as electrons move in resonance with a guiding wave field centered on the particle and generated by its internal vibration. The resulting pilot-wave theory represented the first example of what are now widely known as hidden variable theories, attempts to underpin the statistical theory of quantum mechanics with a rational dynamics. While de Broglie’s pilot-wave theory was successful in rationalizing single-particle diffraction, on the basis of which he won the Nobel prize in 1929, his theory was superseded by the Copenhagen Interpretation, despite its inherent philosophical vagaries (Bacchiagaluppi & Valentini 2009). At the time that de Broglie proposed his pilot-wave theory of quantum dynamics, there was no macroscopic analog to draw from. The first pilot-wave system was recently discovered by Yves Couder and Emmanuel Fort (Couder et al. 2005a; Protière et al. 2006; Eddi et al. 2011), and takes the form of millimetric fluid droplets walking on the surface of a vibrating fluid bath. The relation between this hydrodynamic pilot-wave system and a number of realist models of quantum mechanics has recently been explored by Bush (2015).

By virtue of its accompanying wave field, the walking droplet, or ‘walker’, is a spatially extended object, and exhibits several features previously thought to be exclusive to the microscopic quantum realm (Bush 2010). Central to the walker dynamics is the concept of path memory (Eddi et al. 2011): the wave force imparted to the walker depends on the pilot-wave field generated by its previous bounces. The more long-lived its waves, the longer its path memory. The walker dynamics is thus explicitly non-local in time: prediction of the walker’s future requires knowledge not only of its present state, but of its past.

Eddi *et al.* (2009b) demonstrated that the walkers can tunnel across submerged barriers in a manner reminiscent of quantum tunneling. Fort *et al.* (2010) demonstrated the emergence of orbital quantization for droplets walking in a rotating frame, and developed the dynamic analogy with Landau orbits. Most strikingly, this hydrodynamic pilot-wave system has been shown to exhibit wave-like statistics in three separate geometries. Couder & Fort (2006) examined the diffraction of walkers through single and double slit geometries, and demonstrated a statistical behavior reminiscent of that of single-particle diffraction of electrons and photons (Bach *et al.* 2013). Harris *et al.* (2013a) examined a walker in a confined geometry, and demonstrated that the probability distribution function corresponds to the amplitude of the cavity’s Faraday wave mode, a result reminiscent of electrons in a quantum corral (Crommie *et al.* 1993a,b). Harris & Bush (2014), Oza *et al.* (2014a) and Oza *et al.* (2014b) reexamined walkers in a rotating frame, and demonstrated that at the highest forcing examined, the orbital quantization gives way to chaotic dynamics with wave-like statistics that emerge as the walker drifts between its unstable quantized eigenstates. Similar behavior has been reported for walkers subjected to a central force (Perrard *et al.* 2014): the chaotic dynamics emerging in the limit of high vibrational forcing is characterized by the walker switching between unstable orbital states quantized in energy and angular momentum. Labousse *et al.* (2014) rationalize this behavior through consideration of the energy landscape associated with the walker’s wave field.

When a horizontal fluid layer is subject to a sinusoidal vertical vibration with frequency ω_0 , its free surface becomes unstable to a standing field of Faraday waves when the acceleration amplitude Γ exceeds the Faraday threshold, Γ_F . These waves have half the frequency of the imposed vibration and a wavelength prescribed by the standard water-wave dispersion relation (Benjamin & Ursell 1954). Below the Faraday threshold, millimetric droplets may bounce on the bath surface provided an air layer is sustained between drop and bath during impact (Walker 1978). Below a critical bouncing threshold, the drops will merge with the underlying bath; above it, they will bounce indefinitely (Couder *et al.* 2005b). Just above the bouncing threshold, the drops bounce with the forcing frequency; however, as the forcing amplitude is increased progressively, the bouncing amplitude increases until eventually the bouncing period matches that of the subharmonic Faraday wave field. Resonance is thus achieved between the bouncing droplet and its accompanying wave field, energy is most readily transferred between the two, and one can view the bath as being a damped oscillator forced at resonance. In certain parameter regimes, these resonant bouncers are destabilized by their wave field, and give way to a regular walking state (Couder *et al.* 2005a; Protière *et al.* 2006). The walking drop lands just off center of the descending central peak of its wave field, thus acquiring at each impact a horizontal impulse that propels it forward. A millimetric drop may thus walk steadily across the surface of a vibrating fluid bath by virtue of a resonant interaction with its locally excited Faraday wave field.

Protière *et al.* (2005), Protière *et al.* (2006) and Eddi *et al.* (2008) presented a series of regime diagrams indicating the observed dependence of the droplet behaviour on the forcing acceleration and drop size. Moláček & Bush (2013a) pointed out that, for a given fluid, there are two principal control parameters that prescribe the dynamical behaviour of the system. The first, Γ/Γ_F , is a parameter indicating the distance from the Faraday threshold Γ_F . The second, the vibration number,

$$\Omega = \omega_0 \sqrt{\rho R_0^3 / \sigma} \quad ,$$

indicates the relative magnitude of the imposed vibrational frequency ω_0 and the nat-

ural frequency of the drop of radius R_0 , density ρ and surface tension σ . Moláček & Bush (2013a) and Moláček & Bush (2013b) developed a detailed description of the dynamics accompanying droplet impact. To leading order, the droplet deformation may be neglected, and the role of the interface may be treated as that of a linear spring with a spring constant proportional to the surface tension (Gilet & Bush 2010ab). Through building on a model of quasi-static droplet impact on a rigid superhydrophobic surface (Moláček & Bush 2012), Moláček & Bush (2013a) developed a model that incorporates the influence of droplet deformation and the inertia of the underlying fluid. The interface may then be described in terms of a logarithmic spring whose stiffness increases with depth of penetration. While the relatively low energy bouncing states can be rationalized without considering the influence of the wave field generated by previous impacts, such is not the case for the walking threshold, which depends critically on the destabilizing influence of the wave field. The theoretical developments of Moláček & Bush (2013a) and Moláček & Bush (2013b) have now rationalized the regime diagrams describing the behavior of bouncing drops (Protière et al. 2006; Eddi et al. 2008; Wind-Willassen et al. 2013). Detailed regime diagrams indicating the behavior of the droplets in the $\Omega - \Gamma/\Gamma_F$ plane were presented in Moláček & Bush (2013a), Moláček & Bush (2013b) and Wind-Willassen et al. (2013), who adopted the nomenclature defined in Gilet & Bush (2009b) and Gilet & Bush (2009a). A periodic bouncing state is denoted by $(m, n)^p$ if it bounces m times in n forcing periods. The superscript p represents an ordering of the modes with the same periodicity according to mean mechanical energy, with the highest p denoting the most energetic mode.

In addition to underscoring the importance of the vertical dynamics on the walking, the theoretical developments of Moláček & Bush (2013a) and Moláček & Bush (2013b) have yielded a trajectory equation (Oza *et al.* 2013) that has formed the basis of further theoretical developments, such as the model to be developed herein. In a certain parameter regime delineated and rationalized in Moláček & Bush (2013b) and Wind-Willassen et al. (2013), the walking drops achieve perfect resonance with their Faraday wave field: the combined impact time and time-of flight is precisely equal to the Faraday period, $2/\omega_0$. Such resonant walkers are described in terms of the pilot-wave theory of Oza *et al.* (2013), who explicitly assume the resonance between walker and wave. Doing so allows them to recast the trajectory equation of Moláček & Bush (2013b) into an integral-differential form that is amenable to analysis. The resulting ‘stroboscopic formulation’ allowed for an accurate assessment of the stability of various simple forms of motion, including straight-line walking (Oza *et al.* 2013) as well as orbital motion in both a rotating frame (Harris & Bush 2014; Oza et al. 2014a) and in the presence of a central force (Perrard *et al.* 2014). It also captures much of the rich non-linear behavior arising in the high memory limit, for example, the complex orbits arising in a rotating frame (Oza et al. 2014b).

In the wave models of Eddi et al. (2011), Moláček & Bush (2012) and Oza *et al.* (2013), the wave field created by each impact is described in terms of a Bessel function $J_0(kFr)$ centered on the point of impact, and damped in time at a rate prescribed by the system memory, which depends on both the proximity to the Faraday threshold, and the fluid viscosity. This approximation is adequate to describe a droplet interacting with its own wave field in an unbounded fluid domain, where the short-time ($t \ll T_F$, where T_F is the Faraday period) behavior of the wave field is irrelevant, and the influence of reflections off the boundaries is not considered. However, it explicitly neglects the influence of the transient field generated by the impact, which travels at approximately 10 times the walker speed (Eddi et al. 2011), and may play a significant role in the interaction of multiple walkers, or the interaction of single walkers with topography or

boundaries. As such effects are central to the storyline of the walking drops, arising in bouncing lattices of walkers (Eddi et al. 2009a), tunneling (Eddi et al. 2009b), motion in confined geometries (Harris *et al.* 2013a) and single-particle diffraction (Couder & Fort 2006). Here we develop a more sophisticated model of the wave field accompanying walkers that explicitly captures several time-dependent wave-features generated by each impact.

Our goal is to develop a hydrodynamic pilot-wave model, formulated from first principles, that exhibits the behavior observed in the laboratory experiments. In describing the wave field, our approach is inspired by the potential-flow description of Benjamin & Ursell (1954), with viscous damping being incorporated in the manner outlined by Lamb (1932) and Dias *et al.* (2008) and recently analysed by Ambrose et al. (2012). In §2, we formulate this model, which allows for the self-consistent generation and propagation of a Faraday pilot-wave field. In §3, we present our model results, demonstrating that our system exhibits a number of features of the walking droplets that have not been captured by previous models.

2. Formulation

We proceed by presenting a linear water wave model with viscous damping, along the lines developed by Lamb (1932) and Dias *et al.* (2008). We then briefly review the Faraday instability problem and conclude with a coupled model whereby Faraday waves are generated by a localized, time-dependent surface pressure forcing that models the impacting droplet.

2.1. Governing Equations

In modeling the wave field generated by the bouncing droplet, we consider the three-dimensional free-surface water-wave problem with the following assumptions. We treat the fluid as being of infinite depth, an approximation valid for depths greater than half a wavelength, as is typically the case in the experiments of interest. We assume that viscous dissipation is non-negligible, and explicitly consider both gravitational and surface tension forces. We incorporate the vertical vibration of the bath in terms of a time-dependent gravitational force. Finally, we model the effect of droplet impact in terms of a time-dependent localised pressure force applied at the free surface. With these assumptions, and letting (x, y) denote the horizontal plane and z the vertical direction, the fluid motion is governed by the incompressible Navier-Stokes equations:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} + \mathbf{F}(t), \quad z \leq \eta(x, y, t) \quad (2.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad z \leq \eta(x, y, t), \quad (2.2)$$

where the force due to gravity and the vertical shaking of the fluid is expressed $\mathbf{F} = (0, 0, -g + g\Gamma \cos(\omega_0 t))$. These equations must be solved subject to the appropriate boundary conditions. The condition of no motion at large depths requires that

$$\mathbf{u} \rightarrow 0 \quad \text{as } z \rightarrow -\infty. \quad (2.3)$$

At the free surface, the stress balance and the kinematic condition require, respectively that

$$p \mathbf{n} - \rho \nu \tau \cdot \mathbf{n} = (\sigma \kappa + P_D(x, y, t)) \mathbf{n}, \quad z = \eta(x, y, t), \quad (2.4)$$

$$\eta_t + \mathbf{u} \cdot \nabla (\eta - z) = 0, \quad z = \eta(x, y, t). \quad (2.5)$$

Here $\mathbf{u} = (u, v, w)$ and p are the velocity and pressure fields, ρ is the fluid density and ν the kinematic viscosity. For simplicity we have assumed a laterally unbounded domain, although we remark on boundary effects in the discussion of the Faraday problem below. The free surface displacement is given by η , σ is the surface tension, and $\kappa = \nabla \cdot \mathbf{n}$ is the free surface curvature and \mathbf{n} the outward unit normal. The strain tensor τ is given by $(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$. P_D is the pressure force exerted by the droplet as it impinges on the free surface.

Two approximations are now made. First, as the observed wave slope is small, we linearize the equations. Second, following the approach of Dias *et al.* (2008) developed originally in Lamb (1932), we derive a “weakly dissipative” surface wave model. Specifically, we assume that the waves may be described to leading order as irrotational and inviscid, but that they also have a small rotational component resulting from the vanishing tangential stress free-surface boundary condition.

Neglecting, for the time being, the externally applied pressure from the droplet, we nondimensionalize the system using the wave period T and wavelength λ . The result is

$$\mathbf{u}_t = -\nabla p_d + \epsilon \Delta \mathbf{u}, \quad z \leq 0, \quad (2.6)$$

$$\nabla \cdot \mathbf{u} = 0, \quad z \leq 0, \quad (2.7)$$

$$\mathbf{u} \rightarrow 0, \quad \text{as } z \rightarrow -\infty, \quad (2.8)$$

$$(p - G\eta + B\Delta_H \eta) \mathbf{n} = \epsilon \tau \cdot \mathbf{n}, \quad z = 0, \quad (2.9)$$

$$\eta_t = w, \quad z = 0, \quad (2.10)$$

where \mathbf{F} has been absorbed into the dynamic pressure, p_d . Three nondimensional parameters have been introduced: a reciprocal Reynolds number $\epsilon = \nu T / \lambda^2$, a Bond number $B = \sigma T^2 / \rho \lambda^3$ and $G = g T^2 / \lambda$. Δ_H denotes the horizontal Laplacian. Characteristic values in the experiments are $\epsilon = 0.02$, $B = 0.20$ and $G = 1.23$.

The small-viscosity dissipation model is obtained by introducing the Helmholtz decomposition of the velocity field

$$\mathbf{u} \equiv \nabla \phi + \epsilon \nabla \times \Psi, \quad (2.11)$$

where ϕ results in a potential flow component, and $\Psi = (\psi_1, \psi_2, \psi_3)$ in a vortical component. Specifically, the vortical components of the velocity field are $(\hat{u}, \hat{v}, \hat{w}) = \nabla \times \Psi = (\psi_{3,y} - \psi_{2,z}, \psi_{3,x} - \psi_{1,z}, \psi_{2,x} - \psi_{1,y})$. Substitution into the system (2.6)–(2.10) leads to the following set of equations, valid in the fluid bulk:

$$\phi_t + p = 0, \quad z \leq 0, \quad (2.12)$$

$$\Delta \phi = 0, \quad z \leq 0, \quad (2.13)$$

$$\psi_{j,t} = \epsilon \Delta \psi_j, \quad j = 1, 2, 3, \quad z \leq 0, \quad (2.14)$$

$$\phi, \psi_j \rightarrow 0 \quad \text{as } z \rightarrow -\infty. \quad (2.15)$$

The stress balance equations at $z = 0$ couple the potential and vortical velocity fields. The normal balance yields

$$(p - G\eta + B\Delta_H \eta) = 2\epsilon (\phi_z + \epsilon(\psi_{2,x} - \psi_{1,y}))_z, \quad (2.16)$$

while the tangential stress balances yields

$$0 = 2\phi_{xz} + \epsilon(\psi_{3,y} - \psi_{2,z})_z + \epsilon(\psi_{2,x} - \psi_{1,y})_x, \quad (2.17)$$

$$0 = 2\phi_{yz} + \epsilon(\psi_{1,z} - \psi_{3,x})_z + \epsilon(\psi_{2,x} - \psi_{1,y})_y. \quad (2.18)$$

Lastly, the kinematic condition is

$$\eta_t = \phi_z + \epsilon(\psi_{2,x} - \psi_{1,y}). \quad (2.19)$$

These equations are now simplified by using the boundary layer scaling $\partial_z = O(\epsilon^{-1/2})$ in Ψ and discarding higher order terms. Then, we eliminate Ψ in favour of ϕ and η .

Truncating (2.16) at order ϵ and using (2.12) and (2.13), we obtain the final form of the dynamic boundary condition

$$\phi_t = G \eta - B \Delta_H \eta + 2\epsilon \Delta_H \phi. \quad (2.20)$$

The terms neglected are of order $\epsilon^{3/2}$. The tangential stress balance is now used to replace the viscosity-induced vertical velocity in the kinematic boundary condition. First, we use the boundary layer scaling to eliminate negligible contributions from the vortical flow and so arrive at the approximation

$$2\phi_{zx} = \psi_{2,zz} = \psi_{2,t}, \quad (2.21)$$

$$2\phi_{zy} = -\psi_{1,zz} = -\psi_{1,t}, \quad (2.22)$$

where (2.14) was used in the second equality. Now, use the leading order term of the kinematic condition $\eta_t = \phi_z + O(\epsilon)$ to express ψ_1, ψ_2 in terms of η

$$\psi_2 = 2\eta_x + O(\epsilon), \quad (2.23)$$

$$\psi_1 = -2\eta_y + O(\epsilon). \quad (2.24)$$

Substituting into (2.19) yields the “damped” kinematic condition

$$\eta_t = \phi_z + 2\epsilon \Delta_H \eta. \quad (2.25)$$

Again, terms of order $\epsilon^{3/2}$ arising from the horizontal stress balance were neglected. The damping term is merely the result of accounting for the leading order term of the vertical vortical component at the free surface. Equations (2.20) and (2.25) were also presented in Dias *et al.* (2008), but there were obtained through arguments involving approximations to the damped dispersion relation with $2\nu k^2/|\omega|$ as the small parameter. An analysis of the well-posedness of this formulation was recently given in Ambrose *et al.* (2012).

2.2. Faraday waves

When a vessel containing liquid is vibrated vertically, wave patterns will develop provided the Faraday threshold, Γ_F , is exceeded. Both the Faraday threshold and the form of the waves at the onset of instability will in general depend on both the fluid properties and the shape of the container. The first mathematical modeling of this Faraday instability was presented in Benjamin & Ursell (1954), who considered an inviscid fluid in a container of arbitrary cross section. The influence of viscosity was elucidated by Kumar & Tuckerman (1994) and Kumar (1996), who demonstrate that for a given configuration, the thresholds of the various modes of vibration depend explicitly on the fluid viscosity. We proceed by sketching how the stability analysis can be carried out with the equations derived above, through inclusion of weak viscous effects, in the case of an unbounded domain. Our first objective is to find the forcing threshold at which the planar free surface becomes unstable in our model. We note that all of the bouncing droplet phenomena of interest occur below this threshold, and depend on the proximity to the Faraday threshold (Eddi *et al.* 2011).

Based on the developments of the previous section, we formulate the problem as

$$\Delta\phi = 0, \quad z \leq 0, \quad (2.26)$$

$$\phi_t = g(1 - \Gamma \cos(\omega_0 t)) \eta - \frac{\sigma}{\rho} \Delta_H \eta + 2\nu \Delta_H \phi, \quad z = 0, \quad (2.27)$$

$$\eta_t = \phi_z + 2\nu \Delta_H \eta, \quad z = 0, \quad (2.28)$$

together with $\nabla\phi \rightarrow 0$ as $z \rightarrow -\infty$. For a container of arbitrary cross section, and in the inviscid limit ($\nu = 0$), one needs to satisfy $\phi_n = 0$ on the lateral walls of the container. Hence in the inviscid case, one chooses (Benjamin & Ursell 1954) $\phi = a_m(t)\Phi_m(x, y)e^{k_m z}$ and $k_m > 0$, with eigenfunctions Φ_m satisfying

$$-\Delta_H \Phi_m = k_m^2 \Phi_m, \quad \frac{\partial \Phi_m}{\partial n} = 0 \quad \text{on } S, \quad (2.29)$$

where S is the horizontal shape of the container i.e., the curve corresponding to the intersection of the container boundaries with a horizontal plane. The amplitude equations for the eigenfunction coefficients a_m can then be shown to satisfy the Mathieu equations

$$\frac{d^2 a_m}{dt^2} + (\omega_m^2 - \omega_g^2 \Gamma \cos(\omega_0 t)) a_m = 0, \quad (2.30)$$

$$\omega_m^2 = \omega^2(k_m) = gk_m \left(1 + \frac{\sigma}{\rho g} k_m^2\right), \quad \omega_g^2 = gk_m. \quad (2.31)$$

When $\Gamma = 0$, there are only free oscillations of the liquid surface. Their frequency is given by the classical dispersion relation $\omega(k_m)$. Benjamin & Ursell (1954) delineate the various instability regions of the Mathieu equation for $\Gamma > 0$, their emergence arising for $\Gamma \geq 0$ at frequencies corresponding to integer multiples of half the driving frequency:

$$\omega_m = n \frac{1}{2} \omega_0, \quad n = 1, 2, \dots \quad (2.32)$$

In the inviscid case, the broadest instability tongue is the subharmonic mode, $n = 1$.

If one includes damping of the form we have discussed, and considers an *unbounded* setting to avoid difficulties related to a lateral no-slip condition on cylinder walls, the dynamics will be governed by a damped Mathieu equation. This will result in a finite forcing threshold for the onset of instability. For this unbounded domain, there is a continuum of wave numbers $\mathbf{k} = (k_1, k_2)$, and solutions of the form $\phi = a(\mathbf{k}, t)e^{i\mathbf{k} \cdot \mathbf{x}}e^{|\mathbf{k}|z}$ where $\mathbf{x} = (x, y)$. The corresponding damped Mathieu equation takes the form

$$\frac{d^2 a}{dt^2} + 2\gamma \frac{da}{dt} + [\omega^2 + \gamma^2 - \omega_g^2 \Gamma \cos(\omega_0 t)] a = 0, \quad (2.33)$$

$$\text{where } k = |\mathbf{k}|, \quad \gamma(k) = 2\nu k^2, \quad \omega^2(k) = gk \left(1 + \frac{\sigma}{\rho g} k^2\right) \quad \text{and} \quad \omega_g^2(k) = gk. \quad (2.34)$$

For this equation, the relation between k and Γ for neutral stability of the $n = 1$ mode (i.e. the subharmonic $\frac{1}{2}\omega_0$ mode) is then given approximately (see Appendix A) by the relation

$$\left[\omega^2 + \gamma^2 - \frac{1}{4}\omega_0^2\right]^2 + \omega_0^2 \gamma^2 - \frac{1}{4}\omega_g^4 \Gamma^2 = 0. \quad (2.35)$$

The most unstable wavenumber k can then be obtained by minimizing Γ with respect to k . Figure 1 shows the curve $\Gamma(k)$ for typical values used in experiments together with a comparison to the numerically observed thresholds obtained from time-dependent simulations of the system (2.26)-(2.28).

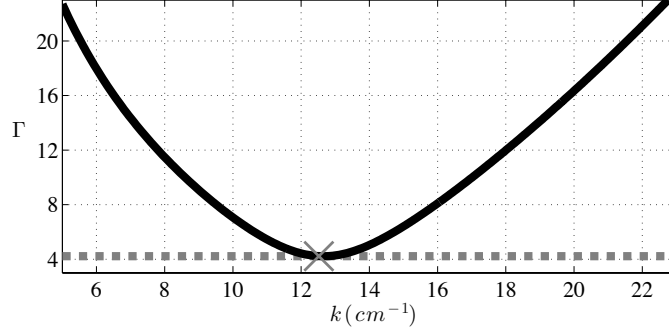


FIGURE 1. Approximate Faraday subharmonic stability curve (black) as predicted by (2.35) for the case $\omega_0 = 80\text{Hz}$, $\nu = 20 \text{ cS}$, $\sigma = 20.6 \text{ dyne/cm}$, $\rho = 0.949 \text{ gm/cm}^3$. The viscosity ν was adjusted to $\nu^* = 0.8025\nu$ so that the minimum of $\Gamma(k)$, as predicted by (2.35), matches the experimental value $\Gamma = 4.22$ reported in Wind-Willassen et al. (2013) shown by the horizontal line. The minimum is achieved at $k_F = 12.64$. The \times indicates the numerically observed value for the instability for the full problem, with $k_* = 12.52 \text{ cm}^{-1}$, at $\Gamma = 4.22$. The inverse Reynolds number for this case, using the Faraday frequency of $\omega_0/2$ and the associated Faraday wavelength is $\epsilon = \nu^* T_f / \lambda_F^2 \approx 0.016$.

2.3. Droplet trajectory and fluid coupling

We now proceed to the fully coupled wave-droplet model. The fluid equations, presented above, must now include the pressure forcing P_D generated by the impacting droplet. The system takes the form

$$\Delta\phi = 0, \quad z \leq 0, \quad (2.36)$$

$$\phi \rightarrow 0, \quad \text{as } z \rightarrow -\infty, \quad (2.37)$$

$$\phi_t = -g(t) \eta + \frac{\sigma}{\rho} \Delta_H \eta + 2\nu \Delta_H \phi - \frac{1}{\rho} P_D(\mathbf{x} - \mathbf{X}(t), t), \quad z = 0, \quad (2.38)$$

$$\eta_t = \phi_z + 2\nu \Delta_H \eta, \quad z = 0, \quad (2.39)$$

where $g(t) = g(1 - \Gamma \cos(\omega_0 t))$.

The motion of the spherical droplet has two distinct stages, flight and impact. During flight, the droplet of mass m is in ballistic motion corrected by the aerodynamic force that may be described in terms of Stokes drag in the horizontal direction (Moláček & Bush 2013b). The drop position is given by $(x, y, z) = (\mathbf{X}(t), Z(t))$ where $Z(t)$ is defined as the vertical position of the base of the droplet. Thus, in a frame moving with the vibrating container, the trajectory of the droplet in flight is governed by

$$m \frac{d^2 Z}{dt^2} = -m g(t), \quad (2.40)$$

$$m \frac{d^2 \mathbf{X}}{dt^2} = -6\pi R_0 \mu_{air} \frac{d\mathbf{X}}{dt}, \quad (2.41)$$

where μ_{air} is the viscosity of air, and R_0 is the drop radius. In the fluid equations, during flight, the applied surface pressure is set to zero, i.e. $P_D = 0$ in (2.38).

The impact on the free surface begins at time $t = t_I$ when $\eta(\mathbf{X}(t_I), t_I) = Z(t_I)$ and $\frac{d(Z-\eta)}{dt}|_{\mathbf{x}=\mathbf{X}} < 0$. During impact, the vertical dynamics is modelled as a nonlinear spring prescribed by the model of Moláček & Bush (2013b):

$$\left(1 + \frac{c_3}{\ln^2 \left| \frac{c_1 R_0}{Z - \bar{\eta}} \right|}\right) m \frac{d^2 Z}{dt^2} + \frac{4\pi\nu\rho R_0 c_2}{3 \ln \left| \frac{c_1 R_0}{Z - \bar{\eta}} \right|} \frac{d}{dt}(Z - \bar{\eta}) + \frac{2\pi\sigma}{\ln \left| \frac{c_1 R_0}{z - \bar{\eta}} \right|} (Z - \bar{\eta}) = -m g(t), \quad (2.42)$$

while the horizontal trajectory is given by

$$m \frac{d^2 \mathbf{X}}{dt^2} + \left(c_4 \sqrt{\rho R_0 / \sigma} F(t) + 6\pi R_0 \mu_{air} \right) \frac{d\mathbf{X}}{dt} = -F(t) \nabla \bar{\eta}|_{\mathbf{x}=\mathbf{X}}. \quad (2.43)$$

Details of the derivation of these equations are given in Moláček & Bush (2013b) but we highlight here some important points. The constants c_1, c_2, c_3, c_4 are fixed: we carry out all single droplet experiments with values similar to those used in Moláček & Bush (2013b) and Wind-Willassen *et al.* (2013) (see Appendix B). Briefly, c_1 prescribes the spring nonlinearity, c_2 the vertical component of damping, c_3 the drops added mass induced during impact and c_4 the skidding friction. The right hand side of (2.43) represents the propulsive force induced by the droplet's impact on the inclined surface at the impact location \mathbf{X} . This impact force $F(t)$ experienced by the droplet is extracted from the vertical dynamics equation as

$$F(t) = \max \left[m \frac{d^2}{dt^2} Z + m g(t), 0 \right]. \quad (2.44)$$

The thresholding to prevent negative (suction) forces ($F(t) < 0$) was introduced on physical grounds in Moláček & Bush (2013b). The impact ends at $t = t_E$, when $\bar{\eta}(\mathbf{X}(t_E), t_E) = Z(t_E)$ and $\frac{d(Z - \bar{\eta})}{dt}|_{\mathbf{x}=\mathbf{X}} > 0$.

In the equations above we have used $\bar{\eta}$ to denote a local “average” of the free surface elevation. The distinction between η and $\bar{\eta}$ is important: when the drop impacts, the surface (through the effect of the pressure P_D) deforms substantially on a spatial scale of the order of the droplet size which is significantly smaller than the wave motion that is induced. In previous models (Oza *et al.* 2013, Moláček & Bush (2013b)) that use a Bessel function approximation to describe the wave field generated by each impact, the free-surface geometry (slope and elevation) is calculated without accounting for the current impact. The wave field generated by an impact is updated only after each new bounce by adding a temporally damped Bessel function $J_0(k_F r)$. These models thus do not explicitly compute the dynamic wave generation and propagation. In contrast, our model computes the full solution to the linear water wave problem. The resulting waveforms will be compared to the Bessel function approximations in §3.3.

We compute the surface elevation $\bar{\eta}$ required for the droplet trajectory equation (2.43) by computing the contributions to the wave field made by previous impacts. In practice, at $t = t_I$ we set $\bar{\eta} = \eta$, and for $t_I < t < t_E$ we solve two surface evolutions: one for $\bar{\eta}, \phi$ in (2.38)-(2.39) with $P_D = 0$ and one for η, ϕ in (2.38)-(2.39), with the pressure forcing given by

$$P_D = \frac{F(t)}{\pi R(t)^2}, \quad (2.45)$$

for $|\mathbf{x} - \mathbf{X}| < R(t)$ and $P_D = 0$ otherwise. $R(t)$ denotes the contact area, obtained by geometric arguments for small penetration depth Z . Specifically, we use

$$R(t) = \min(2|Z|R_0, (R_0/3)^2).$$

We make the further approximation that the pressure forcing is spatially uniform over the impact area $R(t)$. Our experience indicates that the model is not sensitive to this approximation. When $t = t_E$, the “bar” variables are discarded and reinitialised at the next impact.

In summary, the evolution of $\bar{\eta}$ determines the drop dynamics, while the evolution of η accounts for the wave generation. A verification of the consistency of our model is that $\eta(\mathbf{X}(t), t) \approx Z(t)$ during droplet impact, as is evident in the traces reported in figure 3.

3. Numerical modeling and simulations

For numerical efficiency, the horizontally unbounded domain is approximated with a large doubly periodic one, allowing the simple implementation of an accurate Fourier spectral method. The main goals of the numerical simulations are to capture the pilot wave-fields of single and interacting particles, and demonstrate the improvement in wave-field modeling relative to prior Bessel-function-based methods.

The system (2.38)-(2.39) is essentially two-dimensional, requiring only $\phi_z(\mathbf{x}, 0, t)$ (i.e. the irrotational vertical velocity) from (2.36). This calculation is simple in the doubly periodic domain where $\phi_z(\mathbf{x}, 0, t) = \mathcal{F}^{-1} \left[k \hat{\phi}(\mathbf{k}, t) \right]$, where $\hat{\phi} = \mathcal{F}[\phi(\mathbf{x}, 0, t)]$ and \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and its inverse respectively. This “Dirichlet-to-Neumann” map amounts to calculating efficiently the vertical velocity at the free surface of a fluid from the horizontal velocity, when the velocity potential satisfies Laplace’s equation in the bulk. In addition, for accuracy and efficiency, the full evolution equations (2.38)-(2.39) will be solved in Fourier space.

Through a numerical model and its corresponding computational simulations, we show that the coupled wave-droplet system (2.38)-(2.39), (2.42)-(2.43) displays the key features of pilot-wave dynamics, specifically, wave generation and a myriad of bouncing and walking states, reported in laboratory experiments.

3.1. The spectral method

An accurate spectral method is used following the strategy of writing the evolution as a single complex equation used by Milewski & Tabak (1999) and Wang & Milewski (2012). To sketch the method, consider the dimensionless form of the equations:

$$\phi_t = -\mathcal{G}(t) \eta + B \Delta_H \eta + 2\epsilon \Delta_H \phi - B P_d(t), \quad (3.1)$$

$$\eta_t = \phi_z + 2\epsilon \Delta_H \eta, \quad (3.2)$$

where the forcing pressure has been scaled with σ/λ , and time with the period of the subharmonic Faraday mode, resulting in $\mathcal{G} = G + \Gamma \cos(4\pi t)$.

In Fourier space, equations (3.2) and (3.1) are given by

$$\hat{\phi}_t = -[\mathcal{G}(t) + Bk^2] \hat{\eta} - 2\epsilon k^2 \hat{\phi} - B \hat{P}_d(t), \quad (3.3)$$

$$\hat{\eta}_t = k \hat{\phi} - 2\epsilon k^2 \hat{\eta}. \quad (3.4)$$

We perform the change of variables

$$\hat{u} = \hat{\eta} + i \frac{k}{\omega} \hat{\phi}, \quad (3.5)$$

where

$$\omega(k) = \sqrt{k(G + Bk^2)}. \quad (3.6)$$

The system is then easily integrated, for each k , as a single linear, complex equation. The part of the equation that has constant coefficients can be integrated exactly, and then the evolution of the Fourier amplitudes amounts to

$$\frac{d}{dt} \left[\hat{u} e^{(i\omega + 2\epsilon k^2)t} \right] = -i \frac{k}{\omega} \left(\Gamma \cos(4\pi t) \hat{\eta} + B \hat{P}_D(t) \right) e^{(i\omega + 2\epsilon k^2)t}. \quad (3.7)$$

In the absence of both vibration ($\Gamma = 0$) and pressure forcing ($P_D = 0$), no numerical integration is needed. In the presence of these effects, equation (3.7) is solved using a fourth-order Runge-Kutta method. The original variables are recovered from \hat{u} by decomposing expression (3.5) into its Hermitian and skew-Hermitian parts and inverting

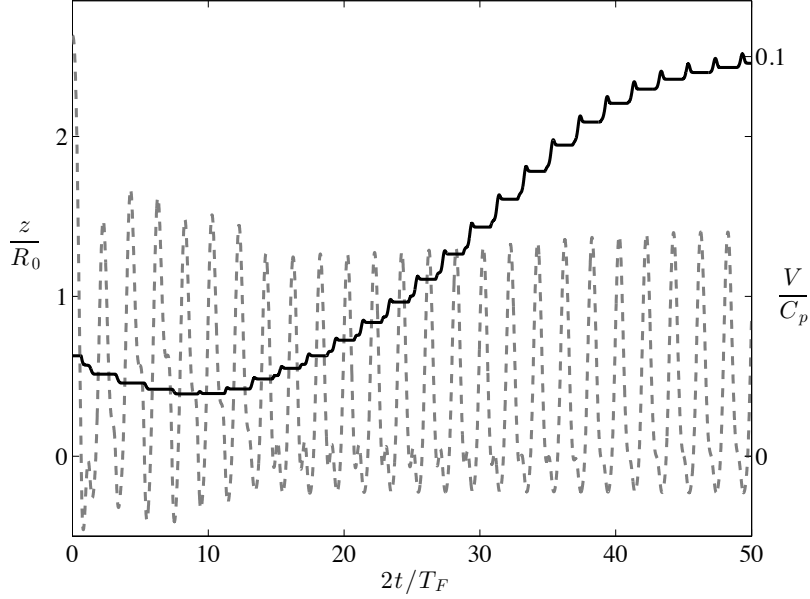


FIGURE 2. Evolution of the droplet's horizontal velocity (V/C_p , black) and elevation above the free surface (z/R_0 , gray), in the laboratory frame of reference. Following its release onto the surface at a low speed, the droplet decelerates, then accelerates towards its steady walking speed. The velocity variations arising over the Faraday period (twice the forcing period) reflect the two stages of the drop motion, impact and free flight.

it. As explained earlier, during contact time, the averaged free surface $\bar{\eta}$ and its slope $\nabla\bar{\eta}$ feed into the droplet dynamics (2.42)-(2.43) and are obtained by integrating the wave equations with $P_D = 0$. The vertical dynamics and trajectory ODEs (2.40)-(2.41), (2.42)-(2.43) are also solved directly by a Runge-Kutta method, and provide the coupling pressure P_D through (2.44) and (2.45). To model accurately the full impact, we must resolve a multiscale problem, with scales ranging from the penetration depth, of $O(0.1\text{mm})$, to the characteristic wavelength of the pilot-wave, of $O(1\text{cm})$. In our single drop computations we typically use 1024×1024 Fourier modes in space.

In what follows, we present a series of numerical simulations confirming that the system of differential equations (2.38)-(2.39), (2.40)-(2.41), (2.42)-(2.43), is capable of generating much of the complex behavior reported in laboratory experiments.

3.2. Dynamics of single drops

We first investigate the bifurcations between different walking and bouncing states as reported by Protière et al. (2006); Eddi et al. (2008); Protière et al. (2006); Moláček & Bush (2013a,b) and Wind-Willassen et al. (2013). In the laboratory experiments, a droplet is released onto a vibrating bath of silicone oil. Depending on the forcing acceleration Γ , the droplet may coalesce, bounce or walk, propelled by its own pilot-wave. We proceed by comparing the experimental observations with the predictions of our model.

Following Moláček & Bush (2013a), we characterize the system in terms of the dimensionless driving acceleration, Γ/Γ_F , and the vibration number, Ω . For given Γ/Γ_F and Ω values, we initialize a computation by releasing a particle onto the fluid surface at a low horizontal speed. Below the walking threshold, for $\Gamma < \Gamma_W(\Omega)$, the particle quickly comes to rest, and settles into a stationary bouncing mode. When $\Gamma > \Gamma_W(\Omega)$

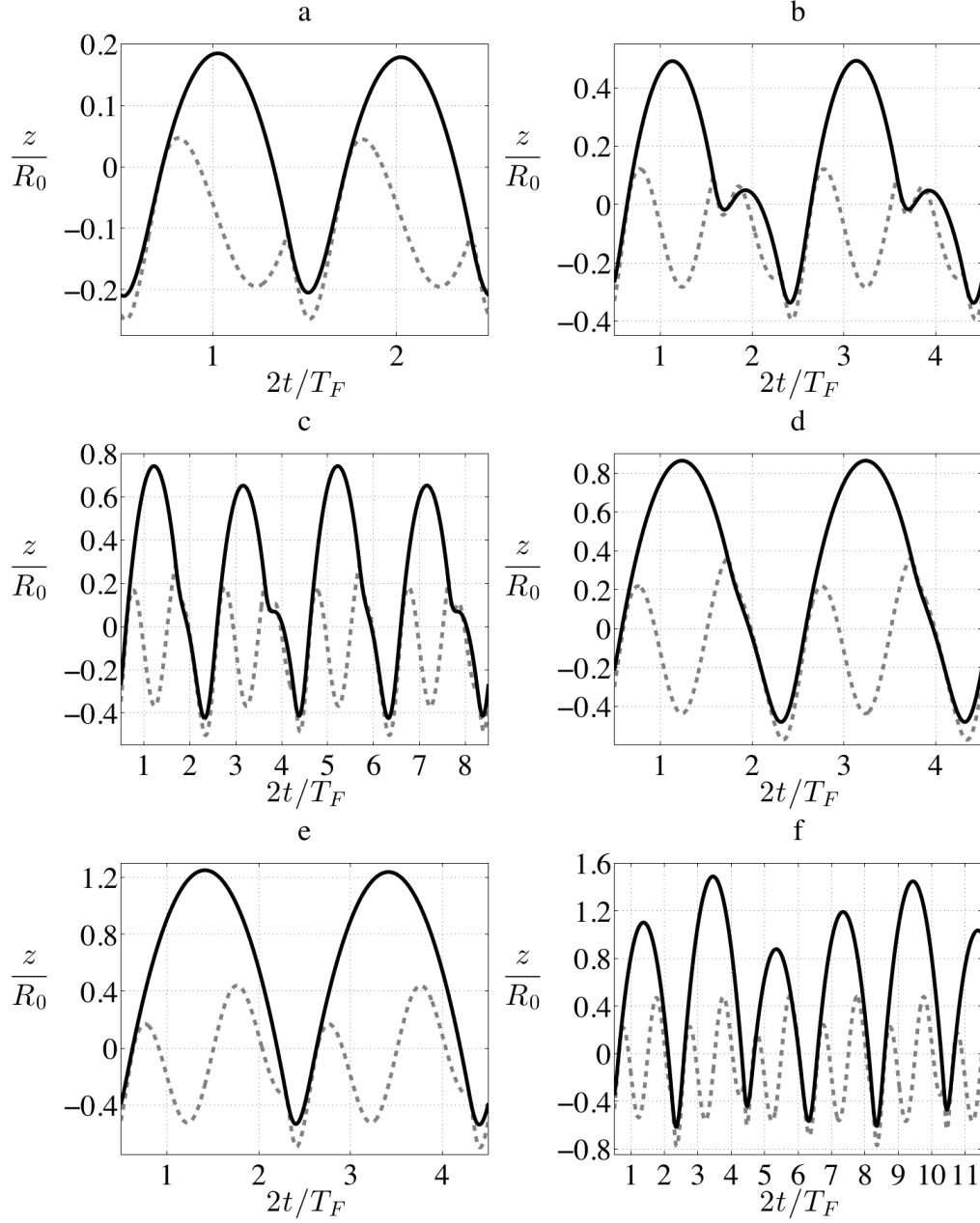


FIGURE 3. Vertical dynamics of the $(m, n)^p$ bouncing and walking modes. The vertical position of the droplet's lowermost point is indicated by the solid line and the height of the underlying bath by the dashed line. The figures indicate elevations relative to the non-vibrating laboratory frame of reference. The horizontal axis is in units of forcing periods ($T_F = 1/80$ s), and the vertical axis in units of drop radii. All experiments shown in this figure correspond to $\Omega = 0.8$, i.e. a drop radius of $R_0 = 0.38$ mm forced by a bath vibrating at 80Hz. (a) $(1, 1)$ bouncing at $\Gamma = 1.6$; (b) $(2, 2)$ bouncing at $\Gamma = 2.3$; (c) $(4, 3)$ bouncing at $\Gamma = 2.9$; (d) $(2, 1)^1$ walking at $\Gamma = 3.4$; (e) $(2, 1)^2$ walking at $\Gamma = 3.8$; (f) chaotic walking at $\Gamma = 4.15$. The Faraday threshold is $\Gamma_F = 4.22$.

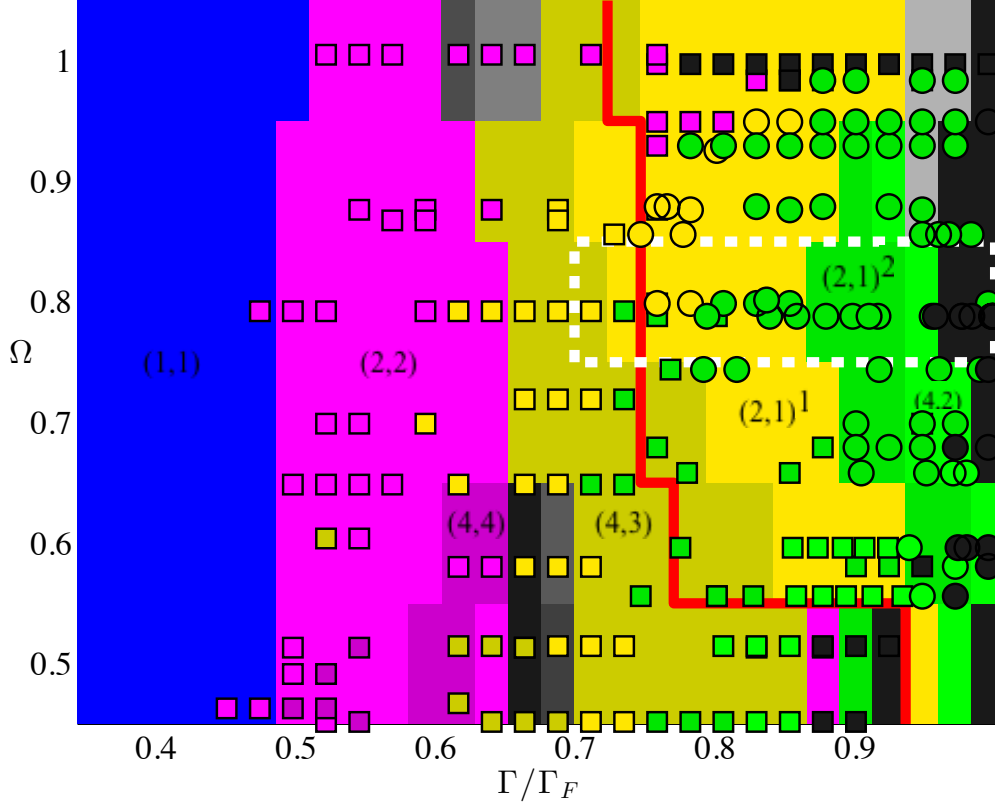


FIGURE 4. Regime diagram indicating the form of the bouncing behavior in the $\Gamma/\Gamma_F - \Omega$ plane, where $\Omega = \omega_o \sqrt{\rho R_0^3 / \sigma}$ is the vibration number. Squares (bouncing modes) and circles (walking modes) are laboratory data reported by Wind-Willassen et al. (2013). The background is tiled in rectangles of size 0.1 by 0.1, coloured according to the $(m, n)^p$ mode predicted to arise at the centre of the tile. The square and round markers follow the same colour scheme as the background. The red line indicates the predicted transition between bouncing and walking predicted by our model. Black regions represent chaotic bouncing or walking, and grey regions indicate that the period of droplet motion is equal to or greater than six forcing periods. The changes in velocity and bouncing phase in the region delimited by the white dashed lines are indicated in figure 5.

the particle accelerates and settles into a steady walking state (see figure 2). In certain cases, particularly when $\Gamma \approx \Gamma_F$, the walking appears to be irregular and chaotic, at least over the times that we compute. We recall that our wave model has only one adjustable parameter, the effective viscosity of the fluid. Using the real viscosity of the fluid overestimates the experimental Faraday threshold by approximately 20%. We thus adjust the viscosity in order to match this threshold (see figure 1). The parameters c_j governing the droplet dynamics (2.42)-(2.43) are similar to those used in Moláček & Bush (2013b); Wind-Willassen et al. (2013), but are slightly modified in order to match the experimentally observed walking threshold, Γ_W (see Appendix B). The walking threshold Γ_W is most sensitive to the skidding friction parameter c_4 and the dynamics is then most sensitive, in decreasing order, to c_1, c_3 and c_2 .

Detailed regime diagrams in the $\Omega - \Gamma/\Gamma_F$ plane were presented in Moláček & Bush (2013a), Moláček & Bush (2013b) and Wind-Willassen et al. (2013). We adopt their

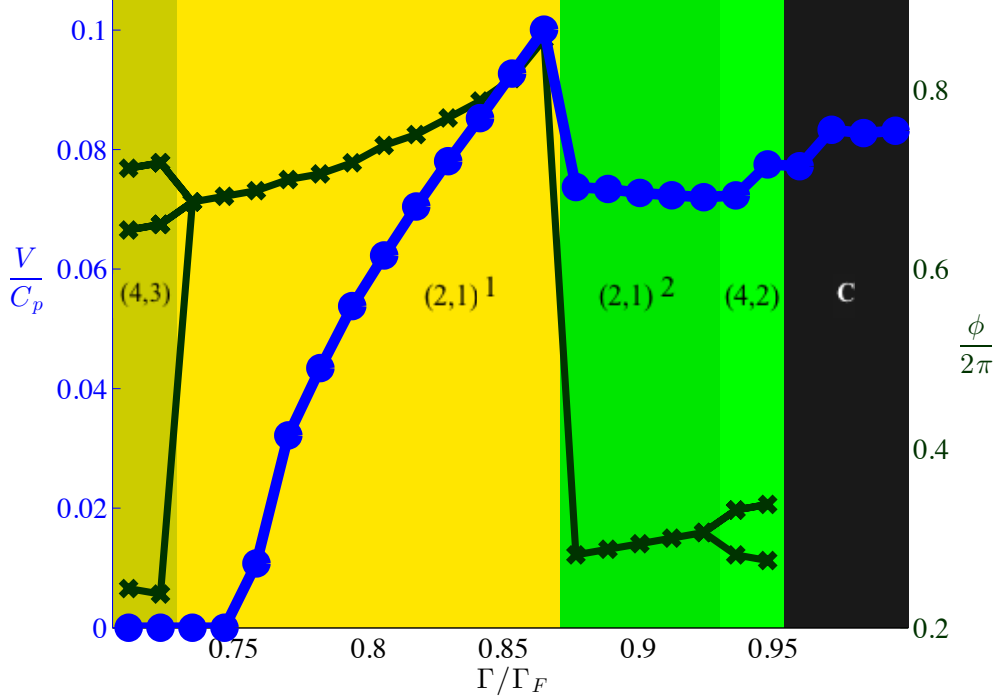


FIGURE 5. The dependence of impact phase ϕ and horizontal speed V on the vibrational forcing Γ/Γ_F for $\Omega = 0.8$, along the region delimited by the dashed white lines in Figure 3. The blue line indicates V/C_p , where C_p is the phase speed of a wave with the Faraday wavelength. The black lines indicate the impact phase $(\phi/2\pi)$. The background colours correspond to the different bouncing and walking modes, as indicated in figure 4. For $\Gamma/\Gamma_F \lesssim 0.75$, the particle is bouncing either in the (4,3) mode or the (2,1) mode. The bouncer then transitions to a walking state. The phase curve takes multiple values in regimes with more than one bounce per period. A characteristic jump in both phase and speed identifies the transition between the $(2,1)^1$ and $(2,1)^2$ walking modes.

nomenclature, labeling different bouncing and walking states with a triplet of integers $(m, n)^p$ where m is number of forcing periods and n the number of bounces in one overall bouncing cycle. If there is a multiplicity of (m, n) modes, p ranks them according to their relative mechanical energies, $p = 1$ being the lowest. For example, (4, 2) signifies a motion that repeats every 4 forcing periods and consists of two different bounces; otherwise, it would be a (2, 1) mode).

In Figure 3, various simulated bouncing modes are displayed. The vertical position of the lowermost point of the droplet and the free surface elevation beneath the droplet centre are shown as a function of time for fixed Ω and several values of Γ . Both bouncing and walking states are represented. At a few points we note a slight inconsistency between the free surface elevation and the droplet position during impact; specifically, the droplet appears to cross the interface. In reality, the droplet and the waves are separated by a thin lubrication layer of air that transmits normal stresses between the two. In our model, the fluid is free to respond to the pressure field of the droplet and we do not impose their relative positions. During most of the contact time the solid and dashed lines coincide. The extent of this discrepancy, that is, the crossing of the drop and bath surfaces, thus provides a measure of the consistency of our model.

Figure 3 also indicates how the system transitions between the various modes. For

example, a fundamental difference between the $(2, 1)^1$ and $(2, 1)^2$ modes (c.f. graphs d and e) is the droplet impact phase. In figure 4, the behaviour of our model is shown in the $\Omega - \Gamma$ plane and compared directly to the experimental results of Wind-Willassen *et al.* (2013) (see their Figure 3). The vibration number Ω was varied in our simulations by changing the droplet radius. The predicted dynamical states are indicated by the background colour and the laboratory experiments are shown by the colour-coded points, squares for bouncers and circles for walkers. Evidently, our numerical model is able to adequately capture the diversity of bouncing modes and their transition points, with only relatively small deviations from the experimental results. The main quantitative differences are that the extent of our predicted walking region (as indicated by the red curve) is relatively large, and that the $(2, 1)^1$ to $(2, 1)^2$ transition occurs at a slightly higher forcing amplitude.

In figure 5, details of the dynamics for a horizontal slice through Figure 4 at $\Omega = 0.8$ are shown. Specifically, it indicates the dependence of the walking speed, impact phase and walking mode on the vibrational forcing. The impact phase and speed show a characteristic decrease at the transition between the $(2, 1)^1$ and $(2, 1)^2$ modes, consistent with the observations and theoretical predictions of Moláček & Bush (2013b). We recall that the variations of the drop's walking speed, arising over the Faraday period due to its distinct phases of impact and free flight, are shown in Figure 2.

3.3. The walker-induced wave field

Previous theoretical models of the wave field have used a superposition of waves generated by preceding impacts:

$$\eta(\mathbf{x}, t) = \sum_n h_n(\mathbf{x}, t), \quad (3.8)$$

where h_n is the wave generated by the n th prior impact at time t_n . Two models for h_n have been used. Eddi *et al.* (2011) proposed a cosine form with both spatial and temporal damping:

$$h_n(\mathbf{x}, t) = \frac{A}{|\mathbf{x} - \mathbf{X}(t_n)|} \cos(k_F |\mathbf{x} - \mathbf{X}(t_n)| + \phi) e^{-|\mathbf{x} - \mathbf{X}(t_n)|/\delta} e^{-(t-t_n)/T_F M_e}, \quad (3.9)$$

in which δ is a decay length parameter. Moláček & Bush (2013b) derived a form valid in the small-drop limit:

$$h_n(\mathbf{x}, t) = \frac{A}{\sqrt{t-t_n}} J_0(k_F |\mathbf{x} - \mathbf{X}(t_n)|) e^{-(t-t_n)/T_F M_e} \cos(\omega_0 t/2), \quad (3.10)$$

where the memory parameter $M_e = T_d/T_F(1 - \Gamma/\Gamma_F)$ depends on the proximity to threshold, as well as the decay time of the unforced waves, T_d . We note that the \sqrt{t} decay factor arising in the narrow-band asymptotic analysis of Moláček & Bush (2013b) is omitted in some other models of the surface waves (e.g. Oza *et al.* (2013)) for the sake of simplicity. Asymptotic values for the amplitude parameter A and damping time $T_d = T_F M_e$ appearing in (3.10) are given in Moláček & Bush (2013b).

Figure 6 illustrates the numerically predicted wave fields generated by a drop impacting the surface in the absence and presence of vibrational forcing. It may be compared directly to the wave fields reported in Figure 4 of Eddi *et al.* (2011). In the absence of vibrational forcing, a transient sweeps radially outward from the point of impact. With vibrational forcing, this transient serves to trigger a standing field of Faraday waves that persists in its wake.

Figure 7 shows the cross-sectional free surface height of a $(2, 1)^1$ bouncer at the moment

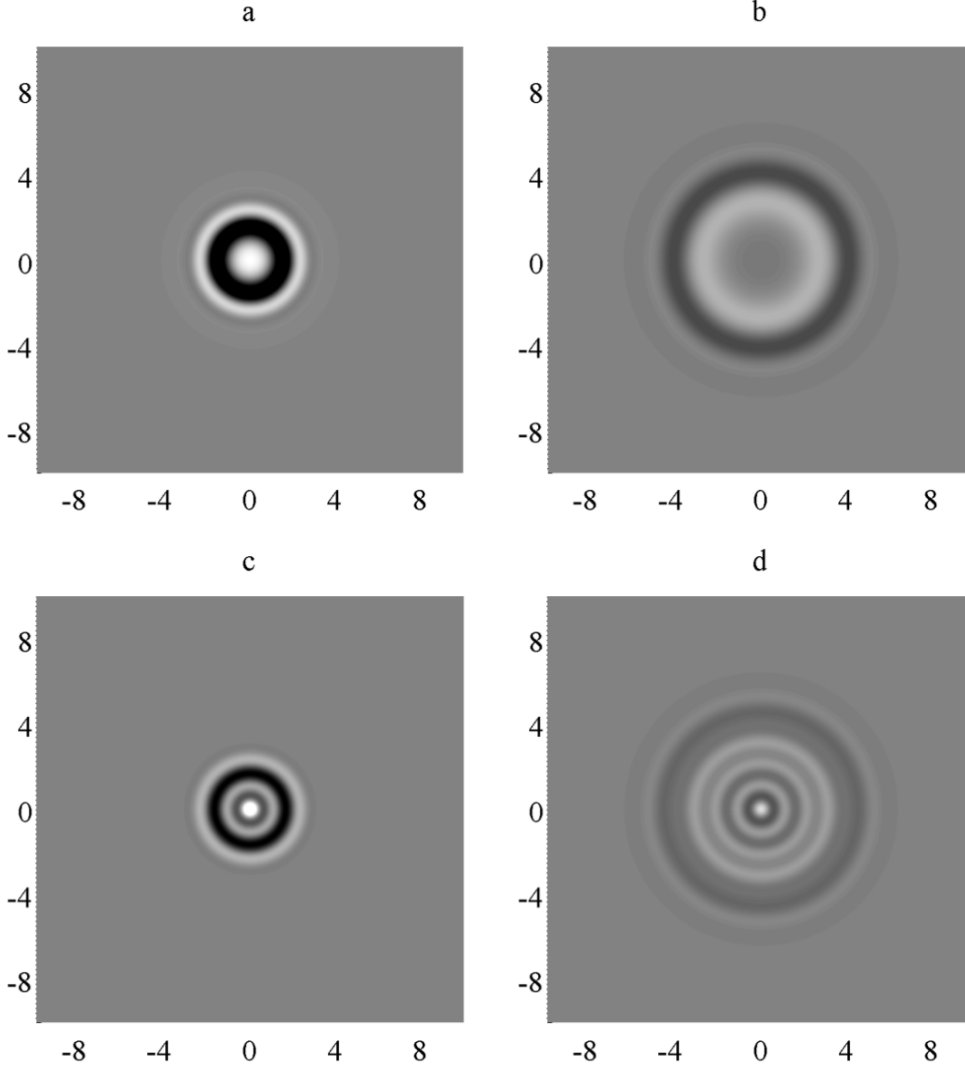


FIGURE 6. The computed wave fields generated by a single drop impact on (a-b) quiescent and (c-d) vibrating fluid baths. For the vibrating bath, $\Gamma = 3.6$, $f = 80Hz$. The wave field was generated by a single impact of a drop of $R_0 = 0.38mm$ and density ten times that of the bath. Images are taken at times (a,c) $t = 2T_F$ and (b,d) $t = 6T_F$ after impact, where T_F is defined on the basis of the vibrating bath simulations.

before impact, as predicted by a number of theoretical models. Figure 8 shows a comparison between our numerical wave profile, and the Bessel approximations of Moláček & Bush (2013b) without and with a spatial damping term. The spatial damping follows from expression (3.10) in Moláček & Bush (2013b) (see their equations (A46) for β_1 and (A47)), which has an $O(r^2/4\beta_1\tau)$ correction term arising from an expansion of $\exp(-r^2/4\beta_1\tau)$. When this correction is applied, each Bessel function is multiplied by a Gaussian radial profile.

In Figure 9, we show the dependence of the spatial decay length of the bouncers and walkers on Γ , as computed by factoring out the $r^{-1/2}$ spatial decay factor of the $J_0(k_F r)$

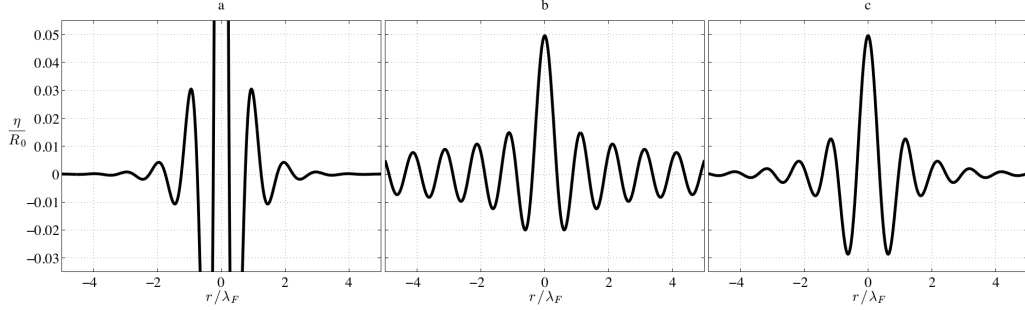


FIGURE 7. Different model predictions of the free surface cross section of a $(2, 1)^1$ bouncer at the moment before impact with $\Gamma = 3.1$. (a) The cosine wave model (3.9) of Eddi et al. (2011) with a spatial damping with decay length $1.6 \lambda_F$ reported by the authors. Note the singularity at $r = 0$. (b) The Bessel model (3.10) of Moláček & Bush (2013b). (c) The result of our computation.

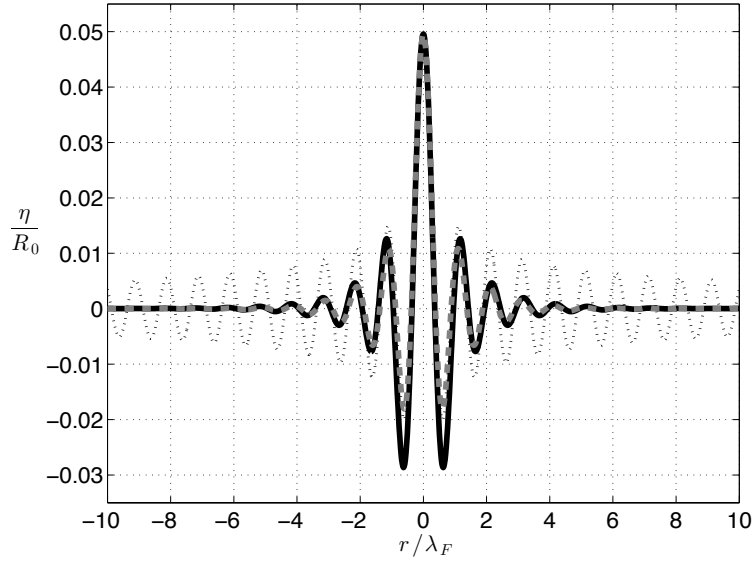


FIGURE 8. Comparison between the wave-field predictions of the present model (black curve) with those of Moláček & Bush (2013b) without (dotted curve) and with (grey dashed curve) the higher order spatial damping correction suggested by Moláček & Bush (2013b) to the Bessel model (3.10).

Bessel function. Specifically, we took the modulus of every maximum and minimum (apart from the central one) and fit it to a curve of the form $A r^{-1/2} \exp(-r/L_d)$. While the spatial decay length increases substantially for $\Gamma/\Gamma_F > 0.98$, for lower Γ it varies slowly between 1.5 and $4 \lambda_F$. We note that the only experimental spatial decay length reported in the literature is the $1.6 \lambda_F$ mentioned by Eddi et al. (2011).

Figure 10 shows the time evolution of the free surface arising from the same $(2, 1)^1$ bouncer as in Figure 7 over the course of a Faraday period. While the Bessel model is constructed as a standing wave, our solution clearly generates a wave that propagates away from the impact centre as evidenced by the V-shaped furrow in Figure 10b and Figure 10c. We note that the full temporal wave field of the Bessel approximation has been used in a variety of situations, for example in chaotic bouncing (Moláček & Bush 2013b).

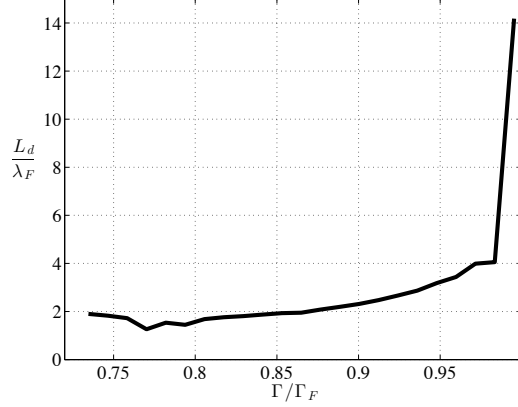


FIGURE 9. Computed dependence of the dimensionless decay length (L_d/λ_F) on the proximity to the Faraday threshold. Here, $\Omega = 0.8$.

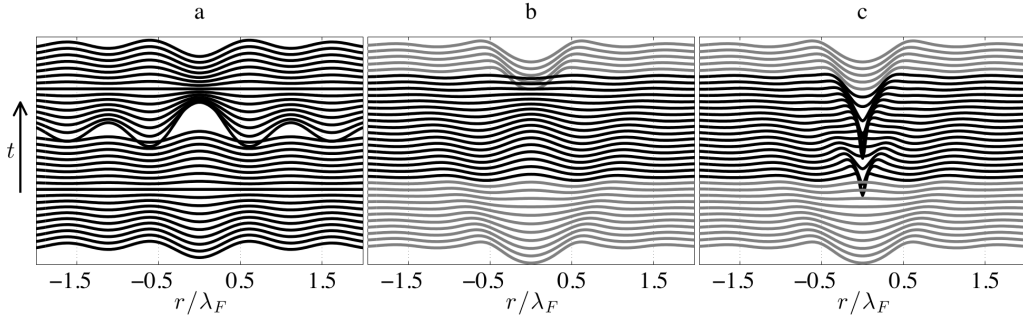


FIGURE 10. The time evolution of the free surface through the course of a Faraday period for the same bouncer as in figure 7. (a) The model of Moláček & Bush (2013b), as described by equation (3.10). The discontinuity arises when the surface is updated by the addition of a new Bessel function. (b) Our model's prediction, $\bar{\eta}$, for the free surface in the absence of the current impact. The dark curves correspond to wave forms arising when the drop is in contact with the bath, the lighter curves to times when the drop is in flight. (c) Our model prediction for the free surface when the current droplet impact is included.

The Doppler shift along a walker's wave field is displayed in Figure 11a. The solid line shows the free surface along the direction of motion, and the dashed line that in the transverse direction. The Doppler shifting is evidenced by the changing distance between crests on the two curves. Figure 11b shows the dependence of the Doppler shift on the walker's mean velocity, along with the experimental results of Eddi et al. (2011). While the Bessel-function-based models have been successful in describing certain aspects of the problem, they cannot capture the Doppler shift due to the moving wave maker. This arises from the fact that the Bessel function $J_0(k_F r)$ can be written as the superposition of plane cosines of fixed wavelength $2\pi/k_F$ in every direction, centered at the origin:

$$J_0(r) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(r \cos \theta) d\theta.$$

Therefore the function $J_0(k_F r)$ in the (x, y) plane does not contain wavelengths other than $2\pi/k_F$. Physically, this is the wavelength observed in all directions as r becomes large. Alternatively, the spectrum $\hat{J}_0(\mathbf{k}/k_F)$ is nonzero only on the circle $k = k_F$ in the (k_1, k_2) plane.

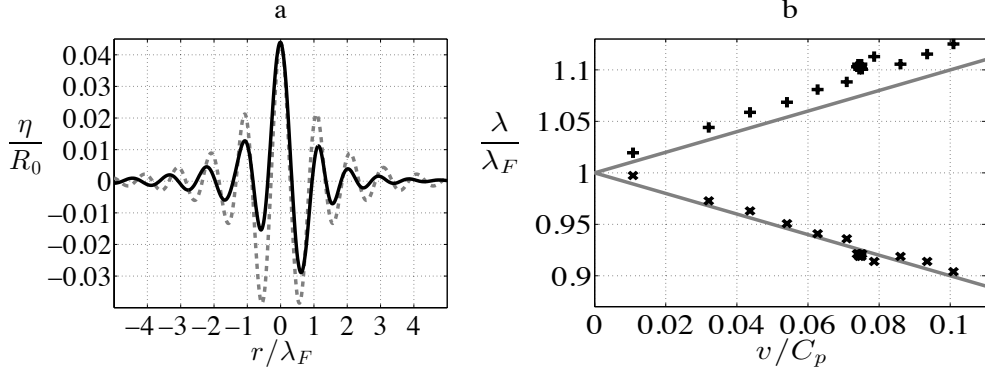


FIGURE 11. (a) The Doppler effect arising at $\Omega = 0.8, \Gamma = 3.6$. The solid line shows the surface along the direction of motion, the dashed line that in the transverse direction. The horizontal axis is in units of Faraday wavelengths, and the vertical axis in units of drop radii. (b) The Doppler effect displayed ahead and behind the walker, with λ/λ_F (vertical axis) in terms of V/C_p . The \times markers depict the wavelength ahead of the walker, the $+$ markers the wavelength behind it. The lines indicate the magnitude of the Doppler effect reported by Eddi et al. (2011). V is the mean walker speed and C_p the phase speed.

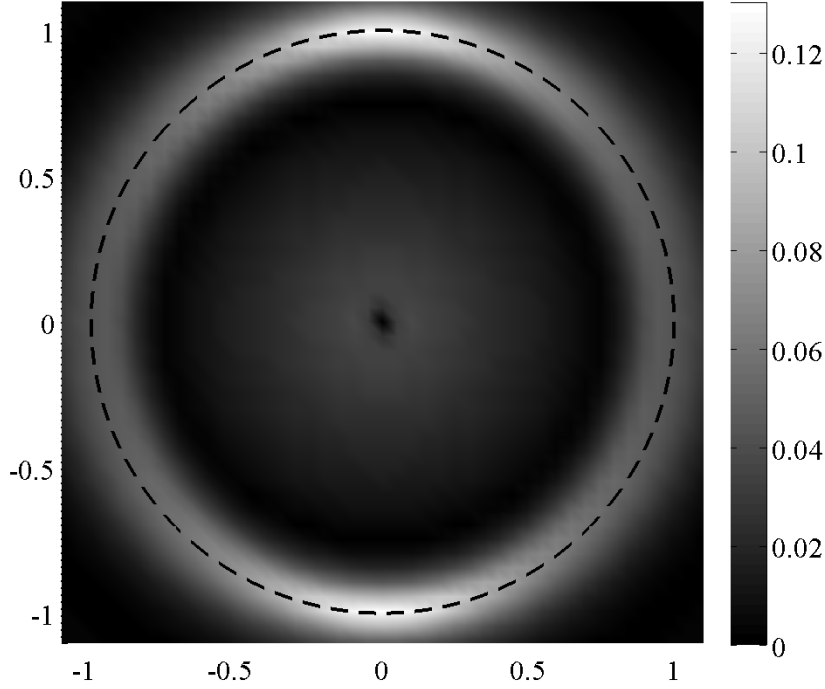


FIGURE 12. Fourier spectrum of the numerical wave field for a walker immediately before the drop impacts the surface. The spectrum is distributed about the dashed circle, which has radius k_F , indicating the support of the spectrum of $J_0(k_F r)$. Axes are in units of Faraday wavelengths.

The free surface elevation of a $(2, 1)$ walker with straight-line trajectory $\mathbf{X} = (V(t - \phi), 0)$ with impacts at $t_n = -2nT_f + \phi$ and observed immediately before an impact at $t = \phi$ is

$$\eta = A \sum_{n=1}^{\infty} \frac{e^{-2n/M_e}}{\sqrt{2nT_f}} \cos \left[\pi \frac{\phi}{T_f} \right] J_0(k_F |(x + 2nVT_f, y)|),$$

and its Fourier transform is given by (using the formula for the Fourier transform of a shifted function):

$$\hat{\eta} = A \cos \left[\pi \frac{\phi}{T_f} \right] \hat{J}_0(\mathbf{k}/k_F) \sum_{n=1}^{\infty} \frac{e^{-2n/M_e}}{\sqrt{2nT_f}} e^{-ik_1 2nVT_f}. \quad (3.11)$$

As this spectrum is a multiple of the spectrum of $\hat{J}_0(\mathbf{k}/k_F)$, it only contains wavelengths $2\pi/k_F$ and so cannot capture a Doppler shift. The angular dependence of the spectrum is given by the sum in equation (3.11). Figure 12 shows the Fourier spectrum of the numerical wave field for a walker immediately before the drop impacts on the surface. By comparison, the discrete Bessel function wavefield spectrum lies on the circle.

Finally, typical wave fields computed from our single droplet simulations are shown in figure 13. These are consistent with the laboratory images presented in Eddi et al. (2011), as well as the stroboscopic model predictions of Oza *et al.* (2013). Figure 13d shows that when the walker moves rapidly to the right, an interference pattern appears in its wake.

3.4. Two-particle interaction

The generalization of our model to numerous walkers is straightforward, so we do not present it in detail. The additional ingredient is the computation of a mean free surface elevation $\bar{\eta}_j$ for each particle during the fluid impact. This elevation incorporates the waves due to the impact of all other particles. Hence, in computing $\bar{\eta}_j$, we take the pressure in (2.38) to be:

$$\sum_{i \neq j} P_{D,i}(\mathbf{x} - \mathbf{X}_i(t), t).$$

We proceed by investigating the interaction between a pair of walkers, as was examined experimentally by Protière et al. (2006), Protière et al. (2008) and Eddi et al. (2012). We confine our attention to two special cases, where the walkers are initially antiparallel and parallel. Protière et al. (2008) report that when the walkers are initially antiparallel, and so approach each other, they may either scatter or lock into orbit, depending on their relative bouncing phase, and the perpendicular distance between their original paths, the so-called impact parameter, d_I .

In the scattering regime, wave forces on the droplets causes them to repel. Conversely, in the orbiting regime, the droplet pair forms a stable and well-defined association bound together by their superposed pilot waves. In figure 14, orbiting and scattering of in-phase droplets are shown. Orbital states obtained by varying Γ for in- and out-of-phase walkers are shown in figure 15. The orbits are stable (i.e. persistent after extended computations) and can have different forms (circular, periodic wobbling or chaotic wobbling) depending on the system parameters, particularly the forcing amplitude Γ . Note that the in-phase orbiters have an orbital diameter of approximately $0.8\lambda_F$; the out-of-phase orbiters approximately $1.3\lambda_F$. This difference may be roughly understood on the grounds that orbiters are more stable when bouncing in the troughs than on the crests of their partner's wave field.

When the walkers are initially parallel, they may lock into a ‘promenade mode’, in

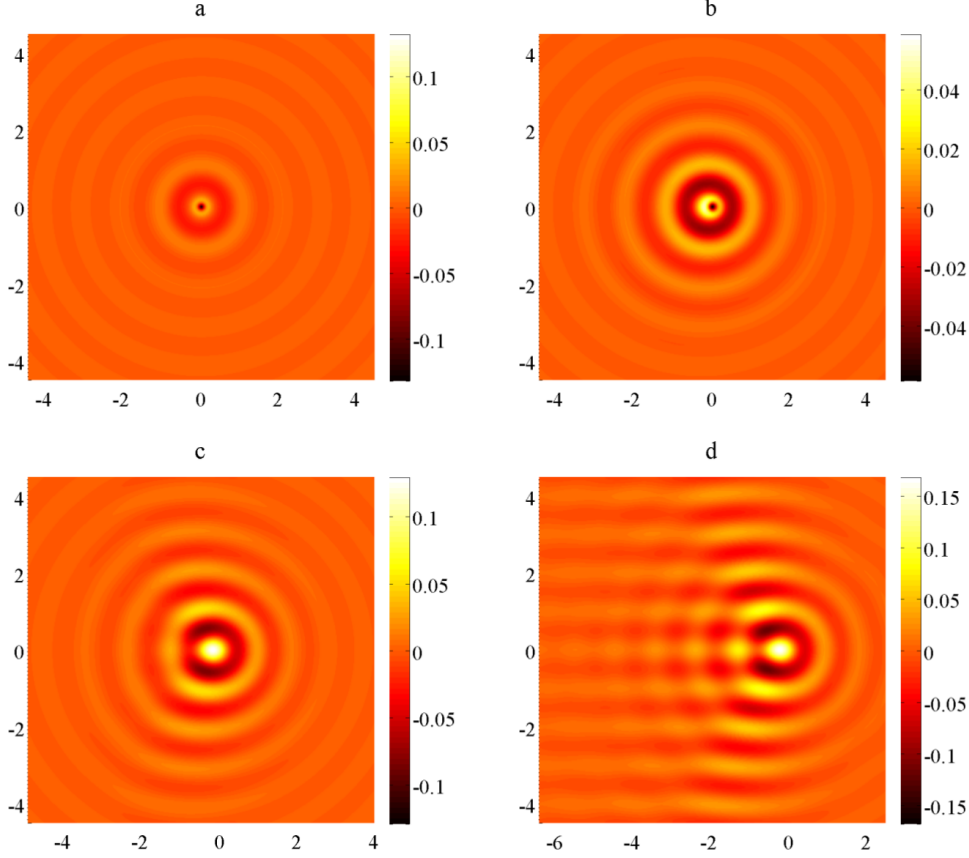


FIGURE 13. Wave fields accompanying bouncing and walking drops. (a) At $\Gamma = 3.1$, the axisymmetric wave field of a bouncer arises. At (b) $\Gamma = 3.3$, (c) $\Gamma = 3.8$, and (d) $\Gamma = 4.15$, walkers of progressively increasing speed arise, their wave fields being progressively more fore-aft asymmetric. Axes are in units of Faraday wavelengths, and the surface elevation (sidebar) is in units of drop radii. In all figures, the drop is at the origin and (except for a) moving to the right. The simulations were performed with drops of $\Omega = 0.8$ (i.e. $R_0 = 0.38\text{mm}$), for $\Gamma_W = 3.15$.

which they walk in unison, but the distance between them oscillates periodically (figure 16). After an initial transient, a well-defined periodically oscillating walking state emerges. This promenade mode and the characteristic oscillations in speed are both readily observed in the laboratory, and have recently been investigated by Borghesi et al. (2014).

4. Conclusions

We have developed a model of pilot-wave hydrodynamics that utilizes a self-consistent treatment of the generation and propagation of the walker's Faraday wave field. Specifically, the walker is described through coupling the droplet and its wave, the vertical motion of the former serving as the source of the latter. The waves are described as deep-water Faraday waves wherein the dissipation occurs in a viscous boundary layer at the free surface. Like the model of Molacek & Bush (2013ab), it explicitly considers the bouncing phase through consideration of a vertical dynamics; thus, it is able to

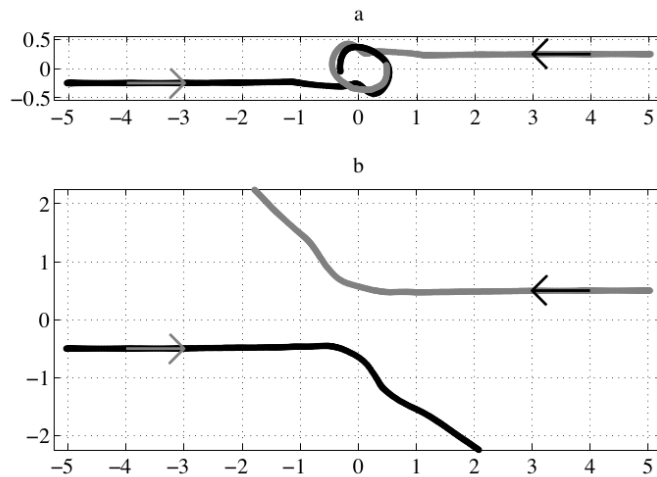


FIGURE 14. Collisions of in-phase droplet pairs. (a) With impact parameter of $d_I = 0.5\lambda_F$, the pair are captured in a periodic orbit with a diameter of approximately $0.5\lambda_F$. (b) With an impact parameter $d_I = \lambda_F$, the droplets scatter. Vertical and horizontal length scales are in units of λ_F .

reproduce the observed dependence of the bouncing and walking modes on the system parameters. This gives it a marked advantage over the stroboscopic model presented by Oza *et al.* (2014), wherein the effects of variable bouncing phase are not considered. Its additional advantage over the model of Molacek & Bush (2013b) is that it captures more accurately the wave field, including the transient wave field that serves as the precursor to the stationary wave field, as may play a significant role in the interaction of walkers with boundaries or other walkers.

We have demonstrated that the model reproduces certain features of walker-walker interactions that have not been comprehensively studied with previous models, including orbiting, scattering and the promenade mode. For these interactions, the dependence of the system behavior on the impact phase has been highlighted. In particular, we have seen that the scattering behavior of a pair of walkers depends on both the impact parameter and the relative phase of the walkers. A quantitative comparison between our model predictions and an ongoing experimental investigation of walker-walker interactions will be presented elsewhere.

Through its relatively complete treatment of the wave field, our model provides a platform for examining the interaction of walkers with boundaries which in experiments are modeled by a transition between deep and shallow regions. The required extension of our model to the case of variable bottom topography is currently underway. This extension will ultimately allow us to explore a number of quantum analog systems, including single-particle diffraction, corrals and tunneling.

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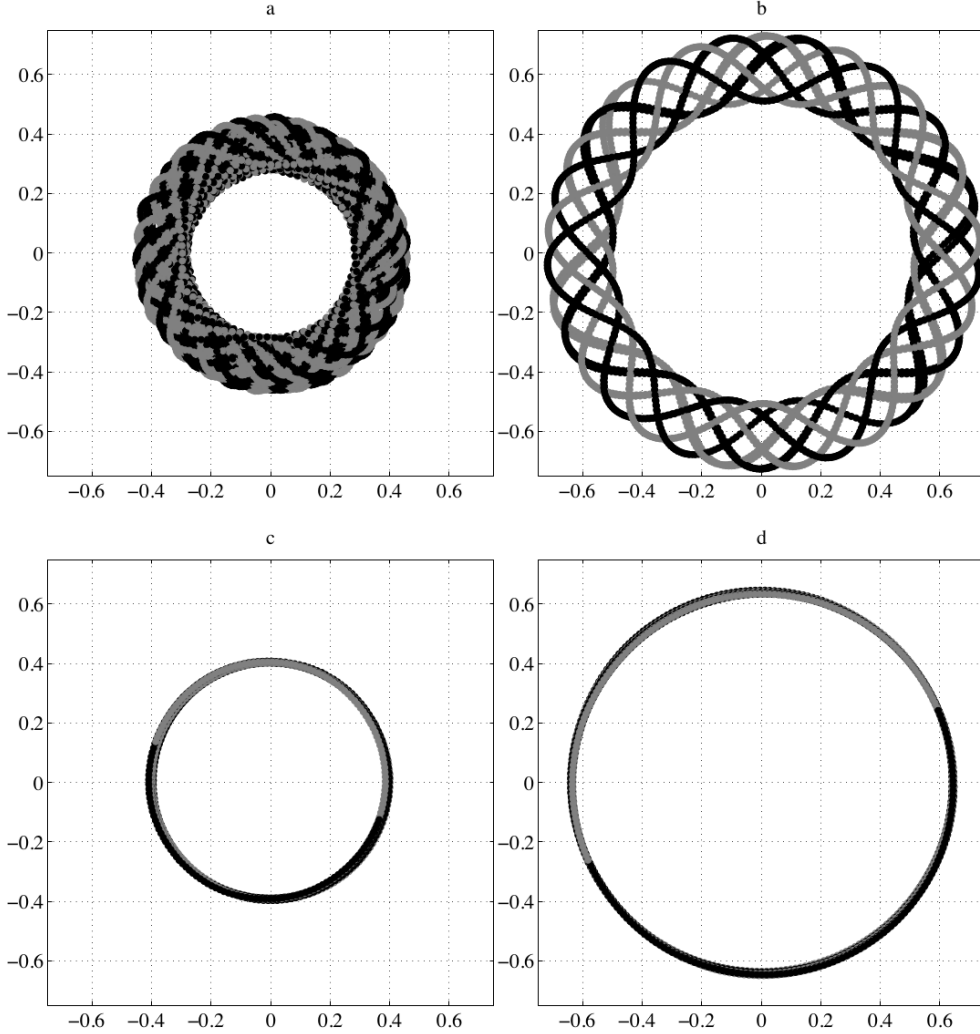


FIGURE 15. Orbital modes arising through walker-walker interactions, specifically from a pair of identical walkers launched with antiparallel velocity. Bouncing in phase with (a) $c_4 = 0.32$, $\Omega = 0.8$, $\Gamma = 3.7$ and (c) $c_4 = 0.3$, $\Omega = 0.7$, $\Gamma = 4$. Bouncing out of phase with (b) $c_4 = 0.32$, $\Omega = 0.8$, $\Gamma = 3.6$ and (d) $c_4 = 0.3$, $\Omega = 0.7$, $\Gamma = 4$. Axes are in units of the Faraday wavelength. Note the offset of orbital radii between the in- and out-of-phase walkers.

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Appendix A. Stability analysis for the vibrating bath

We here present a short calculation that leads to an accurate prediction of the stability threshold and most unstable wave-number for our parametrically driven flow. We shall find the approximate conditions for neutral stability of the subharmonic mode corresponding to $\frac{1}{2}\omega_0$ in the undamped case. The main idea is to truncate the infinite Hill

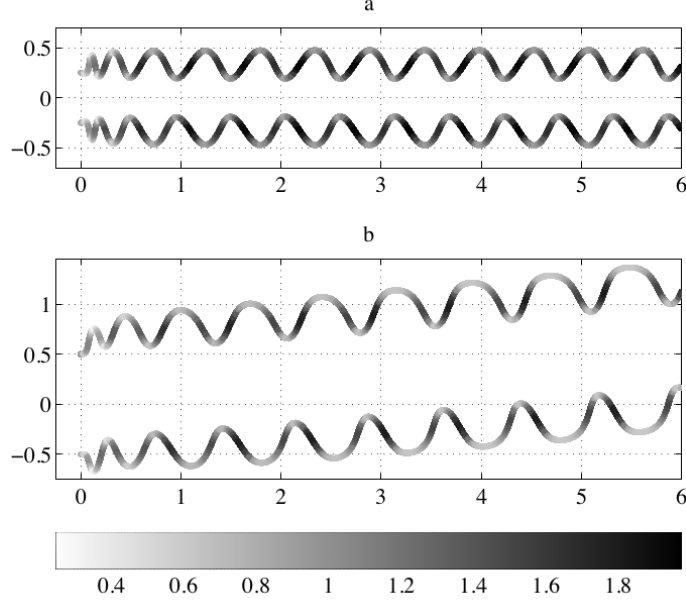


FIGURE 16. Droplet pair forming a stable ‘promenade mode’. Two identical droplets propagate together in a given direction while oscillating in the transverse direction. Axes are in units of the Faraday wavelength. Two walkers bouncing (a) in-phase, and (b) out of phase. The instantaneous speed V/C_p is color-coded according to the underbar. The mean speeds of the promenading pair in (a) and (b) are 0.88 and 0.63 of the single-walker free walking speed, respectively.

matrix for the modes arising from the Floquet analysis of the damped Mathieu equation (see Holmes (1995)) restricted to neutrally stable modes. Consider the damped system of equations for the waves in Fourier space

$$\eta_t = k\phi - 2\nu k^2 \eta, \quad (\text{A } 1)$$

$$\phi_t = -g_\Gamma(t)\eta - \frac{\sigma}{\rho}k^2\eta - 2\nu k^2\phi, \quad (\text{A } 2)$$

where $g_\Gamma = g(1 + \Gamma \sin \omega_o t)$. Proposing a solution of the following neutrally stable form

$$\eta \equiv N^+ e^{i\omega_o t/2} e^{ikx} + N^- e^{-i\omega_o t/2} e^{ikx},$$

$$\phi \equiv M^+ e^{i\omega_o t/2} e^{ikx} + M^- e^{-i\omega_o t/2} e^{ikx},$$

and substituting into the equations yields the truncated system

$$\frac{i\omega_0}{2}N^+ = kM^+ - 2\nu k^2 N^+, \quad (\text{A } 3)$$

$$\frac{i\omega_0}{2}M^+ = -gN^+ + \frac{ig\Gamma}{2}N^- + \frac{\sigma}{\rho}(-k^2 N^+) - 2\nu k^2 M^+, \quad (\text{A } 4)$$

$$-\frac{i\omega_0}{2}N^- = kM^- - 2\nu k^2 N^-, \quad (\text{A } 5)$$

$$-\frac{i\omega_0}{2}M^- = -gN^- - \frac{ig\Gamma}{2}N^+ + \frac{\sigma}{\rho}(-k^2 N^-) - 2\nu k^2 M^-. \quad (\text{A } 6)$$

Terms proportional to $e^{3i\omega_o t/2}$ and higher harmonics have been neglected. However, as we shall see, this approximation is sufficient to obtain accurate predictions. A non-trivial

solution $[N^+, M^+, N^-, M^-]^T$ will arise for nonzero determinant of the truncated Hill matrix:

$$\begin{vmatrix} \left(\frac{i\omega_0}{2} + 2\nu k^2\right) & -k & 0 & 0 \\ g + \frac{\sigma}{\rho}k^2 & \left(\frac{i\omega_0}{2} + 2\nu k^2\right) & -\frac{ig\Gamma}{2} & 0 \\ 0 & 0 & \left(-\frac{i\omega_0}{2} + 2\nu k^2\right) & -k \\ \frac{ig\Gamma}{2} & 0 & g + \frac{\sigma}{\rho}k^2 & \left(-\frac{i\omega_0}{2} + 2\nu k^2\right) \end{vmatrix} = 0.$$

We note that the infinite block diagonal Hill matrix is obtained when all harmonics are included. Hence, neutrally stable oscillations must satisfy

$$\left[\left(\frac{i\omega_0}{2} + 2\nu k^2 \right)^2 + k \left(g + \frac{\sigma}{\rho} k^2 \right) \right] \left[\left(-\frac{i\omega_0}{2} + 2\nu k^2 \right)^2 + k \left(g + \frac{\sigma}{\rho} k^2 \right) \right] - \frac{g^2 \Gamma^2}{4} k^2 = 0,$$

which simplifies to the expression (2.35) in the text. If we fix ω_0 , this gives Γ as a function of k , the neutral curve in the $\Gamma - k$ plane as shown in figure 1.

Appendix B. Physical parameters and constants used in the simulations

Physical parameters:

$$\sigma = 20.6 \text{ dyne/cm} = 20.6 \cdot 10^{-3} \text{ kg/s}^2 \quad (\text{B } 1)$$

$$\rho = 0.949 \text{ g/cm}^3 = 949 \text{ kg/m}^3 \quad (\text{B } 2)$$

$$\nu = \mu/\rho = 0.2 \text{ St} = 0.2 \text{ cm}^2/\text{s} = 2 \cdot 10^{-5} \text{ m}^2/\text{s} \quad (\text{B } 3)$$

$$\mu_{\text{air}} = 1.8 \cdot 10^{-4} \text{ g/(cm} \cdot \text{s)} = 1.8 \cdot 10^{-5} \text{ Kg/(m} \cdot \text{s)} \quad (\text{B } 4)$$

$$g = 980 \text{ cm/s}^2 = 9.8 \text{ m/s}^2 \quad (\text{B } 5)$$

$$\omega_0 = 80 \cdot 2\pi \text{ s}^{-1} \quad (\text{B } 6)$$

$$(\text{B } 7)$$

The system parameters used for the singe-drop simulations were

$$c_1 = 0.7, \quad c_2 = 8, \quad c_3 = 0.7, \quad c_4 = 0.13. \quad (\text{B } 8)$$

For the numerical experiments involving two drops, it was only possible to observe the more complex patterns of interactions (i.e. orbiters and promenaders) by using higher values of c_4 (i.e. $c_4 = 0.3, c_4 = 0.32$).

REFERENCES

- AMBROSE, D.M., BONA, J.L. & NICHOLLS, D.P. 2012 Well-posedness for water waves with viscosity. *Discr. Cont. Dyn. Syst. B*, **17**, 1113-1137.
- BACCHIAGALUPPI G., & VALENTINI A. 2009 *Quantum theory at the crossroads: Reconsidering the 1927 Solvay conference*. Cambridge University Press.
- BACH R., POPE D., LIOU S. & BATELAAN H. 2013 Controlled double-slit electron diffraction. *New J. Phys.* **15**, 033018.
- BENJAMIN, T.B. & URSELL, F. 1954 The stability of the plane free surface of a liquid in vertical periodic motion. *Proc. R. Soc. Lond. A* **225**, 505-515.
- BORGHESI C, MOUKHTAR J, LABOUSSE M, EDDI A, FORT E AND COUDER Y 2014 The interaction of two walkers: wave-mediated energy and force *submitted manuscript*.
- BUSH, J.W.M. 2010 Quantum mechanics writ large. *Proc. Nat. Acad. Sci.* **107**, 17455-17456.
- BUSH, J.W.M. 2015 Pilot-wave hydrodynamics. *Ann. Rev. Fluid Mech.* **47**, 269-292.
- DE BROGLIE, L. 1926. Ondes et mouvements. *Gautier Villars*, Paris.

- DE BROGLIE, L. 1930 An introduction to the study of wave mechanics. *Methuen & Co.* London.
- DE BROGLIE, L. 1956 Une interprétation causale et nonlinéaire de la Mécanique ondulatoire: la théorie de la double solution. *Gautier-Villars* Paris.
- DE BROGLIE, L. 1987 Interpretation of quantum mechanics by the double solution theory. *Annales de la Fondation Louis de Broglie*. **12**, 1-23.
- COUDER Y., PROTIÈRE S., FORT E., & BOUDAUD A. 2005 Walking and orbiting droplets. *Nature* **437**, 208.
- COUDER Y., FORT, E., GAUTIER C-H. & BOUDAUD A. 2005 From bouncing to floating: non-coalescence of drops on a fluid bath. *Phys. Rev. Lett.* **94**, 177801.
- COUDER, Y. & FORT, E. 2006 Single particle diffraction and interference at a macroscopic scale. *Phys. Rev. Lett.* **97**, 154101.
- COUDER, Y., FORT, E. 2012 Probabilities and trajectories in a classical wave-particle duality. *J. Phys.: Conf. Ser.* **361**, 012001.
- CROMMIE, M.F., LUTZ, C.P. & EIGLER, D.M. 1993a Imaging standing waves in a two-dimensional electron gas. *Nature* **363**, 524-527.
- CROMMIE, M.F., LUTZ, C.P. & EIGLER, D.M. 1993b Confinement of electrons to quantum corrals on a metal surface. *Science* **262**, 218-220.
- DIAS, F., DYACHENKO, A. I. & ZAKHAROV, V. E. 2008 Theory of weakly damped free-surface flows: A new formulation based on potential flow solutions. *Phys. Lett. A* **372**, 1297-1302.
- EDDI, A., TERWAGNE, D., FORT, E., & COUDER, Y. 2008 Wave propelled ratchets and drifting rafts. *Europhys. Lett.* **82**, 44001.
- EDDI, A., DECELLE, A., FORT, E. & COUDER, Y. 2009a. Archimedean lattices in the bound states of wave interacting particles. *Europhys. Lett.* **87**, 56002.
- EDDI, A., FORT, E., MOISY, F. & COUDER, Y. 2009b. Unpredictable tunneling of a classical wave-particle association. *Phys. Rev. Lett.* **102**, 240401.
- EDDI, A., SULTAN, E., MOUKHTAR, J., FORT, E., ROSSI, M. & COUDER, Y. 2011 Information stored in Faraday waves: the origin of a path memory, *J. Fluid Mech.* **674**, 433-463.
- EDDI, A., MOUKHTAR, J., PERRARD, J., FORT, E., & COUDER, Y. 2012 Level splitting at a macroscopic scale. *Phys. Rev. Lett.* **108**: 264503.
- FORT, E., EDDI, A., BOUDAUD, A., MOUKHTAR, J. & COUDER, Y. 2010 Path memory induced quantization of classical orbits. *Proc. Natl Acad. Sci. USA* **107**, 17515-17520.
- Gilet, T., & Bush, J.W.M., 2009a The fluid trampoline: droplets bouncing on a soap film. *J. Fluid Mech.* 625: 167–203.
- GILET, T. & BUSH, J.W.M. 2009b Chaotic bouncing of a droplet on a soap film, *Phys. Rev. Lett.* **102**, 014501.
- HARRIS, D.M., MOUKHTAR, J., FORT, E., COUDER, Y., & BUSH, J.W.M. 2013 Wavelike statistics from pilot-wave dynamics in a circular corral. *Phys. Rev. E* **88**, 011001: 1-5.
- HARRIS, D.M. & BUSH, J.W.M. 2013 The pilot-wave dynamics of walking droplets. *Phys. Fluids* **25**:091112.
- HARRIS, D.M. & BUSH, J.W.M. 2014 Droplets walking in a rotating frame: from quantized orbits to multimodal statistics. *J. Fluid Mech.* **739**: 444-464.
- HOLMES, M. H. Introduction to perturbation methods. Texts in Applied Mathematics volume 20. Springer-Verlag, New York, 1995.
- KUMAR, K. & TUCKERMAN, L.S. 1994 Parametric instability of the interface between two fluids. *J. Fluid Mech.* **279**, 49–68.
- KUMAR, K. 1996 Parametric instability of the interface between two fluids. *Proc. Roy. Soc. Lond. A*. **452**, 1113-1126.
- LABOUSSE, M., PERRARD, S., COUDER, Y., & FORT E. 2014 Build-up of selection rules in a memory-based constrained system, *New. J. Phys.*, submitted.
- LAMB, H. 1932 *Hydrodynamics*, Cambridge Univ. Press.
- MILEWSKI, P. A. & TABAK, E. G. 1999 A pseudospectral procedure for the solution of nonlinear wave equations with examples from free-surface flows. *SIAM J. Sci. Comput.* **21(3)**, 1102-1114.
- MOLÁČEK, J. & BUSH, J.W.M. 2012 A quasi-static model of drop impact, *Phys. Fluids* **24**, 127103.
- MOLÁČEK, J. & BUSH, J.W.M. 2013a Drops bouncing on a vibrating bath. *J. Fluid Mech.* **727**, 582-611.

- MOLÁČEK, J. & BUSH, J.W.M. 2013b Drops walking on a vibrating bath: towards a hydrodynamic pilot-wave theory. *J. Fluid Mech.* **727**, 612-647.
- OZA, A.U., ROSALES, R.R. & BUSH, J.W.M. 2013 A trajectory equation for walking droplets: hydrodynamic pilot-wave theory. *J. Fluid Mech.* **737**, 552-570.
- OZA, A.U., HARRIS, D.M., ROSALES, R.R., & BUSH, J.W.M. 2014a Pilot-wave dynamics in a rotating frame: on the emergence of orbital quantization. *J. Fluid Mech.*, **744**, 404-429.
- OZA, A.U., HARRIS, D.M., ROSALES, R.R., & BUSH, J.W.M. 2014b Pilot-wave dynamics in a rotating frame: exotic orbits. *Phys. Fluids* **26**: 082101 .
- PERRARD, S., LABOUSSE, M., MISKIN, M., FORT, E. & COUDER Y. 2014 Self-organization into quantized eigenstates of a classical wave-driven particle. *Nature Comm.* 5:3219.
- PROTIÈRE, S., BOUDAUD, A. & COUDER, Y. 2006 Particle-wave association on a fluid interface. *J. Fluid Mech.* **554**, 85-108.
- PROTIÈRE, S., COUDER, Y., FORT, E. & BOUDAUD, A. 2005. The self-organization of capillary wave sources. *J. Phys.: Condens. Matter* **17**, 3529.
- PROTIÈRE, S., BOHN, S. & COUDER Y. 2008. Exotic orbits of two interacting wave sources. *Phys. Rev. E*. **78**, 36204.
- WALKER J. 1978 Drops of liquid can be made to float on the liquid. What enables them to do so? *Sci. Am.* **238**, 151-158.
- WANG, Z., & MILEWSKI, P.A. 2012 Dynamics of gravity-capillary solitary waves in deep water. *J. Fluid Mech.* **708**, 480-501.
- WIND-WILLASSEN, O., MOLÁČEK, J., HARRIS, D.M. & BUSH, J. W. M. 2013 Exotic states of bouncing and walking droplets. *Phys. Fluids* **25**, 082002.