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A Formula for the Specialization of Skew Schur Functions*

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Abstract. We give a formula for $s_{\lambda/\mu}(1,q,q^2,\ldots)/s_\lambda(1,q,q^2,\ldots)$, which generalizes a result of Okounkov and Olshanski about $f^{\lambda/\mu}/f^\lambda$.

Keywords: skew Schur function, q-analogue, jeu de taquin

1. Introduction

For the notation and terminology below on symmetric functions, see Stanley [6] or Macdonald [4]. Let μ be a partition of some nonnegative integer. A *reverse tableau* of shape μ is an array of positive integers of shape μ which is weakly decreasing in rows and strictly decreasing in columns. Let $\mathrm{RT}(\mu,n)$ be the set of all reverse tableaux of shape μ whose entries belong to $\{1,2,\ldots,n\}$.

Recall that f^λ and $f^{\lambda/\mu}$ denote the number of SYT (standard Young tableaux) of shape λ and λ/μ respectively, and $l(\mu)$ denotes the length of μ. Okounkov and Olshanski [5, (0.14) and (0.18)] give the following surprising formula.

Theorem 1.1. *Let $\lambda \vdash m$, $\mu \vdash k$ with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \le l(\lambda) \le n$. Then*

$$\frac{(m)_k f^{\lambda/\mu}}{f^\lambda} = \sum_{T \in \mathrm{RT}(\mu,n)} \prod_{u\in\mu} \left(\lambda_{T(u)} - c(u)\right), \tag{1.1}$$

where $c(u)$ and $T(u)$ are the content and entry of the square u respectively, and $(m)_k = m(m-1)\cdots(m-k+1)$.

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In this paper, we generalize the above result to a q-analogue. Our main result is the following.

Theorem 1.2. *Let λ and μ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. Then*

$$\frac{s_{\lambda/\mu}\left(1, q, q^2, \ldots\right)}{s_\lambda\left(1, q, q^2, \ldots\right)} = \sum_{T \in \mathrm{RT}(\mu,n)} \prod_{u \in \mu} q^{1-T(u)} \left(1 - q^{\lambda_{T(u)} - c(u)}\right), \tag{1.2}$$

where the right-hand side is defined to be 1 when μ is the empty partition.

2. Proof of the Main Result

For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we define $[n] = 1 - q^n$ and denote by $(n \downharpoonleft k)$ the kth falling q-factorial power, i.e.,

$$(n \downharpoonleft k) = \begin{cases} [n][n-1]\cdots[n-k+1], & \text{if } k = 1, 2, \ldots, \\ 1, & \text{if } k = 0. \end{cases}$$

In particular, we use $[k]!$ to denote $(k \downharpoonleft k)$, and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \frac{[n]!}{[k]![n-k]!}$ for $n \geq k$.

Let λ and μ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. We define

$$t_{\lambda/\mu,n}(q) = s_{\lambda/\mu}\left(1, q, q^2, \ldots\right) \prod_{u \in \lambda/\mu} [n + c(u)]. \tag{2.1}$$

The following lemma is given in [6, Exer. 102, p. 551 and Lem. 7.21.1].

Lemma 2.1. *Let λ and μ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. Then we have*

(a) $t_{\lambda/\mu,n}(q) = \det \left[\left[\begin{matrix} \lambda_i + n - i \\ \lambda_i - \mu_j - i + j \end{matrix}\right]\right]_{i,j=1}^{n}$,

(b) $\prod_{u \in \lambda} [n + c(u)] = \prod_{i=1}^{n} \frac{[\nu_i]!}{[n-i]!}$, *where* $\nu_i = \lambda_i + n - i$.

Lemma 2.2. *Let λ and μ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. Then we have*

$$\frac{s_{\lambda/\mu}\left(1, q, q^2, \ldots\right)}{s_\lambda\left(1, q, q^2, \ldots\right)} = \frac{\det\left[(\lambda_i + n - i \downharpoonleft \mu_j + n - j)\right]_{i,j=1}^{n}}{\det\left[(\lambda_i + n - i \downharpoonleft n - j)\right]_{i,j=1}^{n}}. \tag{2.2}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}\frac{s_{\lambda/\mu}\left(1,q,q^2,\dots\right)}{s_\lambda\left(1,q,q^2,\dots\right)} &= \frac{t_{\lambda/\mu,n}(q)}{t_{\lambda,n}(q)}\prod_{u\in\mu}[n+c(u)] \\ &= \frac{\det\left[\begin{bmatrix}\lambda_i+n-i\\ \lambda_i-\mu_j-i+j\end{bmatrix}\right]_{i,j=1}^n}{\det\left[\begin{bmatrix}\lambda_i+n-i\\ \lambda_i-i+j\end{bmatrix}\right]_{i,j=1}^n}\prod_{j=1}^n\frac{[\mu_j+n-j]!}{[n-j]!} \\ &= \frac{\det\left[\begin{bmatrix}\lambda_i+n-i\\ \lambda_i-\mu_j-i+j\end{bmatrix}[\mu_j+n-j]!\right]_{i,j=1}^n}{\det\left[\begin{bmatrix}\lambda_i+n-i\\ \lambda_i-i+j\end{bmatrix}[n-j]!\right]_{i,j=1}^n} \\ &= \frac{\det\left[(\lambda_i+n-i\downharpoonleft\mu_j+n-j)\right]_{i,j=1}^n}{\det\left[(\lambda_i+n-i\downharpoonleft n-j)\right]_{i,j=1}^n}.\end{aligned}$$

∎

We first consider the denominator of the right-hand side of (2.2).

Lemma 2.3. *We have*

$$\det\left[(\lambda_i+n-i\downharpoonleft n-j)\right]_{i,j=1}^n = \left(\prod_{i=1}^n q^{(i-1)\lambda_i}\right)\prod_{1\le i<j\le n}[\lambda_i-\lambda_j-i+j]. \tag{2.3}$$

Proof. For $j=1,\dots,n-1$, we subtract from the jth column of the determinant on the left-hand side the $(j+1)$th column, multiplied by $[\lambda_1+j]$. Then for all $j<n$, the (i,j)th entry becomes

$$(\lambda_i+n-i\downharpoonleft n-j-1)([\lambda_i+j+1-i]-[\lambda_1+j]). \tag{2.4}$$

In particular, the $(1,j)$th entry becomes 0 for $j<n$. Therefore, the determinant on the left-hand side becomes

$$\left(\prod_{i=2}^n[\lambda_1-\lambda_i-1+i]q^{\lambda_i}\right)\det\left[(\lambda_{i+1}+n-i-1\downharpoonleft n-j-1)\right]_{i,j=1}^{n-1},$$

and then the result follows by induction. ∎

The following lemma is almost the same as [5, Lemma 2.1], just lifted to the q-analogue.

Lemma 2.4. *Let $x,\ y\in\mathbb{Z}$ with $x+1\neq y$ and $k\in\mathbb{N}$. Then we have*

$$\frac{(y\downharpoonleft k+1)-(x+1\downharpoonleft k+1)}{-q^y+q^{x+1}} = \sum_{l=0}^k q^{-l}(y\downharpoonleft l)(x-l\downharpoonleft k-l).$$

Proof. We have

$$\begin{aligned}
&\left(-q^{y}+q^{x+1}\right)\sum_{l=0}^{k} q^{-l}(y\downharpoonright l)(x-l\downharpoonright k-l)\\
&=\sum_{l=0}^{k}\left(-q^{y-l}+q^{x+1-l}\right)(y\downharpoonright l)(x-l\downharpoonright k-l)\\
&=\sum_{l=0}^{k}(y\downharpoonright l)(x-l\downharpoonright k-l)[y-l]-\sum_{l=0}^{k}(y\downharpoonright l)(x-l\downharpoonright k-l)[x+1-l]\\
&=\sum_{l=1}^{k+1}(y\downharpoonright l)(x-l+1\downharpoonright k-l+1)-\sum_{l=0}^{k}(y\downharpoonright l)(x-l+1\downharpoonright k-l+1)\\
&=(y\downharpoonright k+1)-(x+1\downharpoonright k+1).
\end{aligned}$$

∎

For two partitions μ and ν, we write $\mu \succeq \nu$ if $\mu_i \geq \nu_i \geq \mu_{i+1}$ for all $i \in \mathbb{N}$, or equivalently ν is obtained from μ by removing a horizontal strip. Thus given a reverse tableau $T \in \mathrm{RT}(\mu, n)$, we can regard it as a sequence

$$\mu = \mu^{(1)} \succeq \mu^{(2)} \succeq \cdots \succeq \mu^{(n+1)} = \emptyset,$$

where $\mu^{(i)}$ is the shape of the reverse tableau consisting of entries of T not less than i.

Let μ/ν be a skew diagram. We define

$$(x \downharpoonright \mu/\nu) = \prod_{u\in\mu/\nu} [x - c(u)]. \tag{2.5}$$

This is a generalization of the falling q-factorial powers. Now we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.2, it is equivalent to prove that

$$\frac{\det[(\lambda_i+n-i\downharpoonright \mu_j+n-j)]_{i,j=1}^{n}}{\det[(\lambda_i+n-i\downharpoonright n-j)]_{i,j=1}^{n}} = \sum_{T\in\mathrm{RT}(\mu,n)}\prod_{u\in\mu} q^{1-T(u)}\left[\lambda_{T(u)}-c(u)\right]. \tag{2.6}$$

Since Lemma 2.2 still holds when $\mu \not\subseteq \lambda$, in which case both sides of (2.2) are equal to 0, we just assume $l(\mu) \leq l(\lambda) \leq n$ in (2.6). The proof of (2.6) is by induction on n. The case $n = 0$ is trivial, which is equivalent to the statement $\frac{1}{1} = 1$. For the induction step ($n > 0$), it suffices to prove that

$$\begin{aligned}
&\frac{\det[(\lambda_i+n-i\downharpoonright \mu_j+n-j)]_{i,j=1}^{n}}{\det[(\lambda_i+n-i\downharpoonright n-j)]_{i,j=1}^{n}}\\
&=\sum_{\substack{\nu\preceq\mu\\ l(\nu)<n}} q^{-|\nu|}(\lambda_1\downharpoonright \mu/\nu)\frac{\det\left[\left(\lambda_i^{\uparrow}+n-1-i\downharpoonright \nu_j+n-1-j\right)\right]_{i,j=1}^{n-1}}{\det\left[\left(\lambda_i^{\uparrow}+n-1-i\downharpoonright n-1-j\right)\right]_{i,j=1}^{n-1}},
\end{aligned} \tag{2.7}$$

where $\lambda^{\uparrow}$ denotes the partition obtained from λ by removing λ_1.

To see the sufficiency, let $T^{\uparrow}$ be the reverse tableau obtained from a given $T \in \mathrm{RT}(\mu, n)$ by removing all entries equal to 1 and decreasing remaining entries by 1. Let ν be the shape of $T^{\uparrow}$. Then we have $\nu \preceq \mu$ and $l(\nu) < n$. On the other hand, given partitions ν and μ with $\nu \preceq \mu$ and $l(\nu) < n$, then for $T^{\uparrow} \in \mathrm{RT}(\nu, n-1)$, we can uniquely recover $T \in \mathrm{RT}(\mu, n)$ from $T^{\uparrow}$ in a reverse way. Thus for a fixed ν with $\nu \preceq \mu$ and $l(\nu) < n$, we have

$$\sum_{\substack{T \in \mathrm{RT}(\mu,n)\\ \mathrm{shape}(T^{\uparrow})=\nu}} \prod_{u\in\mu} q^{1-T(u)} \left[\lambda_{T(u)} - c(u)\right] \tag{2.8}$$

$$= \sum_{T^{\uparrow}\in \mathrm{RT}(\nu,n-1)} (\lambda_1 \downharpoonleft \mu/\nu) \prod_{u\in\nu} q^{-1} q^{1-T^{\uparrow}(u)} \left[\lambda^{\uparrow}_{T^{\uparrow}(u)} - c(u)\right]$$

$$= q^{-|\nu|} (\lambda_1 \downharpoonleft \mu/\nu) \sum_{T^{\uparrow}\in \mathrm{RT}(\nu,n-1)} \prod_{u\in\nu} q^{1-T^{\uparrow}(u)} \left[\lambda^{\uparrow}_{T^{\uparrow}(u)} - c(u)\right]$$

$$= q^{-|\nu|} (\lambda_1 \downharpoonleft \mu/\nu) \frac{\det\left[\left(\lambda^{\uparrow}_i + n-1-i \downharpoonleft \nu_j + n-1-j\right)\right]_{i,j=1}^{n-1}}{\det\left[\left(\lambda^{\uparrow}_i + n-1-i \downharpoonleft n-1-j\right)\right]_{i,j=1}^{n-1}}, \tag{2.9}$$

here the last equality follows from induction hypothesis. By summing (2.8) and (2.9) respectively over all partitions ν with $\nu \preceq \mu$ and $l(\nu) < n$, we then obtain (2.6) from (2.7).

Consider the numerator of the right-hand side of (2.7),

$$\det[(\lambda_i + n - i \downharpoonleft \mu_j + n - j)]. \tag{2.10}$$

For $j = 1, 2, \ldots, n-1$, we subtract from the jth column of (2.10) the $(j+1)$th column, multiplied by $(\lambda_1 - \mu_{j+1} + j \downharpoonleft \mu_j - \mu_{j+1} + 1)$. Then for all $j < n$, the (i, j)th entry of (2.10) becomes

$$\begin{aligned}(\lambda_i + n - i \downharpoonleft \mu_{j+1} + n - j - 1)((\lambda_i - \mu_{j+1} + j + 1 - i \downharpoonleft \mu_j - \mu_{j+1} + 1)\\ - (\lambda_1 - \mu_{j+1} + j \downharpoonleft \mu_j - \mu_{j+1} + 1)).\end{aligned} \tag{2.11}$$

In particular, the first row of (2.10) becomes

$$(0, \ldots, 0, (\lambda_1 + n - 1 \downharpoonleft \mu_n)).$$

We can now apply Lemma 2.4, where we set

$$x = \lambda_1 - \mu_{j+1} + j - 1, \quad k = \mu_j - \mu_{j+1}, \quad y = \lambda_i - \mu_{j+1} + j + 1 - i.$$

Then (2.11) becomes

$$\begin{aligned}-(\lambda_i + n - i \downharpoonleft \mu_{j+1} + n - j - 1)[\lambda_1 - \lambda_i + i - 1] q^{\lambda_i - \mu_{j+1} + j + 1 - i}\\ \cdot \sum_{l=0}^{\mu_j - \mu_{j+1}} q^{-l} (y \downharpoonleft l)(x - l \downharpoonleft k - l).\end{aligned} \tag{2.12}$$

Let $\nu_j = l + \mu_{j+1}$. Since

$$\begin{aligned} q^y - q^{x+1} &= q^{\lambda_i - \mu_{j+1} + j + 1 - i} \cdot [\lambda_1 - \lambda_i + i - 1], \\ (x - l \downharpoonleft k - l) &= (\lambda_1 - \nu_j + j - 1 \downharpoonleft \mu_j - \nu_j), \\ (y \downharpoonleft l) &= (\lambda_i - \mu_{j+1} + j + 1 - i \downharpoonleft \nu_j - \mu_{j+1}), \\ (\lambda_i + n - i \downharpoonleft \mu_{j+1} + n - j - 1)(y \downharpoonleft l) &= (\lambda_i + n - i \downharpoonleft \nu_j + n - j - 1), \end{aligned}$$

(2.12) becomes

$$-[\lambda_1 - \lambda_i + i - 1] q^{\lambda_i - \mu_{j+1} + j + 1 - i} \sum_{\nu_j = \mu_{j+1}}^{\mu_j} q^{\mu_{j+1} - \nu_j} (\lambda_1 - \nu_j + j - 1 \downharpoonleft \mu_j - \nu_j) \cdot (\lambda_i + n - i \downharpoonleft \nu_j + n - j - 1).$$

Expand the determinant (2.10) by the first row,

$$(\lambda_1 + n - 1 \downharpoonleft \mu_n) \det \Bigg[[\lambda_1 - \lambda_{i+1} + i] q^{\lambda_{i+1} + j - i} \sum_{\nu_j = \mu_{j+1}}^{\mu_j} q^{-\nu_j} \cdot (\lambda_1 - \nu_j + j - 1 \downharpoonleft \mu_j - \nu_j)(\lambda_{i+1} + n - i - 1 \downharpoonleft \nu_j + n - j - 1) \Bigg]_{i,j=1}^{n-1}. \tag{2.13}$$

For any chosen value of ν_j $(1 \le j \le n-1)$ in the range from μ_{j+1} to μ_j, $\nu = (\nu_1, \ldots, \nu_{n-1})$ is a partition, and we have $\nu \preceq \mu$. Furthermore, when ν_j $(1 \le j \le n-1)$ ranges from μ_{j+1} to μ_j, ν ranges over all partitions with $\nu \preceq \mu$ and $l(\nu) < n$. Therefore, (2.13) equals

$$(\lambda_1 + n - 1 \downharpoonleft \mu_n) \sum_{\substack{\nu \preceq \mu \\ l(\nu) < n}} \det \Big[[\lambda_1 - \lambda_{i+1} + i] q^{\lambda_{i+1} + j - i - \nu_j} \cdot (\lambda_1 - \nu_j + j - 1 \downharpoonleft \mu_j - \nu_j)(\lambda_{i+1} + n - i - 1 \downharpoonleft \nu_j + n - j - 1) \Big]_{i,j=1}^{n-1}. \tag{2.14}$$

For the determinant in (2.14), we can extract $[\lambda_1 - \lambda_{i+1} + i] q^{\lambda_{i+1} - i}$ from the ith row and extract $q^{j - \nu_j} (\lambda_1 - \nu_j + j - 1 \downharpoonleft \mu_j - \nu_j)$ from the jth column by multilinearity for $1 \le i, j \le n-1$. Then (2.14), which is equal to (2.10), becomes

$$\left(\prod_{i=1}^{n-1} [\lambda_1 - \lambda_{i+1} + i] q^{\lambda_{i+1}} \right) \sum_{\substack{\nu \preceq \mu \\ l(\nu) < n}} \left(\prod_{j=1}^{n-1} q^{-\nu_j} \right) (\lambda_1 \downharpoonleft \mu / \nu) \cdot \det[(\lambda_{i+1} + n - i - 1 \downharpoonleft \nu_j + n - j - 1)]_{i,j=1}^{n-1}. \tag{2.15}$$

On the other hand, by Lemma 2.3 we have

$$\det[(\lambda_i+n-i\,\lfloor n-j)]_{i,j=1}^{n}$$
$$=\prod_{i=1}^{n-1}\left([\lambda_1-\lambda_{i+1}+i]q^{\lambda_{i+1}}\right)\det[(\lambda_{i+1}+n-i-1\,\lfloor n-j-1)]_{i,j=1}^{n-1}. \tag{2.16}$$

Combining (2.15) and (2.16) together, we then obtain (2.7), which implies Theorem 1.2. ∎

Theorem 1.1 can be recovered from Theorem 1.2 by setting $q=1$. To show that, we need the following result given in [6, Prop. 7.19.11].

Lemma 2.5. *Let* $|\lambda/\mu|=m$. *Then*

$$s_{\lambda/\mu}\left(1,q,q^2,\ldots\right)=\frac{\sum_T q^{\mathrm{maj}(T)}}{[m]!},$$

where T *ranges over all SYTs of shape* λ/μ, *and* $\mathrm{maj}(T)$ *is the major index of* T.

Proof of Theorem 1.1: Divide both sides of (1.2) by $(1-q)^{|\mu|}$ and then set $q=1$. Then the right-hand side of (1.2) becomes

$$\sum_{T\in\mathrm{RT}(\mu,n)}\prod_{u\in\mu}\left(\lambda_{T(u)}-c(u)\right). \tag{2.17}$$

Since

$$\left.\sum_T q^{\mathrm{maj}(T)}\right|_{q=1}=f^{\lambda/\mu},$$

when T ranges over all SYTs of shape λ/μ, we know by Lemma 2.5 that

$$\left.\frac{s_{\lambda/\mu}\left(1,q,q^2,\ldots\right)}{(1-q)^{|\mu|}s_\lambda\left(1,q,q^2,\ldots\right)}\right|_{q=1}$$
$$=\left.\frac{[m]!\sum_{T_1}q^{\mathrm{maj}(T_1)}}{(1-q)^{\mu}[m-k]!\sum_{T_2}q^{\mathrm{maj}(T_2)}}\right|_{q=1} \tag{2.18}$$
$$=\frac{(m)_k f^{\lambda/\mu}}{f^\lambda},$$

where T_1 and T_2 range over all partitions of shape λ/μ and λ, respectively. Combining (2.17) and (2.18) together, we then obtain Theorem 1.1. ∎

Corollary 2.6. *The rational function*

$$\frac{s_{\lambda/\mu}\left(1,q,q^2,\ldots\right)}{(1-q)^{|\mu|}s_\lambda\left(1,q,q^2,\ldots\right)}$$

is a Laurent polynomial in q *with nonnegative integer coefficients.*

Proof. Given $T \in \mathrm{RT}(\mu, n)$, if $\lambda_{T(u_0)} < c(u_0)$ for some $u_0 \in \mu$, then

$$\prod_{u \in \mu} \left[\lambda_{T(u)} - c(u)\right] = 0. \tag{2.19}$$

In fact, while u moves from right to left along rows of T, $\lambda_{T(u)}$ is weakly decreasing, and $c(u)$ is decreasing by 1 in each step. Let u_1 be the leftmost square in the row containing u_0. Since $\lambda_{T(u_1)} \geq c(u_1)$ and $\lambda_{T(u_0)} < c(u_0)$, we have $\lambda_{T(u_2)} = c(u_2)$ for some square u_2, which implies Equation (2.19).

On the other hand, by Theorem 1.2 we have

$$\frac{s_{\lambda/\mu}\left(1, q, q^2, \dots\right)}{(1-q)^{|\mu|} s_\lambda\left(1, q, q^2, \dots\right)} = \sum_{T \in \mathrm{RT}(\mu, n)} \prod_{u \in \mu} q^{1-T(u)} \cdot \frac{\left[\lambda_{T(u)} - c(u)\right]}{1-q}.$$

Then the result follows after omitting the sum terms that equal to 0 on the right-hand side. ∎

For the special case when $\mu = 1$, we give a simple formula for $s_{\lambda/1}/(1-q)s_\lambda$ in Corollary 2.7 below. Before giving a combinatorial proof of this result, we first introduce some notation.

The acronym SSYT stands for a semistandard Young tableau where 0 is allowed as a part. *Jeu de taquin* (jdt) is a kind of transformation between skew tableaux obtained by moving entries around, such that the property of being a tableau is preserved. For example, given a tableau T of shape λ, we first delete the entry $T(i, j)$ for some box (i, j). If $T(i, j-1) > T(i-1, j)$, we then move $T(i, j-1)$ to box (i, j); otherwise, we move $T(i-1, j)$ to (i, j). Continuing this moving process, we eventually obtain a tableau of shape $\lambda/1$. On the other hand, given a tableau of shape $\lambda/1$, we can regard $(0,0)$ as an empty box. By moving entries in a reverse way, we then get a tableau of shape λ with an empty box after every step. For more information about jdt, readers can refer to [6, Ch. 7, App. I].

The following result was first obtained by Kerov [2, Thm. 1 and (2.2)] (after sending $q \mapsto q^{-1}$) and by Garsia and Haiman [1, (I.15), Thm. 2.3] $\left(\text{setting } t = q^{-1}\right)$ by algebraic reasoning. For further information see [3, p. 9].

Corollary 2.7. *We have*

$$\frac{s_{\lambda/1}\left(1, q, q^2, \dots\right)}{(1-q) s_\lambda\left(1, q, q^2, \dots\right)} = \sum_{u \in \lambda} q^{c(u)}. \tag{2.20}$$

Proof. We define two sets in the following way:

$$T_{\lambda/1} = \{(T, k) \,|\, T \text{ is an SSYT of shape } \lambda/1, \text{ and } k \in \mathbb{N}\},$$

$$T_\lambda = \{(T, u) \,|\, T \text{ is an SSYT of shape } \lambda, \text{ and } u \in \lambda\}.$$

Since we can rewrite (2.20) as

$$s_{\lambda/1}\left(1, q, q^2, \dots\right) \cdot \sum_{i \geq 0} q^i = s_\lambda\left(1, q, q^2, \dots\right) \cdot \sum_{u \in \lambda} q^{c(u)},$$

it suffices to prove that there is a bijection $\varphi\colon \mathcal{T}_\lambda \to \mathcal{T}_{\lambda/1}$, say $\varphi(T,u) = (T_\varphi, k)$, such that $|T| + c(u) = |T_\varphi| + k$, where $|T|$ and $|T_\varphi|$ denote the sum of the entries in T and T_φ respectively.

We define φ in the following way. Given $(T,u) \in \mathcal{T}_\lambda$, let $k = T(u) + c(u)$. To obtain T_φ, we first delete the entry $T(u)$ from T, and then carry out the jdt operation. Since T is an SSYT, we have $k \geq 0$, and thus the definition is reasonable.

On the other hand, given $(T_\varphi, k) \in \mathcal{T}_{\lambda/1}$, we carry out the jdt operation to T_φ step-by-step in the reverse way. Denote by u_t the empty box and T_t the tableau obtained after t steps. If we get an SSYT by filling u_t with $k - c(u_t)$ in T_t, then we call u_t a *nice* box.

We first show that a nice box exists. For the sake of discussion, if (i, j) is not a box of a tableau T, then we define $T(i, j) = -\infty$ if $i < 0$ or $j < 0$, and $T(i, j) = +\infty$ if $i \geq 0$ and $j \geq 0$. The existence is obvious if the initial empty box u_0 is nice. If there is no integer t such that $k - c(u_t)$ is less than the adjacent entry left or not greater than the adjacent entry above in T_t, then by filling the last empty box u_{t_0} with $k - c(u_{t_0})$, we get an SSYT, which implies that u_{t_0} is a nice box. Otherwise, let t be the smallest integer such that $k - c(u_t)$ is less than the adjacent entry left or not greater than the adjacent entry above. Then we claim that u_{t-1} is a nice box. Assume that $u_t = (i, j)$. Since T_{t-1} and T_t satisfy the conditions of SSYT except for the empty box, we have $T_{t-1}(i, j-1) \leq T_{t-1}(i, j) \leq T_{t-1}(i-1, j+1)$ if $u_{t-1} = (i-1, j)$, and $T_{t-1}(i-1, j) < T_{t-1}(i, j) < T_{t-1}(i+1, j-1)$ if $u_{t-1} = (i, j-1)$. In the first case, we have $k - c(u_t) \leq T_{t-1}(i, j)$, thus $k - c(u_{t-1}) = k - c(u_t) - 1 < T_{t-1}(i, j) \leq T_{t-1}(i-1, j+1)$. In the latter one, we have $k - c(u_t) < T_{t-1}(i, j)$, so $k - c(u_{t-1}) = k - c(u_t) + 1 \leq T_{t-1}(i, j) < T_{t-1}(i+1, j-1)$. By assumption, $k - c(u_{t-1})$ is not less than the entries left and greater than the entry above in T_{t-1}. Therefore, we get an SSYT by filling u_{t-1} with $k - c(u_{t-1})$ in T_{t-1} in both cases, which completes the proof of the existence.

Next we show the uniqueness of the nice box. Let $u = (i, j)$ be the first nice box and let T be the corresponding SSYT. If there exists another nice box $u' = (i', j')$, and T' is the corresponding SSYT, then we have $i' \geq i$ and $j' \geq j$. Since T' is an SSYT, we must have $T'(i', j') \geq T'(i, j) + i' - i$. Since T is an SSYT, we have $T'(i, j) > k + i - j$ when $j' = j$, and $T'(i, j) \geq k + i - j$ when $j' > j$. In either case we get a contradiction, since $T'(i', j') = k + i' - j'$ by the definition of T'. ∎

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